Fuzzy Portfolio Selection: a comparative study

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Abstract

In this paper, we carry out the numerical study of a fuzzy portfolio selection model where the objective is to minimize the downside risk and the rates of returns on securities are approximated by means of LR-fuzzy numbers of trapezoidal form. Data from 96 securities over 195 month are used to compare the selected portfolios with a simple utility function and with the out-of-sample data as well as to investigate their true performance.

Keywords: Fuzzy returns, Interval-valued expectation, Downside risk, Possibilistic mean-variance, Investment analysis.

1 Introduction

The classical portfolio selection problem was formulated by Markowitz in the 1950s as a quadratic programming problem (MV) in which the risk variance is minimized and the investment diversification is treated in computational terms [6]. It is well known that the portfolio models initiated by Markowitz gave rise to a variety of regression models, included the extensively used CAPM. Different models coexist to select the best portfolio according to their respective objective functions in the framework of risk-return trade-off.

The portfolio selection problem deals with finding an optimal investment strategy to form a satisfying portfolio, taking into account the uncertainty involved in the behavior of financial markets. The investors are assumed to strike a balance between maximizing the return and minimizing the risk of their investment.

2 Portfolio selection models

In the classic probabilistic approach the return of each asset is a random variable and the variance or standard deviation is a measure of risk. The mean-variance formulation is as follows:

$$
\begin{aligned}
(MV)\quad & Min\, x^T Q x \\
& s.t. \sum_{j=1}^{n} \bar{r}_j x_j \geq \rho \\
& \sum_{j=1}^{n} x_j = 1 \qquad\qquad (1)\\
& 0 \leq x_j \leq u_j \quad j = 1,\ldots,n
\end{aligned}
$$

where x_j is the portfolio allocation of security j, and n is the number of securities. $\bar{r}_j$ is the average return in security j over the entire period T, $Q=[\sigma_{ij}]$ is the variance-covariance matrix and $\sigma_{ij} = \left(1/T\right)\sum_{t=1}^{T}(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)$ is the covariance of securities i and j; being r_{jt} the return of security j over period t. ρ is the minimum expected return required by a particular investor. The non-negative constraint over the decision variables does not allow short selling of securities and the allocation should not exceed an upper bound u_j.

This classical MV model is always valid if the expected return is multivariate normally distributed and the investor is averse to risk and always prefers lower risk. It must be pointed out that for a large number of securities it might take some time to find optimal solutions, because the calculation of the variance-covariance matrix is time consuming.

Fuzzy Set Theory has been widely used to solve many practical problems including financial risk management. Let us recall some portfolio selection models based on fuzzy decision theory. Tanaka and Guo [7] use possibility distributions to model uncertainty on the expected returns. In Inuiguchi and Ramik [4], the portfolio selection problem exemplifies the advantages and disadvantages of different fuzzy mathematical programming approaches. Carlsson, Fullér and Majlender [3] present an algorithm for finding an optimal portfolio with highest utility score under possibility distributions. They used the following utility score for a risky portfolio P with a rate of return r_P:

$$U(P) = \overline{M}(r_P) - 0.005A\overline{\sigma}^2(r_P) \qquad (2)$$

where A is an index of the investor's risk aversion (A≅2.46), and $\overline{M}(r_P)$ and $\overline{\sigma}^2(r_P)$ are the possibilistic (crisp) mean value and variance of r_P, respectively.

In León, Liern, Marco, Segura and Vercher [5], the rates of returns are approximated by means of fuzzy numbers and the risk is modeled with a downside risk function. Our proposal does not require the estimation of the joint distribution of asset returns, but instead we approach the decision problem by using some fuzzy tools. Firstly, we work on the assumption that the uncertainty in the returns is modeled by means of fuzzy quantities, $\widetilde{R}_j$, j=1,…,n. Secondly, we calculate the mean interval and the fuzzy downside risk for a given portfolio $P(x) = \{x_1, x_2, \ldots, x_n\}$ and find the optimum of the next fuzzy portfolio selection problem:

$$\begin{aligned} &Min \quad \widetilde{w}(P(x)) \\ &s.t. \quad \widetilde{E}(\sum_{j=1}^{n} \widetilde{R}_j x_j) \geq \rho_0 \\ &\qquad \sum_{j=1}^{n} x_j = 1 \\ &\qquad l_j \leq x_j \leq u_j, \quad j = 1,\ldots,n \end{aligned} \qquad (3)$$

where ρ_0 is the given total return that must be achieved. The fuzzy downside risk $\widetilde{w}(P(x))$ with respect to the mean interval is defined by the mean semi-absolute deviation, which only penalizes the negative deviations of the expected return:

$$\widetilde{w}(P) = \widetilde{E}\left(\max\left\{ 0, \widetilde{E}\left(\sum_{j=1}^{n} \widetilde{R}_j x_j \right) - \sum_{j=1}^{n} \widetilde{R}_j x_j \right\} \right) \qquad (4)$$

