Fuzzy relation equations with dual composition

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Abstract

In this paper we investigate the System of fuzzy relation equations with inf $\rightarrow$ composition. We base on the knowledge of he system of fuzzy relation equations with $sup - *$-composition. First, the general notions are explained and then are shown the smallest and maximal solutions. Finally is investigated a simple criterion of solvability for a special case of fuzzy sets.

Keywords: System of fuzzy relation equations, $sup - *$-composition and inf $\rightarrow$ composition, solvability of fuzzy relation equation system, maximal solution.

1 Introduction

System of fuzzy relation equations is traditionally understood with respect to the $sup - *$-composition where $*$ is usually a continuous t-norm. This comes back to early Zadeh's papers [11, 12] where he introduced the compositional rule of inference and considered it's realization with help of $sup - min$-composition.

The $sup - *$ is not the only possible composition between two fuzzy relations. In [10], Sanchez suggested the dual (in some sense) composition expressed by $inf \rightarrow$ operations. The properties of this composition have been investigated in [2, 1] where the necessary and sufficient condition of solvability and the characterization of a set of all solutions have been presented. However, the $inf \rightarrow$ composition is not so popular in applications as its counterpart and therefore, only basic things regarding the solvability of systems of fuzzy relation equations are known. The purpose of this contribution is to show that there are many things which are common for both systems with $sup - *$ and $inf \rightarrow$-composition operations. For example, in a specific, but a very often case where input and output fuzzy sets are fuzzy points, the necessary and sufficient condition of solvability is the same for both systems. However, in general it is not true that the one system is solvable if and only if the dual one is solvable as well.

2 Preliminaries

2.1 Residuated lattice

We choose a complete residuated lattice as the basic algebra of operations.

Definition 1
A residuated lattice is an algebra

$$\mathcal{L} = \langle L, \vee, \wedge, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle$$

with four binary operations and two constants such that

- $\langle L, \vee, \wedge, \mathbf{0}, \mathbf{1} \rangle$ *is a lattice where the ordering* $\leq$ *defined using operations* $\vee, \wedge$ *as usual, and* $\mathbf{0}, \mathbf{1}$ *are the least and the greatest elements, respectively;*
- $\langle L, *, \mathbf{1} \rangle$ *is a commutative monoid, that is,* $*$ *is a commutative and associative operation with the identity* $a * \mathbf{1} = a$*;*
- *the operation* $\rightarrow$ *is a residuation operation with respect to* $*$*, i.e.*

$$a * b \leq c \quad \text{iff} \quad a \leq b \rightarrow c. \qquad (1)$$

A residuated lattice is complete if it is complete as a lattice.

The following binary operation of biresiduation can be additionally defined:

$$x \leftrightarrow y = (x \to y) \wedge (y \to x).$$

2.2 Fuzzy sets and fuzzy relations

We accept here a mathematical definition of a fuzzy set. In the rest of this paper we suppose that a complete residuated lattice $\mathcal{L}$ with a support L is fixed, and $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary non-empty sets. Then a *fuzzy set* or better, a fuzzy subset A of $\mathbf{X}$, is identified with a function $A : \mathbf{X} \longrightarrow L$. This function is known as *membership function* of the fuzzy set A. The set of all fuzzy subsets of $\mathbf{X}$ is denoted by $\mathcal{F}(\mathbf{X})$, so that

$$\mathcal{F}(\mathbf{X}) = \{A \mid A : \mathbf{X} \longrightarrow L\} = L^{\mathbf{X}}.$$

For two fuzzy subsets A and B of $\mathbf{X}$ we write $A = B$ or $A \leq B$ if $A(x) = B(x)$ or $A(x) \leq B(x)$, holds for all $x \in \mathbf{X}$, respectively. A fuzzy set $A \in \mathcal{F}(\mathbf{X})$ is called *normal* if $A(x_0) = \mathbf{1}$ for some $x_0 \in \mathbf{X}$.

The algebra of operations over fuzzy subsets of $\mathbf{X}$ is introduced as an induced residuated lattice on $L^{\mathbf{X}}$. This means that each operation from $\mathcal{L}$ induces the corresponding operation on $L^{\mathbf{X}}$ taken pointwise.

