A reflection on the use of *And*

Sergio Guadarrama
Departamento de Informática
Universidad Rey Juan Carlos (Spain)
sergio.guadarrama@urjc.es

Eloy Renedo
Departamento de Inteligencia Artificial
Universidad Politecnica de Madrid (Spain)
erenedo@dia.fi.upm.es

Abstract

In this paper we want to explore some of the different uses in the language of the conjunction *and* and how they are related with the corresponding theoretical models [1, 8, 9].

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1 Introduction

In the language we have different uses of *and*, and therefore different meanings. However in the classical logic the models of *and* are independent of its use. In this paper it is proposed a broader framework in order to capture different uses of *and* ([5, 6]). That is, models with properties incorporating features proper of each use.

In the first section there are set out the needed definitions: sets, operations, relation, and so on. In the following section, a list of definitions of different uses of *and* with some instructive examples extracted from the dictionaries are shown. Finally there are described "coherent" models, that is, models that capture the conjunctive nature of the *and*'s uses.

2 Definitions

Given a universe U of elements, a relation R is a subset of $U \times U$, that is, $R \subseteq U \times U$.

Definition 2.1 *Given a universe U and a binary operation $* : U \times U \to U$, the $*$-relation R^l_* left induced by the operation $*$, is defined by*

$$(a, b) \in R^l_*, \text{ if it exists } c \in U, \text{,such that, } a = b * c$$

or equivalently

$$\forall\, a, b \in U : (a * b, a) \in R^l_*.$$

The $*$-relation R^r_* right induced by the operation $*$ is defined by

$$(a, b) \in R^r_*, \text{ if it exists } c \in U, \text{such that, } a = c * b$$

or equivalently

$$\forall\, a, b \in U : (a * b, b) \in R^r_*.$$

Remark 2.2 *If the operation $*$ is commutative, then the relations R^l_* and R^r_* induced by $*$ are identical, and we will denote it by $R_* = R^l_* = R^r_*$.*

Example 2.3 *Let U be [0,1] and $*$ be min, then the left induced $*$-relation R^l_{min} is*

$$(a, b) \in R^l_{min} \text{ if } \exists c \in [0, 1]; a = min(b, c) \Leftrightarrow a \leq b,$$

that coincides with the right induced relation $R^r_{min} = R^l_{min} = R_{\leq}$, because the min is commutative.

Example 2.4 *Let U be $\mathbb{Z}^+$, and $*$ the mod operation. The left induced relation R^l_{mod} in $\mathbb{Z}^+$ is:*

$$(a, b) \in R^l_{mod} \text{ if } \exists c \in \mathbb{Z}^+; a = \text{mod}(b, c) \Leftrightarrow a \leq b.$$

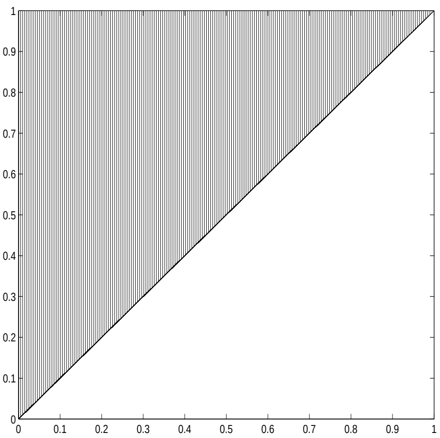
Although mod operation is not commutative, the right induced relation R^r_{mod} is equal to $R^l_{mod} = R_{\leq}$.

Example 2.5 *Let* $(U, \cdot, +, ')$ *be a boolean algebra and* $*$ *the non commutative operation (material implication)* $*(a,b) = a' + b$*. Then, left and right relations come given by* $a \geq b'$ *and* $a \geq b$ *respectively.*

Definition 2.6 *Given a poset* L *equipped with an order* $\preceq$ *we can define the relations* $E_{\preceq}$ *and* $E_{\succeq}$*, such that,* $(x,y) \in E_{\preceq} \Leftrightarrow x \preceq y$*, and,* $(x,y) \in E_{\succeq} \Leftrightarrow y \preceq x$*, then a binary operation* $*$ *in* $(L, \preceq)$ *is:*

