Normal forms based fuzzy systems

Martina Daňková
University of Ostrava
IRAFM
30. dubna 22
701 03 Ostrava
Czech Republic
Martina.Dankova@osu.cz

Abstract

Normal forms have been introduced as approximating relations for fuzzy relations. Moreover, they can be viewed as a formalization of a collection of fuzzy IF-THEN rules. Therefore, investigation of approximating abilities of normal forms together with determination of an appropriate inference engine contributes to building of fuzzy systems.

Keywords: Fuzzy approximation, Normal forms, Fuzzy systems.

1 Introduction

It is possible to implement approximate knowledge about some function in the form of a sequence of IF-THEN rules used as the basis for the control of some physical system. Such control laws (rules) may be viewed as a qualitative characterization of variables using fuzzy predicates and functional dependencies between variables using conditional sentences with fuzzy predicates.

Special formulas, introduced by I. Perfilieva in [8] and called normal forms, naturally formalize such a system of fuzzy IF-THEN rules and, as it was shown in [16], they are related to the field of fuzzy approximation. By the term fuzzy approximation, we mean the theory studying properties of special functions, relations or formulas (having an approximating character) aggregating local information about a given function. The approach from [16] has been generalized in [5], where the basis for fuzzy approximations has been proposed. Note that the fuzzy interpolation methods (see [6, 18, 14]) create a particular subclass of fuzzy approximation methods.

In this contribution, we are interested in fuzzy control systems which use fuzzy approximation. Due to [10, 9], it is necessary to distinguish between the logical deduction and the inference for fuzzy approximation. In fuzzy logic, the notion of inference is closely related to composition of fuzzy relations (fuzzy sets, or both). We will introduce different compositions for different fuzzy approximations. More specifically, we will consider compositions, which produce appropriate outputs based on the knowledge in the form of fuzzy approximating formulas. Since we assume only a special kind of composition, consequently, we will prefer the later notion. Moreover, we will prove properties relating to the behaviour of such fuzzy systems.

2 Preliminaries

2.1 BL-algebras

The operations used in sequel will form a linearly ordered *BL-algebra* $\mathcal{L}$ on a set L which is an algebra

$$\mathcal{L} = \langle L, \wedge, \vee, *, \rightarrow, \mathbf{0}, \mathbf{1} \rangle \quad (1)$$

with four binary operations and two constants such that

(i) $(L, \vee, \wedge, \mathbf{0}, \mathbf{1})$ is a lattice,

(ii) $(L, *, \mathbf{1})$ is an Abelian monoid,

(iii) $*$ and $\rightarrow$ form an adjoint pair, i.e.

$$z \leq (x \rightarrow y) \quad \text{iff} \quad x * z \leq y$$

for all $x, y, z \in L$,

(iv) moreover, for all $x, y \in L$

$$x*(x \to y) = x \wedge y,$$

(v) and finally $\mathcal{L}$ is linearly ordered, i.e. for each pair $x, y : x \wedge y = x \quad \text{or} \quad x \wedge y = y$.

In particular, when $L = [0, 1]$, the operation $*$ stands for a continuous t-norm. And apparently each continuous t-norm determines a BL-algebra on the unit interval $[0, 1]$ with its standard linear ordering.

2.2 Extensionality of fuzzy relations

Below, we will work with fuzzy relations having special property called extensionality. This property can be viewed as an analogy to Lipschitz continuity, see [17] or [13] for the case of the extensionality w.r.t. similarity (reflexive, symmetric and $*$-transitive) relation. And for the characterization of a generalized extensionality w.r.t. a reflexive binary fuzzy relation, we refer to [4]. For the sake of brevity, we will use the following notation

$$R(\mathbf{x}, \mathbf{y}) = R_1(x_1, y_1) * \cdots * R_n(x_n, y_n), \tag{2}$$

where $R_1, \ldots, R_n$ are binary L-fuzzy relations on M and $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in M^n$. The following definition is taken from [4].