A (binary) *fuzzy relation* on $\mathbf{X} \times \mathbf{Y}$ is a fuzzy subset of the Cartesian product. $\mathcal{F}(\mathbf{X} \times \mathbf{Y})$ denotes the set of all binary fuzzy relations on $\mathbf{X} \times \mathbf{Y}$.

Let $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ and $S \in \mathcal{F}(\mathbf{Y} \times \mathbf{Z})$, then the fuzzy relation $T = R \circ S$ on $\mathbf{X} \times \mathbf{Z}$

$$T(x, z) = \bigvee_{y \in \mathbf{Y}} (R(x, y) * S(y, z))$$

is called a $sup - *$-composition of R and S, and the fuzzy relation $Q = R \circ' S$ on $\mathbf{X} \times \mathbf{Z}$

$$Q(x, z) = \bigwedge_{y \in \mathbf{Y}} (R(x, y) \to S(y, z))$$

is called an $inf \to$-composition of R and S.

In particular, if A is a unary fuzzy relation on $\mathbf{X}$ or simply a fuzzy subset of $\mathbf{X}$ then the $sup - *$ ($inf \to$)-composition between A and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$ is the fuzzy subset of $\mathbf{Y}$ defined by

$$\begin{aligned} (A \circ R)(y) &= \bigvee_{x \in \mathbf{X}} (A(x) * R(x, y)) \\ ((A \circ' R)(y) &= \bigwedge_{x \in \mathbf{X}} (A(x) \to R(x, y))). \end{aligned}$$

2.3 Systems of fuzzy relation equations

Let $n \geq 1$. A $sup - *$-system of fuzzy relation equations

$$A_i \circ R = B_i, \quad 1 \leq i \leq n, \tag{2}$$

or

$$\bigvee_{x \in \mathbf{X}} (A_i(x) * R(x, y)) = B_i(y), \quad 1 \leq i \leq n,$$

where $A_i \in \mathcal{F}(\mathbf{X}), B_i \in \mathcal{F}(\mathbf{Y})$ and $R \in \mathcal{F}(\mathbf{X} \times \mathbf{Y})$, is considered with respect to unknown fuzzy relation R.

Analogously, we will consider an $inf \to$-system of fuzzy relation equations

$$A_i \circ' R = B_i, \quad 1 \leq i \leq n, \tag{3}$$

or

$$\bigwedge_{x \in \mathbf{X}} (A_i(x) \to R(x, y)) = B_i(y), \quad 1 \leq i \leq n,$$

with the same description of parameters and also with respect to unknown fuzzy relation R.

Because solutions of (2) and (3) may not exist, in general, the problem to investigate necessary and sufficient, or only sufficient conditions for solvability arises. This problem has been widely studied in the literature with respect to the system (2), and some nice theoretical results have been obtained. Let us recall [9, 7, 10] with necessary and sufficient conditions, or [3, 4] with sufficient conditions.

Two types of fuzzy relations have been always assumed with respect to the problem of solvability of (2) and (3), namely

$$\check{R}(x, y) = \bigvee_{i=1}^{n} (A_i(x) * B_i(y)) \tag{4}$$

considered in Mamdani & Assilian [6], and

$$\hat{R}(x,y) = \bigwedge_{i=1}^{n} (A_i(x) \to B_i(y)) \tag{5}$$

first considered in Sanchez [10].

3 Complete set of solutions

The following statement is the general condition of solvability.

Theorem 1
The system (3) is solvable if and only if the fuzzy relation $\check{R}$ is its solution. If the system (3) is solvable then $\check{R}$ is its smallest solution.

The proof of this theorem can be obtained from the proof of the analogous statement concerning the solvability of the system (2) (see e.g. [5]).

It is easy to see that if system (3) is solvable then the set of solutions forms a $\wedge$-semi-lattice, i.e. a fuzzy relation $R_1 \wedge R_2$ is a solution to (3) whenever R_1 and R_2 are its solutions. Moreover, this semi-lattice has the least element and in the case of finite universes $\mathbf{X}$ and $\mathbf{Y}$, it has maximal elements (see [1]).