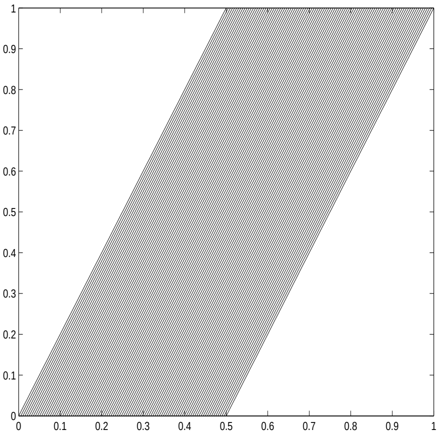
- *a left conjunction if:*
$$E_{\preceq} \subset R^l_*$$
- *a right conjunction if:*
$$E_{\preceq} \subset R^r_*$$
- *a conjunction if:*
$$E_{\preceq} = R^l_* = R^r_*$$
- *a left disjunction if:*
$$E_{\succeq} \subset R^l_*$$
- *a right disjunction if:*
$$E_{\succeq} \subset R^r_*$$
- *a disjunction if:*
$$E_{\succeq} = R^l_* = R^r_*$$

Example 2.7 *Let it be* $[0,1]$ *endowed with the usual order* $([0,1], \leq)$ *and* $* = T$*, a continuos t-norm ([4, 10])in* $[0,1]^{[0,1]^2}$*, then the inducted relation by* T *is a conjunction:*



R_T

With $* = am$*, the arithmetic mean operation, the inducted relation is not a conjunction:*



R_{avg}

Remark 2.8 *In any lattice* $(L, \preceq, \cdot, +)$*, the operator* $\cdot$ *is a conjunction and the operator* $+$ *is a disjunction since for all* $x, y \in L, x = x \cdot y, y = x + y$ *is equivalent to* $x \preceq y$*, and therefore,* $E_{\preceq} = E.$ *and* $E_{\succeq} = E_+$*.*

In what follows, given an universe U and a lattice of truth values $(L, \preceq, \cdot, +)$, any mapping $\mathcal{T} : U \to L$ that assign to each element of U a truth value of L we will name as a truth assignment.

Definition 2.9 *Given an universe* U *and a relation* R *on* $U \times U$*, we can define the mapped relation* E_R *on* $L \times L$ *through a truth assignment* $\mathcal{T}$ *as:*
$$(a,b) \in R \Rightarrow (\mathcal{T}(a), \mathcal{T}(b)) \in E_R$$
And given an operation $*$ *on* U*, with the induced relations* R^l_* *and* R^r_**, we can define the left and the right mapped relations* E^l_* *and* E^r_* *through a truth assignment* $\mathcal{T}$ *as:*
$$(a,b) \in R^l_* \Rightarrow (\mathcal{T}(a), \mathcal{T}(b)) \in E^l_*$$
$$(a,b) \in R^r_* \Rightarrow (\mathcal{T}(a), \mathcal{T}(b)) \in E^r_*$$

Definition 2.10 *Given an universe* U *and a truth assignment* $\mathcal{T}$*, we can say that an operation* $\& : U \times U \to U$ *behaves as:*

- *a left* $\mathcal{T}$*-conjunction if its left mapped relation* $E^l_{\&}$ *is a left conjunction on* $(L, \preceq)$
$$E_{\preceq} \subseteq E^l_{\&}$$

- *a right $\mathcal{T}$-conjunction if its right mapped relation $E^l_\&$ is a right conjunction on $(L, \preceq)$*

$$E_\preceq \subseteq E^r_\&$$

- *a $\mathcal{T}$-conjunction if its mapped relations $E^l_\& = E^r_\& = E_\&$ and it is a conjunction on $(L, \preceq)$*

$$E_\preceq = E_\&$$

or equivalently

$$\mathcal{T}(a\&b) \preceq \mathcal{T}(a) \cdot \mathcal{T}(b)$$

- *idem for $\mathcal{T}$-disjunctions ...*

3 Some uses of *and*

Let us show some uses of the conjunction *and* extracted from English dictionaries (see [2, 3, 7]).

3.1 Copulative (also)

Used to join words, phrases, sentences or parts together. Examples:

- I have socks and shoes.
- I live in Madrid and she lives in Barcelona.
- We have many flowers and plants.
- John is tall and rich.

3.2 Copulative (in addition to)

Used with numbers. Examples:

- One hundred and ten.
- Three and two are five.
- She walked one mile and half.

3.3 Copulative (very)

Used to join the same word, making their meaning stronger. Examples:

- She walked miles and miles (increase).
- I tried and tried (repetition)
- He talked and talked (continuation)

3.4 Copulative (distinction)

Used to make distinctions within the same word. Example:

- There are lawyers and lawyers!