Definition 1 *Let M be some nonempty set of objects. An L-fuzzy relation $P \underset{\sim}{\subset} M^n$ (i.e. $P : M^n \to L$) is extensional w.r.t. $R_1, \ldots, R_n$ (or shortly R) on M^n if*

$$R(\mathbf{x}, \mathbf{y}) \leq P(\mathbf{x}) \to P(\mathbf{y}), \tag{3}$$

for each $\mathbf{x}, \mathbf{y} \in M^n$.

The general treating of the extensionality property allows us to capture the nature of P better than restricting only to R being similarity relation. For example, we may work with fuzzy relations which are extensional w.r.t. fuzzy orderings (reflexive, $*$-antisymmetric and $*$-transitive), see [2] also for applications.

2.3 Normal forms

Let us recall notions of disjunctive and conjunctive normal forms (introduced in [15] and generalized in [3]). Let $f(\mathbf{x})$ be an n-ary L-fuzzy relation on a nonempty set of objects M, R is given by (2) and $\mathbf{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_k\}$, where $\mathbf{c}_i = [c_{i_1}, \ldots, c_{i_n}] \in M^n$ for all $i = 1, \ldots, k$.

Then the discrete disjunctive and conjunctive normal forms for f are given as follows:

$$f_{\mathrm{DNF},k}(\mathbf{x}) = \bigvee_{i=1}^{k} (R(\mathbf{c}_i, \mathbf{x}) * f(\mathbf{c}_i)), \tag{4}$$

$$f_{\mathrm{CNF},k}(\mathbf{x}) = \bigwedge_{i=1}^{k} (R(\mathbf{x}, \mathbf{c}_i) \to f(\mathbf{c}_i)), \tag{5}$$

respectively.

Note, (4) and (5) are L-fuzzy relations that are called normal forms due to [15, 17].

In the case of extensional L-fuzzy relation f, the following relationship can be proved

$$f_{\mathrm{DNF},k}(\mathbf{x}) \leq f(\mathbf{x}) \leq f_{\mathrm{CNF},k}(\mathbf{x}), \tag{6}$$

for all $\mathbf{x} \in M^n$. Moreover, the precision of approximation of f by normal forms for f can be estimated as follows:

$$C(\mathbf{x}) \leq f_{\mathrm{D(C)NF}}(\mathbf{x}) \leftrightarrow f(\mathbf{x}), \tag{7}$$

where

$$C(\mathbf{x}) = \bigvee_{i=1}^{k} (R(\mathbf{x}, \mathbf{c}_i) * R(\mathbf{c}_i, \mathbf{x})), \tag{8}$$

for all $\mathbf{x} \in M^n$.

In the theory of approximations, we usually approximate a given function by another one, so that both functions belong to the same class of functions, e.g. class of continuous functions. Here, classes of L-fuzzy relations that we are going to approximate, are determined by the extensionality property.

We will prove that approximating relations expressed by normal forms are extensional w.r.t. specially constructed relations.

Theorem 1 *If $f \underset{\sim}{\subset} M^n$ is extensional w.r.t. R then $f_{\mathrm{DNF},k}$ is extensional w.r.t. $D_1, \ldots, D_n$ given by*

$$D_i(x, y) = \bigwedge_{j=1}^{k} (R_i(c_{i_j}, x) \to R_i(c_{i_j}, y)), \qquad (9)$$

and $f_{\mathrm{CNF},k}$ is extensional w.r.t. $E_1, \ldots, E_n$ given by

$$E_i(x, y) = \bigwedge_{j=1}^{k} (R_i(y, c_{i_j}) \to R_i(x, c_{i_j})). \qquad (10)$$