We suppose that $L = [0,1]$ in the rest of this section. If

1. operation $\to$ has continuous second partial mappings,
2. t-norm $*$ is continuous and satisfy the conditional cancellation law,
3. universes $\mathbf{X}$ and $\mathbf{Y}$ are finite,

then we can obtain the maximal solutions in the following way.

Theorem 2
Let

$$\bigwedge_{x \in \mathbf{X}} (A(x) \to R^y(x)) = B^y \tag{6}$$

be a solvable equation (where $A \in \mathcal{F}(\mathbf{X}), B^y \in [0,1]$ and $R^y(x) \in \mathcal{F}(\mathbf{X})$ is unknown fuzzy set) and $\check{R}^y(x)$ is its smallest solution. Let $D^y = \{d \in \mathbf{X} | \neg A(d) \leq B^y\}$. Then

$$O^y = \{M^y_d(x) | d \in D^y\} \tag{7}$$

is set of the maximal solutions of this equation. Where

$$M^y_d(x) = \begin{cases} A(x) * B^y, & \text{if } x = d \text{ and } B^y < 1 \\ 1, & \text{elsewhere.} \end{cases}$$

PROOF:

$B^y = 1$: then $O^y = \{M^y = (1,1,\ldots,1)\}$ and $\bigwedge(A(x) \to 1) = 1$, hence this element is the solution of the equation (6). And sure it is maximal solution too, because for all fuzzy set $R(x)$: $R(x) \leq M^y(x) = 1$ holds.

$B^y \neq 1$: We take an element $M^y_{\bar{x}}(x) \in O^y$ (i.e. $\bar{x} \in D^y$), then

$$\bigwedge_{x \in \mathbf{X}} (A(x) \to M^y_{\bar{x}}(x)) = $$
$$= \begin{cases} \forall x \neq \bar{x} \text{ holds } A(x) \to 1 = 1 \\ \text{for } \bar{x} \text{ holds } A(\bar{x}) \to (A(\bar{x}) * B^y) \end{cases}$$

therefore

$$\bigwedge_{x \in \mathbf{X}} (A(x) \to M^y_{\bar{x}}(x)) = $$
$$= A(\bar{x}) \to (A(\bar{x}) * B^y) = $$
$$= \bigvee \{C \in [0,1] | A(\bar{x}) * C \leq A(\bar{x}) * B^y\}$$

Then

1. $A(\bar{x}) * B^y = 0$ then $A(\bar{x}) \to 0 \leq B^y$ that holds from assumption $\bar{x} \in D^y$,
2. $A(\bar{x}) * B^y > 0$ then $A(\bar{x}) * C \leq A(\bar{x}) * B^y$ hence $C \leq B^y$. This relation results from continuity of t-norm and validity the conditional cancellation law.

Therefore

$$A(\bar{x}) \to (A(\bar{x}) * B^y) \leq B^y.$$

Because generally the inequality

$$A(\bar{x}) \to (A(\bar{x}) * B^y) \geq B^y$$

holds in the residuated lattice, we get the equality

$$\begin{aligned} A(\bar{x}) \to (A(\bar{x}) * B^y) &= B^y \\ \bigwedge_{x \in \mathbf{X}} (A(x) \to M^y_{\bar{x}}(x)) &= B^y. \end{aligned}$$

Now we prove, that $M^y_{\bar{x}}(x) \in O^y$ is a maximal solution. Let $B^y \neq 1$ and $R(x)$ is a solution of the equation (6). Then there is a proper element $M^y_{\bar{x}} \in O^y$: $R(x) \leq M^y_{\bar{x}}(x)$, i.e. $\forall x \neq \bar{x}$: $R(x) \leq 1 = M^y_{\bar{x}}(x)$, that sure holds and for $\bar{x}$: $R(\bar{x}) \leq M^y_{\bar{x}}(\bar{x}) = A(\bar{x}) * B^y = \check{R}(\bar{x})$ and $\check{R}(\bar{x}) \leq R(\bar{x})$ hence $R(\bar{x}) = M^y_{\bar{x}}(\bar{x}) = A(\bar{x}) * B^y = \check{R}(\bar{x})$ and just this equality we want to prove.

