An Algorithm to compute a T-transitive approximation of a reflexive and symmetric fuzzy relation

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Abstract

There are fast known algorithms to compute the transitive closure of a fuzzy relation, but there are only a few different algorithms that compute T-transitive low approximations of a fuzzy relation. A fast method to compute a T-transitive low approximation of a reflexive and symmetric fuzzy relation is given for any continuous t-norm, spending $O(n^3)$ time.

Keywords: Fuzzy relation; transitive closure; low T-transitive approximation; T-indistinguishability.

1 Introduction

Clustering applications often need to compute fuzzy equivalence relations from reflexive and symmetric fuzzy relations. Similarities and T-indistinguishabilities fuzzy relations are used to represent the concept of equality and neighbourhood, generalising the classical equivalence relations. Some of its applications are found in some classification and clusterization methods, allowing to distinguish and 'classify' objects. All the α-cuts of a Min-transitive fuzzy relation (which are the classical relations defined by the couples of elements that have a degree of relationship greater or equal than α) are transitive relations [Zadeh; 1971]. This property does not stand for any other t-norm different from the minimum. Only the t-norm minimum satisfies that always the Min-transitive closure operation commutes with the transitive closure of the α-cut classical relation.

The algorithms that compute T-transitive fuzzy relations are useful in several braches of artificial intelligence. The transitivity property of fuzzy relations can be understood as a threshold of a degree of relation (for example equality) between two elements, when a degree of relation between those elements with a third one of an universe in discourse is known. The classical concept of transitivity is generalised in fuzzy logic by the T-transitivity property of a fuzzy relations, where T is a triangular t-norm.

Sometimes it is important to held the T-transitivity property of a fuzzy relation, and the colleted knowledge is represented in a non T-transitive relation that can be changed by another T-transitive relation as close as possible. Given a fuzzy relation R (or a directed graph) it is well known the concept of T-transitive closure R^T, which is the lowest T-transitive fuzzy relation that contains a given fuzzy relation.

On the last years some authors have developed new and fast algorithms, especially to compute the Min-transitive closure of fuzzy relations, however there exists a few methods to compute several low T-transitive approximation of a given fuzzy relation, but in general it does not exist a unique low T-transitive approximation.

This paper proposes a fast method to compute a T-transitive low approximation of a reflexive and symmetric fuzzy relation, for any t-norm T.

2 Preliminaries

Let E = {e_1, ..., e_n} be a finite set.

Given a fuzzy relation R: E×E → [0, 1] it is called e_{ij} to the relation degree value for elements e_i and e_j in E. So $e_{ij} = R(e_i, e_j)$.

A fuzzy relation R is **reflexive** if $e_{ii} = 1$ for $1 \le i \le n$.

The relation R is **symmetric** if $e_{ij} = e_{ji}$ for $1 \le i, j \le n$.

A binary operation T: [0, 1] × [0, 1] → [0, 1] is a **t-norm** [Schweizer & Sklar; 1983] if it satisfies the following axioms:

1. T(1, x) = x
2. T(x, y) = T(y, x)
3. T(x, T(y, z)) = T(T(x, y), z)
4. If x ≤ x' and y ≤ y' then T(x, y) ≤ T(x', y').

Definition 2.1. Let T be a triangular t-norm. A fuzzy relation R: E×E → [0, 1] is **T-transitive** if

T(R(a, b), R(b, c)) ≤ R(a, c) for all a, b, c in E.

So, $T(e_{ik}, e_{kj}) \le e_{ij}$ for all i, j, k from 1 to n.

Definition 2.2. A reflexive and symmetric fuzzy relation is called a **proximity** relation. A **similarity** is a reflexive, symmetric and min-transitive fuzzy relation.

A **T- indistinguishability** is a reflexive, symmetric and T-transitive fuzzy relation.

Definition 2.3 The relation A **includes** the relation B, (and it is denoted A ⊒ B) if $a_{ij} \ge b_{ij}$ for all i, j.

Definition 2.4. The **residual implicator** J^T of a t-norm T, or quasi-inverse of T, is the binary operation J^T: $[0, 1]^2 \to [0, 1]$, defined by

$J^T(x, y) = \mathrm{Sup}\{z \in [0, 1]; T(x, z) \le y\}$.