Different definitions of the average of a fuzzy number can be used. Dubois and Prade [1] introduced the mean value of a fuzzy number, $\widetilde{E}(\widetilde{R}_j)$, as a closed interval bounded by the expectations calculated from its lower and upper probability mean values. Alternatively, Carlsson and Fullér [2] defined the interval-valued possibilistic mean of a fuzzy number, $M(\widetilde{R}_j)$, which is consistent with the extension principle and is also based on the set of level cuts. They showed the relationship between these two interval-valued expectations for LR-fuzzy numbers with strictly decreasing reference functions, that is $M(\widetilde{R}_j) \subset \widetilde{E}(\widetilde{R}_j)$. In our approach we will use the definition provided by Dubois and Prade [1].

All the above models do not consider the transaction costs, but if the number of securities that will be included in the portfolio is large, this cost increases. In fact, given the transaction costs it might be unprofitable to split the budget into many small blocks of shares. If we reformulate the above problems as integer, such as bounds on the number of selectable shares in the portfolio, the difficulty of resolution increases dramatically. Then, the application of Soft Computing techniques provides solutions than may be advantageous to the decision maker (see, for instance, Verdegay [8]).

3 Data and computational results

In this paper we carry out numerical experiments of our fuzzy model (3) by using securities from the Spanish Stock Market. In particular, we have considered the weekly returns on 96 assets traded in the Madrid Stock Exchange between January 2001 and September 2004. We have taken the observations of the Wednesday prices as an estimate of the weekly prices. Hence, the return on the security j during the week t is defined by means of the following formulae:

$$r_{jt}=(p_{(j+1)t}-p_{jt})/p_{jt} \qquad (5)$$

where p_{jt} is the price of the security j on the Wednesday of the t^{th} week, for t=1,...,T.

Although we want to include all shares traded in Madrid, we excluded some of them for the following reasons: some companies were not listed at the starting period, but entered at different dates afterwards, and others left out the market for various reasons over the examined period. It is important to mention though that none of the most traded shares are excluded.

We compare the numerical performance of our fuzzy model (3) with that of the Carlsson; Fullér and Majlender (2) possibilistic approach to selecting portfolios with highest utility score and with the portfolios provided by the probabilistic mean-variance model (1).

For the probabilistic model we have estimated the average vector of returns and the elements of the variance-covariance matrix through historical data. On the other hand, if we consider the return observations as a sample, its percentiles inform us about the possibility distribution of the returns, then we have decided to set the core of the trapezoidal fuzzy number as an interval $[P_{25}, P_{75}]$ and the quantities P_{25}-P_5 and P_{95}-P_{75} as the left and right spreads, where P_k is the k-th percentile of the sample.

In order to test the performance of the three models for a short- and medium-term we have consider two experiments:

(I) using weekly returns between January 2001 and March 2004 (T=170 observations) for selecting the portfolios, while the observations until September 2004 were left out to test the true performance;

(II) using the data till June 2004 (T=182 observations) and keeping the last 13 observations for checking the obtained returns.

We assume that the investors wish that no share will receive more than 25% of their budget. In addition, the short selling is not allowed.

Table 1 and Table 2 show the returns and risk measures for the optimal portfolios selected by the three models for experiments I and II, respectively. For each portfolio we have calculated all mean and risk measures, also for different values of ρ and ρ_0, for the MV model and the fuzzy model. It must be notice the low values for all the respective means. In fact it is not possible to increase a lot these parameter values for models (1) and (3) because those formulations would lead to an infeasible instance.

Table 1: Mean and risk measures of the selected portfolios for experiment I

	MV Model ρ=0	ρ=0,0075	Fuzzy Model ρ0=0	ρ0=0,0075	Model (2)
#securities	27	8	4	49	4
Probab. variance	0,000078	0,000370	0,000140	0,000257	0,000836
probab. E(P)	0,003966	0,007500	0,004822	0,004895	0,006050
U(P)	0,004008	0,007362	0,005199	0,006000	0,009120
possib. Mean	0,004014	0,007373	0,005203	0,006014	0,009142
Possib. Variance	0,000452	0,000923	0,000273	0,001091	0,001815
w(P(x))	0,049716	0,070834	0,038732	0,077053	0,099440
fuzzy mean	0,004540	0,008697	0,005778	0,007500	0,012344

Notice that the portfolios selected for fuzzy models usually involve fewer non-zero components and hence reduce the number of small transactions that the MV model solutions imply.