We assume $R(x)$ is a solution of the equation (6) and $\forall x \neq \bar{x}$: $R(x) = 1 = M^y_{\bar{x}}(x)$ and for $\bar{x}$: $R(\bar{x}) > M^y_{\bar{x}}(\bar{x}) = A(\bar{x}) * B^y = \check{R}(\bar{x})$.

Then

$$\begin{aligned} B^y &= \bigwedge_{x\in X} (A(x) \to M^y_{\bar{x}}(x)) = A(\bar{x}) \to M^y_{\bar{x}}(\bar{x}) \\ &< A(\bar{x}) \to R(\bar{x}) = \bigwedge_{x\in X} (A(x) \to R(x)) \end{aligned}$$

That is in conflict with statement that $R(x)$ is the solution of equation (6).

We get the inequality $A(\bar{x}) \to M^y_{\bar{x}}(\bar{x}) < A(\bar{x}) \to R(\bar{x})$ from the assumption $M^y_{\bar{x}}(\bar{x}) < R(\bar{x})$ and continuity of t-norm. □

Theorem 3
Let

$$\bigwedge_{x\in \mathbf{X}} (A_i(x) \to R^y(x)) = B^y_i \quad i = 1, 2, \ldots, N \quad (8)$$

be the solvability system of equations (where $A_i \in \mathcal{F}(\mathbf{X}), B^y_i \in [0,1]$ *and* $R^y(x) \in \mathcal{F}(\mathbf{X})$ *is unknown fuzzy set) and* $\check{R}^y(x) = \bigvee_{i=1}^N A_i(x) * B^y_i$ *is the smallest solution. Then*

$$\begin{aligned} P^y &= \{\tilde{R}^y(x) | \tilde{R}^y(x) = \bigwedge_{i=1}^N S^y_i(x); \\ & S^y_i(x) \in O^y_i,\ S^y_i(x) \geq \check{R}^y(x)\}, \end{aligned} \quad (9)$$

is the set of the maximal solutions of this system, where O^y_i *is the set of the maximal solution of* i*-th equation.*

PROOF: We have following relations:

$$\begin{aligned} &\bigwedge_{x\in X} (A_j(x) \to \tilde{R}^y(x)) = \\ &= \bigwedge_{x\in X} \left(A_j(x) \to \bigwedge_{i=1}^N S^y_i(x) \right) = \\ &= \bigwedge_{x\in X} \bigwedge_{i=1}^N (A_j(x) \to S^y_i(x)) = \\ &= \left(\bigwedge_{x\in X} \bigwedge_{i=1, i\neq j}^N (A_j(x) \to S^y_i(x)) \right) \wedge B^y_j \leq \\ &\leq B^y_j \end{aligned}$$

and

$$\begin{aligned} \bigwedge_{x\in X} (A_j(x) \to \tilde{R}^y(x)) &\geq \bigwedge_{x\in X} \left(A_j(x) \to \check{R}^y(x)\right) \\ &= B^y_j \end{aligned}$$

so

$$\bigwedge_{x\in X} (A_j(x) \to \tilde{R}^y(x)) = B^y_j.$$

Now we prove that $\tilde{R}^y(x) \in P^y$ is the maximal solution. Let $\exists j : B^y_j \neq 1$ (case $\forall j : B^y_j = 1$ is trivial) further let $R(x)$ be a solution of the system (8), then there is an appropriate element $\tilde{R}^y(x) \in P^y$: $R(x) \leq \tilde{R}^y(x)$.