3.5 Copulative (despite)

Used to express surprise or some contradiction. Example:

- You're a vegetarian and you eat fish!
- You're tired and you are working.

3.6 Consecutive (then)

In this case the *and* has the meaning of *then* and links statements that are consecutive. It can express a temporal sequence, p *before than* q , can denote a consequence, p *cause that* q, or can be a necessity, p *in order to* q.

- He came and went. (before than)
- I was late and she got angry. (cause that)
- I will go and see him. (in order to)

4 Coherent Models

Any model of *and* must capture its conjunctive nature and the concrete meaning of its use.

4.1 Conjunctive nature

Definition 4.1 *Given a set of statements S and an operator $\& : S \times S \to S$ that given two statements p_1, p_2, gives the &-statement $\&(p_1, p_2) =$ "p_1 and p_2". Then a truth assignment $\mathcal{T} : S \to L$ is coherent with a use of* and *if its & associated operator behaves as $\mathcal{T}$-conjunction on L; and incorporates as properties the proper features of the use.*

- The mapped relation $E_\&$ is defined as

$$(p\&q, p) \in R_\& \Rightarrow (T(p\&q), T(p)) \in E_\&$$

$$(p\&q, q) \in R_\& \Rightarrow (T(p\&q), T(q)) \in E_\&$$

- and & is $\mathcal{T}$-conjunction if

  $(T(p\&q), T(p)) \in E_{\&} \Rightarrow (T(p\&q), T(p)) \in E_{\preceq}$

  $(T(p\&q), T(q)) \in E_{\&} \Rightarrow (T(p\&q), T(q)) \in E_{\preceq}$

  or equivalently

$$\forall p, q \in S, \mathcal{T}(p\&q) \preceq \mathcal{T}(p) \cdot \mathcal{T}(q)$$

In the following examples we will take the truth lattice L as $([0,1], \leq, min, max)$

Example 4.2 *Let be p = "I have socks" and q = "I have shoes" then $p\&q$ = "I have socks and shoes".*

- *Given the truth assignment $\mathcal{T}_1(p) = 1$, $\mathcal{T}_1(q) = 1$, $\mathcal{T}_1(p\&q) = 0$ then this truth assignment is coherent with the conjunction &, since*

$$\mathcal{T}_1(p\&q) = 0 \leq min(\mathcal{T}_1(p), \mathcal{T}_1(q)) = 1$$

- *Given the truth assignment $\mathcal{T}_2(p) = 0$, $\mathcal{T}_2(q) = 1$, and being $\mathcal{T}_2(p\&q) = 0.5$ then this truth assignment is* not *coherent with the conjunction &, since*

$$\mathcal{T}_2(p\&q) = 0.5 \not\leq min(\mathcal{T}_2(p), \mathcal{T}_2(q)) = 0$$

Remark 4.3 *Usually in fuzzy logic, it is assumed that the assignment of truth values to the &-statements is functionally expressible by an operator $\wedge : L \times L \to L$, such that,*

$$\mathcal{T}(p\&q) = \wedge(\mathcal{T}(p), \mathcal{T}(q))$$

Example 4.4 *Let be p = "John is tall" and q = "John is rich" then $p\&q$ = "John is tall and rich".*

- *Given the truth assignment $\mathcal{T}_1(p) = 0.8$, $\mathcal{T}_1(q) = 0.5$ and be $\mathcal{T}_1(p\&q) = prod(\mathcal{T}_1(p), \mathcal{T}_1(q)) = 0.4$ then this truth assignment is coherent with the conjunction &, since*

$$\mathcal{T}_1(p\&q) = 0.4 \leq min(\mathcal{T}_1(p), \mathcal{T}_1(q)) = 0.5$$

- *Given the truth assignment $\mathcal{T}_2(p) = 0.8$, $\mathcal{T}_2(q) = 0.5$, and being $\mathcal{T}_2(p\&q) = am(\mathcal{T}_2(p), \mathcal{T}_2(q)) = 0.65$ then this truth assignment is not coherent with the conjunction &, since*

$$\mathcal{T}_2(p\&q) = 0.65 \not\leq min(\mathcal{T}_2(p), \mathcal{T}_2(q)) = 0.5$$

4.2 The case of the copulative *and* with the meaning of *also*

In this case the *and* is commutative and the statements p and q must be distinct, because otherwise it will change its meaning (see 3.3 or 3.4).

$$\mathcal{T}(p\&q) = \mathcal{T}(q\&p)$$

Also it looks reasonable to assume that the & have a monotonic behavior with respect to $(L, \preceq)$.