In particular:

$$J^{Min}(x,y) = \begin{cases} 1, & if\ x \le y, \\ y, & if\ x > y \end{cases}$$

$$J^{Prod}(x,y) = \begin{cases} 1, & if\ x \le y, \\ \dfrac{y}{x}, & if\ x > y \end{cases}$$

$$J^W(x,y) = \min\{1 - x + y, 1\}$$

For a left-continuous t-norm T, and any x, y ∈ [0, 1], it is known that $T(x, J^T(x, y)) \le y$, i.e. $J^T(x, y)$ is the greatest solution, z, of the inequality T(x, z) ≤ y. In case of a continuous t-norm T and y ≤ x, then $J^T(x, y)$ is the greatest solution, z, of the equation T(x, z) = y.

Lemma 1. If the rows and corresponding columns of a fuzzy T-indistinguishability relation R are permuted, the resulting relation is also a fuzzy T-indistinguishability relation denoted $P_\pi(R)$, where π is a permutation.

Proof. It is trivial. $P_\pi(R)$ is reflexive and symmetric. If $e_{ij} \ge T\{e_{ik}, e_{kj}\}$ for all i, j, k then $e_{rs} = e_{\pi(i)\,\pi(j)} \ge T\{e_{\pi(i)\,k}, e_{k\pi(j)}\} = T\{e_{rk}, e_{ks}\}$ for all r, s, k. □

Example 1. Let π be the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$. Some examples of T-indistinguishabilities A and their permutated T-indistinguishabilities $P_\pi(A)$ are the following.

$$T = \text{Min},\ A = \begin{pmatrix} 1 & 0.9 & 0.5 & 0.6 \\ 0.9 & 1 & 0.5 & 0.6 \\ 0.5 & 0.5 & 1 & 0.5 \\ 0.6 & 0.6 & 0.5 & 1 \end{pmatrix}$$

$$\text{and } P_\pi(A) = \begin{pmatrix} 1 & 0.9 & 0.6 & 0.5 \\ 0.9 & 1 & 0.6 & 0.5 \\ 0.6 & 0.6 & 1 & 0.5 \\ 0.5 & 0.5 & 0.5 & 1 \end{pmatrix}$$

$$T = \text{Prod},\ A = \begin{pmatrix} 1 & 0.9 & 0.5 & 0.6 \\ 0.9 & 1 & 0.5 & 0.54 \\ 0.5 & 0.45 & 1 & 0.3 \\ 0.6 & 0.54 & 0.3 & 1 \end{pmatrix}$$

$$\text{and } P_\pi(A) = \begin{pmatrix} 1 & 0.9 & 0.6 & 0.5 \\ 0.9 & 1 & 0.54 & 0.45 \\ 0.6 & 0.54 & 1 & 0.3 \\ 0.5 & 0.45 & 0.3 & 1 \end{pmatrix}$$

$$T = W,\ A = \begin{pmatrix} 1 & 0.9 & 0.5 & 0.6 \\ 0.9 & 1 & 0.4 & 0.5 \\ 0.5 & 0.4 & 1 & 0.1 \\ 0.6 & 0.5 & 0.1 & 1 \end{pmatrix}$$

$$\text{and } P_\pi(A) = \begin{pmatrix} 1 & 0.9 & 0.6 & 0.5 \\ 0.9 & 1 & 0.5 & 0.4 \\ 0.6 & 0.5 & 1 & 0.1 \\ 0.5 & 0.4 & 0.1 & 1 \end{pmatrix}$$

3 Construction of a fuzzy T-indistinguishability relation from two T-subindistinguishabilities.

A point of view of the proposed algorithm in this paper is the solution for the problem of constructing a T-indistinguishability with some properties from two T-indistinguishabilities. The problem can also be seen in a reverse way, trying to decompose a T-indistinguishability in two T-subindistinguishabilities.

The solution given in this section introduces the algorithm that constructs a low T-transitive approximation of a reflexive and symmetric fuzzy relation, which is known to be a T-indistinguishability.

Let C and D be two T-indistinguishability fuzzy relations having dim(C) = n_1, dim(D) = n_2.

The method tries to construct a T-indistinguishability relation R(F; C, D) with the following form:

R(F; C, D) = $\begin{pmatrix} \boxed{C} & \boxed{F^T} \\ \boxed{F} & \boxed{D} \end{pmatrix}$ having dim(R) = $n_1 + n_2 = n$.

A method for computing the values e_{ij} in F, is the assignation of the $n_1 \times n_2$ values in any order, but obtaining them from an interval [a, b] where

b = min{min(C), min(D), min(values already generated in F)}.

a = max $\{T(e_{ik}, e_{kj})\}$ for all k such that $e_{ik} \in$ F is already computed.