Table 2: Mean and risk measures of the selected portfolios for experiment II

	MV Model ρ=0	ρ=0,0025	Fuzzy Model ρ0=0	ρ0=0,0075	Model (2)
#securities	27	27	4	5	4
Probab. variance	0,000075	0,000075	0,000136	0,000199	0,000793
probab. E(P)	0,003806	0,003791	0,004603	0,005166	0,005557
U(P)	0,003677	0,003666	0,004693	0,006421	0,008916
possib. Mean	0,003682	0,003672	0,004697	0,006427	0,008938
Possib. Variance	0,000433	0,000434	0,000274	0,000492	0,001743
w(P(x))	0,048509	0,048594	0,038739	0,052102	0,097468
fuzzy mean	0,004055	0,004049	0,005015	0,007500	0,012015

From the investor's point of view, the moments when the investments were made, March 2004 and June 2004 could broadly be described as being of neutral tendency. The highest utility levels U(P) is achieved by the portfolio provided by model (2).

4 Out-of-sample performance of portfolios

In this section we examine the true performance of the above portfolios during the following twelve weeks and six months of the investment.

It must be pointed out that none of these portfolio selection models claim to predict the future returns satisfactorily. But if the investors believed in these models, they should expect to receive a positive return, given the fact that all portfolios have very low values of their respective measures of risk.

We have computed the expected total a posteriori returns of all portfolios at the end of the investment period, if the investment had been recovered during any of the following twelve weeks for the first experiment. The results appear in Fig. 1. Notice that the more aggressive a portfolio, the higher its true losses.

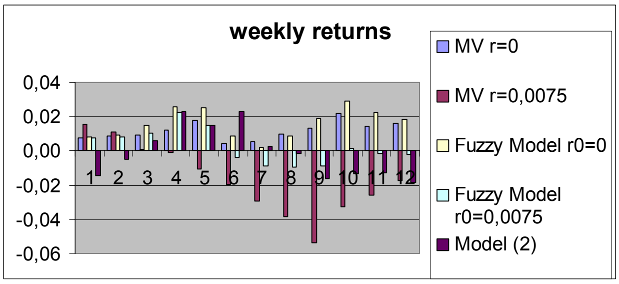

Figure 1: A posteriori returns between 07/04/2004 and 23/06/2004 for portfolios in Table 1

For this first scenario the best results are obtained for the portfolios selected with the lowest requirement for the expected returns, both for short term investment (between 1 and 12 weeks) and for medium-term periods of investment (up to six months). The results of the monthly investments appear in Fig. 2, for the first Wednesday of each month from March to September 2004.

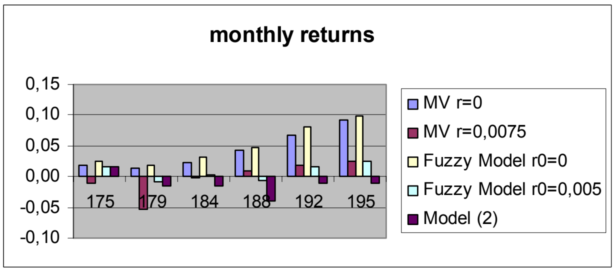

Figure 2: Monthly a posteriori returns for portfolios in Table 1

For the second experiment the portfolios were constructed at the end of June 2004. Fig. 3 shows that the best results correspond to the fuzzy model (3), the greatest profit being 0,076 in the last week (22/09/2004/). Again, greater values of the target rate provoke risky portfolios that would provide less profit at that period of time in this particular market.

In this period the a posteriori performance of the portfolios selected for the MV model is similar for different values of ρ, but that is not the case for the fuzzy model (3), for which the demand for more total expected return (ρ_0=0,0075) implies lowest profits.

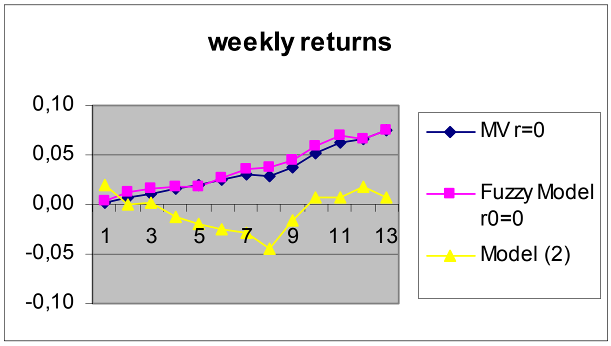

Figure 3: Weekly a posteriori returns for portfolios in Table 2

In summary, all models yield higher utility levels for high expected return values, but the out-of-sample performance is better for the lowest risky portfolios associated with the lowest mean returns.

Conclusions

We develop a fuzzy linear programming approach to the portfolio selection analysis on the Spanish stock market. This approach derives from linkages between rates of returns modeled by fuzzy numbers and a fuzzy downside risk function.

The comparative study with classical and fuzzy portfolio selection models is made with respect to the out-of-sample returns of the selected portfolios for different horizons, showing the good performance of our fuzzy approach.

An extension to the active management problem (a buy & hold policy) is possible with slight changes by repeating the selection process every month.

Acknowledgments

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