$$\mathcal{T}(q_1) \preceq \mathcal{T}(q_2) \Rightarrow \mathcal{T}(p\&q_1) \preceq \mathcal{T}(p\&q_2)$$

With truth values in $([0,1], \leq)$ we can require that:

$$\mathcal{T}(p) = 1, \mathcal{T}(q) = 1 \Rightarrow \mathcal{T}(p\&q) = 1$$

Example 4.5 *In the examples 4.2 and 4.4 now can have the following coherent truth assignments:*

- *$\mathcal{T}_1(p) = 1$, $\mathcal{T}_1(q) = 1$, and being $\mathcal{T}_1(p\&q) = min(\mathcal{T}_1(p), \mathcal{T}_1(q))$ then*

$$\mathcal{T}_1(p\&q) = 1 = min(\mathcal{T}_1(p), \mathcal{T}_1(q))$$

- *$\mathcal{T}_2(p) = 0.8$, $\mathcal{T}_2(q) = 0.5$, and being $\mathcal{T}_2(p\&q) = W(\mathcal{T}_2(p), \mathcal{T}_2(q))$ then*

$$\mathcal{T}_2(p\&q) = 0.3 \leq min(\mathcal{T}_2(p), \mathcal{T}_2(q))$$

4.3 The case of the copulative *and* with the meaning of *very*

In this case the statements are the same and its truth value have a restrictive behavior and usually is not idempotent.

$$\mathcal{T}(p\&p) \preceq \mathcal{T}(p)$$

Example 4.6 *(See 3.3)*

- *Let be p = "He talked" and $p\&p$ = "He talked and talked" then the truth assignment $\mathcal{T}_1(p) = 0.8$ and $\mathcal{T}_1(p\&p) = \mathcal{T}_1(p)^2$ is coherent, since*

$$\mathcal{T}_1(p\&p) = 0.64 \leq \mathcal{T}_1(p)$$

- *Let be p = "She walked miles" and $p\&p$ = "She walked miles and miles" then the truth assignment $\mathcal{T}_2(p) = 1$ and $\mathcal{T}_2(p\&p) = \mathcal{T}(p)/2$ is coherent, since*

$$\mathcal{T}_1(p\&p) = 0.5 \leq \mathcal{T}_1(p)$$

4.4 The case of the copulative *and* with the meaning of *then*

This case assumes that there is a relation between the statements. If we have an operator $\rhd : U \times U \to U$ that represents the relation *"p then q"*, then the truth assignment to be coherent must verify that

$$\mathcal{T}(p\&q) \leq \mathcal{T}(p \rhd q)$$

Example 4.7 *(See 3.6)*

- *Let be $p =$ "He came", $q =$ "He went", $p\&q =$ "He came and went" and $p \rhd q =$ "He came before he went", then the truth assignment $\mathcal{T}_1(p) = 1$, $\mathcal{T}_1(q) = 1$, and $\mathcal{T}_1(p\&q) = 1$ is coherent only if $\mathcal{T}_1(p \rhd q) = 1$, since*

$$1 = \mathcal{T}_1(p\&q) \leq \mathcal{T}_1(p \rhd q)$$

  In the case that he came, he went and he went before came, the truth assignment $\mathcal{T}_1(p) = 1$, $\mathcal{T}_1(q) = 1$, and $\mathcal{T}_1(p \rhd q) = 0$ is coherent only if $\mathcal{T}_1(p\&q) = 0$, since

$$\mathcal{T}_1(p\&q) \leq \mathcal{T}_1(p \rhd q) = 0$$

- *Let be $p =$ "I was late", $q =$ "She got angry", $p\&q =$ "I was late and she got angry" and $p \rhd q =$ "She got angry because I was late", then the truth assignment $\mathcal{T}_2(p) = 0.8$, $\mathcal{T}_2(q) = 0.7$, and $\mathcal{T}_2(p \rhd q) = 0.5$ is coherent if*

$$\mathcal{T}_2(p\&q) \leq \mathcal{T}_2(p \rhd q) = 0.5$$

  In the case that she got angry because other causes then $\mathcal{T}_2$ will be coherent if

$$\mathcal{T}_2(p\&q) \leq \mathcal{T}_2(p \rhd q) = \mathcal{T}_2(q) = 0.7$$

5 Conclusions

Proper models of *and* must capture its conjunctive nature as well as the concrete meaning of its use. The pros and cons of each model and how to get the coherency between models and uses have been shown. This framework helps to study and to expand the representations of the use of words in language.

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