F^T is the symmetric part of the computed F.

So the computed values in F satisfy two conditions:

Condition 1: $e_{ij} \leq$ min{min(C), min(D), min(values already computed in F)}. (1)

Condition 2: $e_{ij} \geq$ max $\{T(e_{ik}, e_{kj})\}$ for all k such that $e_{ik} \in$ F is already computed. (2)

Lemma 2. If C and D are two T-indistinguishabilities fuzzy relations, then the fuzzy relation R(F; C, D) given by the previous method is also a T- indistinguishability fuzzy relation.

Proof. As C and D are symmetric and reflexive fuzzy relations, R is also symmetric and reflexive fuzzy relation.

It is proved that $e_{ij} \geq \max_k\{T\{e_{ik}, e_{kj}\}\}$ for all i, j, k to prove the T-transitivity property of R in four separated cases for the values e_{ij} in C, D, F and F^T.

Case 1: $e_{ij} \in$ C, it is, if $1 \leq i \leq n_1$, $1 \leq j \leq n_1$.

As C is T-transitive, $e_{ij} \geq \max_k\{T\{e_{ik}, e_{kj}\}\}$ for all i, j, $k \leq n_1$.

If $k > n_1$ then $e_{ik} \in F^T$ and $e_{kj} \in$ F. By condition 1, the values of e_{ij} in F and F^T are lower than the values in C, so $e_{ij} \geq \min\{e_{ik}, e_{kj}\} \geq T\{e_{ik}, e_{kj}\}$ for any t-norm.

Case 2: $e_{ij} \in$ D, it is, $n_1 < i \leq n$, $n_1 < j \leq n$.

As D is T-transitive, $e_{ij} \geq \max_k\{T\{e_{ik}, e_{kj}\}\}$ for all i, j, $k > n_1$.

If $k \leq n_1$ then $e_{ik} \in$ F and $e_{kj} \in F^T$. By condition 1, the values of e_{ij} in F and F^T are lower than values in D, so $e_{ij} \geq \min\{e_{ik}, e_{kj}\} \geq T\{e_{ik}, e_{kj}\}$ for any t-norm..

Case 3: $e_{ij} \in$ F, it is, $n_1 < i \leq n$, $1 \leq j \leq n_1$.

By the application of condition 2 in the construction method, if $k \leq n_1$ then $e_{ij} \geq$ max $\{T(e_{ik}, e_{kj})\}$ for all k.

Case 4: $e_{ij} \in F^T$, it is $i \leq n_1 < j$. The proof is similar to the one in case 3.

So the fuzzy relation R(F; C, D) is a T-indistinguishability. □

Example 2. Let C = $\begin{pmatrix} 1 & 0.9 \\ 0.9 & 1 \end{pmatrix}$ and D = (1) two reflexive and symmetric T-transitive fuzzy relations for any t-norm T.

The fuzzy relation R(F; C, D) is constructed assigning values to a_{31} and a_{32} verifying conditions 1 and 2.

Let *T* = *Min*. The values in F can be chosen in the interval [0, 0.9]. For example let a_{31}=0.6 is taken then R(F; C, D) = $\begin{pmatrix} 1 & 0.9 & 0.6 \\ 0.9 & 1 & 0.6 \\ 0.6 & 0.6 & 1 \end{pmatrix}$, which is a Min-indistinguishability.

Let *T* = *Prod*. The first value of F, for example a_{31}, must be chosen in the interval [0, 0.9]. For example let a_{31}=0.6 is taken, then by condition (2) a_{32} must be chosen greater than 0.6 * 0.9 = 0.54 and by (1) it should be taken lower than 0.6. So a_{32} can be chosen in the interval [0.54, 0.6]. For example, let a_{32} be 0.54, then

$$R(F; C, D) = \begin{pmatrix} 1 & 0.9 & 0.6 \\ 0.9 & 1 & 0.54 \\ 0.6 & 0.54 & 1 \end{pmatrix}$$ is a Prod-indistinguishability.

Let $T = W$ be the Łukasiewicz t-norm. R(F; C, D) is generated assigning values to a_{31} and a_{32}.