First we show two properties of elements from the set P^y. Since

$$\begin{aligned} B^y_j &= \bigwedge_{x\in X} \left(A_j(x) \to \tilde{R}^y(x)\right) \leq \\ &\leq A_j(\bar{x}) \to A_j(\bar{x}) * B^y_j = B^y_j, \end{aligned}$$

we get the equation

$$\bigwedge_{x\in X} \left(A_j(x) \to \tilde{R}^y(x)\right) = A_j(\bar{x}) \to (A_j(\bar{x}) * B^y_j).$$

For all $x \in X$

$$\begin{aligned} \check{R}^y(x) &= \bigvee_{i=1}^N (A_i(x) * B^y_i) = A_j(x) * B^y_j \geq \\ &\geq A_i(x) * B^y_i \quad \forall i = 1, \ldots, N \end{aligned}$$

holds. Since we take only the elements $S^y_i(x) \in O^y_i$ that $S^y_i(x) \geq A_j(x) * B^y_j$ and this satisfy only $A_j(x) * B^y_j$ or 1. Hence for all $x \in X$

$$\tilde{R}^y(x) = \begin{cases} 1 & \text{or} \\ A_j(x) * B^y_j = \check{R}^y(x). \end{cases} \quad (10)$$

Let exist a fuzzy set $R(x)$ solving the equation (8). Then have to exist $\tilde{R}^y(x) \in P^y$ that $R(x) \leq \tilde{R}^y(x)$.

This we prove by contradiction. Let $R(x)$ solve (8) and let for some fuzzy set $\tilde{R}^y(x) \in P^y$ following relations hold: for all $x \neq \bar{x}$ $R(x) = \tilde{R}^y(x)$ and $R(\bar{x}) > \tilde{R}^y(\bar{x})$. Then three cases can set in.

1. Exist just one j that

$$\bigwedge_{x\in X} (A_j(x) \to R(x)) = A_j(\bar{x}) \to R(\bar{x}). \quad (11)$$

We choose a fuzzy set from P^y satisfying $\tilde{R}^y(\bar{x}) = A_j(\bar{x}) * B_j^y$. Then we get

$$\begin{aligned} B_j^y &= \bigwedge_{x\in X} \left(A_j(x) \to \tilde{R}^y(x)\right) = \\ &= A_j(\bar{x}) \to A_j(\bar{x}) * B_j^y < \\ &< A_j(\bar{x}) \to R(\bar{x}) = \quad (12) \\ &= \bigwedge_{x\in X} (A_j(x) \to R(x)) \end{aligned}$$

and that is contradiction with the statement, $R(x)$ is a solution of (8). Strong inequality in (12) we get from continuity .

2. Exist several equation for which (11) holds. Then it suffices to choose only one and to proceed as well as above (in the first case).

3. If exist no one such equation, i.e.: for all $i = 1, \ldots, N$

$$\bigwedge_{x\in X} (A_i(x) \to R(x)) < A_i(\bar{x}) \to R(\bar{x}).$$

From the relation: for all $x \neq \bar{x}$ equality $R(x) = \tilde{R}^y(x)$ holds, then we get

$$\begin{aligned} B_i^y &= \bigwedge_{x\in X, x\neq\bar{x}} (A_i(x) \to R(x)) = \\ &= \bigwedge_{x\in X, x\neq\bar{x}} \left(A_i(x) \to \tilde{R}^y(x)\right) \end{aligned}$$

hence

$$\begin{aligned} B_i^y &= \bigwedge_{x\in X, x\neq\bar{x}} \left(A_i(x) \to \tilde{R}^y(x)\right) \wedge 1 = \\ &= \left(\bigwedge_{x\in X} \left(A_i(x) \to \tilde{R}^y(x)\right)\right) \wedge \\ &\qquad \wedge (A_j(\bar{x}) \to 1) \end{aligned}$$

Therefore we can take easily $\tilde{R}^y(\bar{x}) = 1$ and so we get the contradiction with the assumption, that $R(\bar{x}) > \tilde{R}^y(\bar{x}) = 1$. □