By condition (1) a_{31} or a_{32} should belong to the interval [0, 0.9]. Suppose a_{31}=0.6, then by condition (2), $a_{32} \geq W\{a_{31}, a_{12}\} = a_{32} + a_{12} - 1 = 0.6 + 0.9 - 1 = 0.5$, and by condition (1) a_{32} should be in the interval [0.5, 0.6]. For example, let a_{32} be 0.5, then

$$R(F; C, D) = \begin{pmatrix} 1 & 0.9 & 0.6 \\ 0.9 & 1 & 0.5 \\ 0.6 & 0.5 & 1 \end{pmatrix}$$ is a W-indistinguishability.

The Min-indistinguishabilities fuzzy relations are T-indistinguishabilities fuzzy relations for any t-norm T. As W ≤ Prod ≤ Min, Min-indistinguishabilities and Prod-indistinguishabilities are also W-indistinguishabilities fuzzy relations.

Lemma 3. If B is a T-indistinguishability fuzzy relation then there exits a permutation P_π and two T-indistinguishability fuzzy relation C and D, such that $B = P_\pi(R(F; C, D))$.

Proof:

Let B be a fuzzy relation with dimension n. The decomposition of B can be done choosing the boxes in the last step of the T-indistinguishabilities construction method. The proof that those boxes are T-indistinguishabilities is trivial by construction. □

Lemmas 2. and 3. provide an idea to build any given T-indistinguishability by making boxes of subindistinguishabilities (or adding elements to the universe of the cartesian product) with two disjoint subrelations that are also T-indistinguishabilities and choosing the elements of F. In every step it is given a T-indistingushability of a greater dimension that is a subrelation of the final T-indistinguishability.

4 Algorithm to compute a low T-transitive approximation of a proximity fuzzy relation

Let A be a proximity fuzzy relation.

Step 1. Set B initially blank

Step 2. Let U(A) be the set of elements of the upper triangular matrix of A sorted in a decreasing order.

Step 3. Set $b_{ii} = 1$, for all $1 \leq i \leq n$.

Step 4. While there is a blank in B do

Let a_{rs} be the highest value of the list U(A).

If b_{rs} is blank,

$I = \{j; b_{rj}$ is not blank in B$\}$ and $I' = \{i; b_{is}$ is not blank in B$\}$.

$H(A) = \{a_{ij}, i\in I, j\in I'$ sorted in increasing order$\}$.

While H(A) is not empty

Let a_{ij} be the smallest element not computed in H(A)

Set $b_{ij} = b_{ji} = \min\{a_{ij}, \min_k\{J^T(b_{ik}, b_{kj}), J^T(b_{jk}, b_{ki})\}$.

Delete the top element from H(A).

Delete the top element from U(A).

Example 3: Given the following approximation fuzzy relation:

$$A = \begin{pmatrix} 1 & 0.9 & 1 & 0.6 \\ 0.9 & 1 & 0.6 & 1 \\ 1 & 0.6 & 1 & 0.8 \\ 0.6 & 1 & 0.8 & 1 \end{pmatrix}$$

The algorithm is applied to compute its low T-transitive approximation B, where T is the Min, Prod and W triangular norm.

Step 1: Set B to be blank

Step 2: Let U(A) be the set of elements of the upper triangular matrix of A sorted in a decreasing order. U(A) = {1; 1; 0.9; 0.8; 0.6; 0.6}.

Step 3: Set b_{ii} =1 for all i.

Step 4: We take the greatest value of U(A), $a_{13} = 1$. Let I = {j; b_{1j} that are not blank values in matrix B} = {1} and let I' = {i; b_{i4} that are not blank in matrix B} = {3). The values $b_{31} = b_{13} = a_{13} = 1$ are inserted in B.

The following greatest element in U(A) is 1 = a_{24}. I = {j; b_{2j} are not blank in B} = {2} and I' = {i; b_{i3} is not blank in B } = {4). The values b_{24} and b_{42} are inserted in B, having $b_{24} = b_{42} = 1$.

$$B = \begin{pmatrix} 1 & & 1 & \\ & 1 & & 1 \\ 1 & & 1 & \\ & 1 & & 1 \end{pmatrix}$$

The following non blank greatest element in U(A) is $0.9 = a_{12}$. I = {j; b_{1j} are not blank in B} = {1, 3} and I' = {i; b_{i3} is not blank in B } = {2, 4}. The values b_{12}, b_{13}, b_{42}, b_{43} and its symmetric values are inserted in B. The list H(A) = {a_{12}, a_{42}, a_{43}, a_{13}}, but $a_{14} = a_{32} = 0.6 < a_{34} = 0.8 < a_{12} = 0.9$. If the first element in H(A) is $a_{14} = 0.6 = b_{14}$, then $b_{32} = b_{23} = 0.6$, $b_{34} = b_{43} = \min\{a_{34}, \min\{J^T(b_{31}, b_{14}), J^T(b_{32}, b_{24}\}\} = \min\{0.8, \min\{J^T(1, 0.6\}, J^T(1, 0.6)\}\} = 0.6$, $b_{12} = b_{21} = \min\{a_{12}, \min\{J^T(b_{13}, b_{32}\}, J^T(b_{32}, b_{31}\}, J^T(b_{14}, b_{42}\}, J^T(b_{42}, b_{14}\}\} = \min\{0.9, \min\{1, 0.6, 0.6, 1\}\} = 0.6$, so:

$$B_T = \begin{pmatrix} 1 & 0.6 & 1 & 0.6 \\ 0.6 & 1 & 0.6 & 1 \\ 1 & 0.6 & 1 & 0.6 \\ 0.6 & 1 & 0.6 & 1 \end{pmatrix}.$$

5 Conclusions

Its is given a algorithm to compute a low T-transitive approximation of a reflexive and symmetric fuzzy relation, given any triangular norm T.

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References

[1]. D. Boixader, On the relationship between T-transitivity and approximate equality Fuzzy Sets and Systems 33 (2003) 6 – 69.

[2]. B. De Baets, H. De Meyer, Transitive approximation of fuzzy relations by alternating closures and openings, Soft Computing 7 (2003) 210-219.

[3]. F. Esteva, P. Garcia, L. Godo, R. O. Rodriguez, Fuzzy approximation relations, modal structures and possibilistic logic, Mathware and Soft Computing 5 (2-3) (1998) 151-166.

[4]. L. Garmendia, C. Campo, S. Cubillo, A. Salvador, A Method to Make Some Fuzzy Relations T-Transitive. International Journal of Intelligence Systems. Vol. 14, Nº 9, (1999) 873 – 882.

[5]. L. Garmendia, A. Salvador, On a new method to T-transitivize fuzzy relations, Information Processing and Management of Uncertainty in Knowledge - based Systems, IPMU 2000. (2000) 864 – 869.

[6]. L. Garmendia, A. Salvador On a new method to T-transitivize fuzzy relations, in Technologies for Constructing Intelligent Systems 2, Springer. Edited by Bouchon-Meunier, B., Gutierrez-Rios, J., Magdalena, L., Yager, R. R, (2000) 251 – 260.

[7]. H. Hashimoto, Transitivity of generalised fuzzy matrices, Fuzzy Sets and Systems. Vol. 17, no. 1, (1985) 83-90.

[8]. J. Jacas, J. Recasens, Fuzzy T-transitive relations: eigenvectors and generators, Fuzzy Sets and Systems 72 (1995) 147–154.

[9]. J. Jacas, J. Recasens, Decomposable indistinguishability operators, Proceedings of the Sixth IFSA Congress, Sao Paulo, 1995.

[10]. H. S. Lee, An optimal algorithm for computing the max–min transitive closure of a fuzzy similarity matrix , Fuzzy Sets and Systems 123 (2001) 129–136.

[11]. J. Montero, J. Tejada, On fuzzy transitivity, Instituto Nacional de Estadística 111 (1986) 49-58.

[12]. S. Ovchinnikov, Representations of Transitive Fuzzy Relations, in Aspects of Vagueness, H. J. Skala, S. Termini and E. Trillas (Eds.), Reidel Pubs. (1984) 105-118.

[13]. B. Schweizer, A. Sklar, Probabilistic Metric Spaces, North-Holland, New York, (1983).

[14]. E. Trillas, L. Valverde, An inquiry into indistinguishability operators, in Aspects of Vagueness, H. J. Skala, S. Termini y E. Trillas (Eds.), Reidel Pubs. (1984) 231-256.

[15]. L. Valverde., On the structure of F-indistinguishability operators, Fuzzy Sets and Systems 17 (1985) 313–328.

[16]. M. Wagenknecht, On transitive solutions of fuzzy equations, inequalities and lower approximation of fuzzy relations. Fuzzy Sets and Systems 75 (1995) 229-240.

[17]. L. Zadeh, A., Similarity relations and fuzzy orderings, Inform. Sci. 3 (1971) 177–200.