Theorem 4
Let

$$\bigwedge_{x\in \mathbf{X}} (A_i(x) \to R(x,y)) = B_i(y), \quad (13)$$

$i = 1, 2, \ldots, N$, be the system of equations (where $A_i \in \mathcal{F}(\mathbf{X}), B_i(y) \in \mathcal{F}(\mathbf{Y})$ and $R(x,y) \in \mathcal{F}(\mathbf{X}\times\mathbf{Y})$ is unknown fuzzy relation), which is solvability and $\check{R}(x,y) = \bigvee_{i=1}^{N} A_i(x) * B_i(y)$ is the smallest solution. Then

$$\mathbf{R} = \{\tilde{R}(x,y) | \tilde{R}(x,y) = (\tilde{R}^{y_1}(x), \ldots, \tilde{R}^{y_m}(x)); \\ \tilde{R}^{y_j}(x) \in P^{y_j}\} \quad (14)$$

(where P^{y_j} is a set of the maximal solutions of the system (13) for fixed element $y_j \in \mathbf{Y}$) is the set of the maximal solutions of this system of equations.

PROOF: The proof is the immediate consequence of previous theorems. □

4 Solvability with respect to $\hat{R}$

In [4], the necessary and sufficient condition of the solvability of (2) with respect to $\check{R}$ is given. We will show in this section that the same condition is necessary and sufficient too for the solvability of (3) with respect to $\hat{R}$.

Theorem 5
Let fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$, $1 \leq i \leq n$, be normal. Then the fuzzy relation $\hat{R}$ is a solution to (3) if and only if for all $i, j = 1, \ldots, n$ the following inequality

$$\bigvee_{x\in\mathbf{X}} (A_i(x) * A_j(x)) \leq \bigwedge_{y\in\mathbf{Y}} (B_i(y) \leftrightarrow B_j(y)) \quad (15)$$

holds.

The proof can be obtained from the proof of the analogous statement concerning the solvability of the system (2) with respect to $\check{R}$ (see [4]).

Although solvability and $\hat{R}$-solvability of system (3) are not in general equivalent, this is true under the assumption about semi-partitioning of $\mathbf{X}$. The theorem given below proves this fact.

We put restrictions on fuzzy sets $A_1, \ldots, A_n \in \mathcal{F}(\mathbf{X})$ assuming that they are normal and form a semi-partition of $\mathbf{X}$. For this, we recall the definition of a semi-partition (see [8]).

Definition 2
Normal fuzzy sets $A_1, \ldots, A_n \in \mathcal{F}(\mathbf{X})$ form a semi-partition of $\mathbf{X}$ if $\forall i, \forall j$

$$\bigvee_{x \in \mathbf{X}} (A_i(x) * A_j(x)) \leq \bigwedge_{x \in \mathbf{X}} (A_i(x) \leftrightarrow A_j(x)). \quad (16)$$

Theorem 6
Let fuzzy sets $A_1, \ldots, A_n \in \mathcal{F}(\mathbf{X})$ be normal and form a semi-partition of $\mathbf{X}$. Then system (3) is solvable if and only if it is $\hat{R}$-solvable.

5 A simple criterion of solvability

If fuzzy sets A_i and B_i are so called fuzzy points or classes of equivalence of some fuzzy equivalence, we expect to have simpler conditions for solvability of (3) in general as well as in particular, with respect to $\hat{R}$.

It turned out that the condition of solvability of (3) which we are going to introduce in this section, is the same as the condition of solvability of (2) under the same assumptions (see[7]). The proof is also similar and therefore, it is omitted.

Theorem 7
Let fuzzy sets $A_i \in \mathcal{F}(\mathbf{X})$ and $B_i \in \mathcal{F}(\mathbf{Y})$, $1 \leq i \leq n$, be normal (so that there exist $x_i \in \mathbf{X}$ and $y_i \in \mathbf{Y}$ which make true the following: $A_i(x_i) = 1$, $B_i(y_i) = 1$). Further, let fuzzy equivalence E on $\mathbf{X}$ and fuzzy equivalence F on $\mathbf{Y}$ exist so that all the fuzzy sets A_i are fuzzy points with respect to x_i and E, and all the fuzzy sets B_i are fuzzy points with respect to y_i and F, i.e.

$$(\forall x) A_i(x) = E(x_i, x) \quad \text{and} \quad (\forall y) B_i(y) = F(y_i, y).$$

Then the system (3) is solvable if and only if

$$(\forall i)(\forall j) A_i(x_j) \leq B_i(y_j). \quad (17)$$

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