PROOF: We will prove the extensionality of $f_{\mathrm{DNF},k}$. From the extensionality of f, it follows

$$R(\mathbf{c}_i, \mathbf{x}) * f(\mathbf{c}_i) \leq (R(\mathbf{c}_i, \mathbf{x}) \to R(\mathbf{c}_i, \mathbf{y})) \to (R(\mathbf{c}_i, \mathbf{y}) * f(\mathbf{c}_i)),$$

which gives

$$\bigvee_{i=1}^{k} R(\mathbf{c}_i, \mathbf{x}) * f(\mathbf{c}_i) \leq \bigwedge_{i=1}^{k} (R(\mathbf{c}_i, \mathbf{x}) \to R(\mathbf{c}_i, \mathbf{y})) \to \bigvee_{i=1}^{k} R(\mathbf{c}_i, \mathbf{y}) * f(\mathbf{c}_i),$$

and finally

$$\bigwedge_{i=1}^{k} (R(\mathbf{c}_i, \mathbf{x}) \to R(\mathbf{c}_i, \mathbf{y})) \leq f_{\mathrm{DNF},k}(\mathbf{x}) \to f_{\mathrm{DNF},k}(\mathbf{y}),$$

and using $\bigwedge_{i,j=1}^{k} (a_i \& a_j) = \bigwedge_{i=1}^{k} a_i \& \bigwedge_{j=1}^{k} a_j$, we prove the extensionality of $f_{\mathrm{DNF},k}(\mathbf{x})$ w.r.t. D. Analogously, we can prove the extensionality of $f_{\mathrm{CNF},k}(\mathbf{x})$ w.r.t. E. □

The following corollary says that the normal forms for extensional fuzzy relations w.r.t. a transitive R are extensional w.r.t. R as well.

Corollary 1 *If $f \underset{\sim}{\subset} M^n$ is extensional w.r.t. transitive R then $f_{\mathrm{D(C)NF},k}$ is extensional w.r.t. R.*

PROOF: Follows from Theorem 1 and transitivity of R, i.e.

$$R_i(c_{i_j}, x) * R_i(x, y) \leq R_i(c_{i_j}, y), \qquad (11)$$

$$R_i(x, y) \leq R_i(c_{i_j}, x) \to R_i(c_{i_j}, y), \qquad (12)$$

$$R_i(x, y) \leq \bigwedge_{j=1}^{k} (R_i(c_{i_j}, x) \to R_i(c_{i_j}, y)), \qquad (13)$$

$$R_i(x, y) \leq D_i(x, y). \qquad (14)$$

for all $i = 1, \ldots, n$ and $x, y \in M$. Analogously $R_i(x, y) \leq E_i(x, y)$. □

3 Normal form + inference rule ⇒ fuzzy system

A partial knowledge about some function $F : M^n \to [a, b]$, where $[a, b] \subset \mathbb{R}$, may be formalized by normal forms of the form (4) or (5). It relates to the following set of fuzzy IF-THEN rules:

$$\text{IF } (\mathbf{x} \text{ is } \mathcal{R}_i) \text{ THEN } (y = F_i), \qquad (15)$$

for $i = 1, \ldots, k$. This k rules create the rule-base of a fuzzy control system over some real physical system. The expression "$\mathbf{x}$ is $\mathcal{R}_i$" is interpreted as $R(\mathbf{c}_i, \mathbf{x})$ or $R(\mathbf{x}, \mathbf{c}_i)$, and moreover, "$y = F_i$" are interpreted as $f(\mathbf{c}_i) = T(F(\mathbf{c}_i))$, where $T : [a, b] \to L$ for all $i = 1, \ldots, k$.

Let us define compositions of an input $A \underset{\sim}{\subset} M^n$ and normal forms (4), (5) as follows:

$$B_{\mathrm{DNF}} = \bigvee_{\mathbf{x} \in M^n} (A(\mathbf{x}) * f_{\mathrm{DNF},k}(\mathbf{x})), \qquad (16)$$

$$B_{\mathrm{CNF}} = \bigwedge_{\mathbf{x} \in M^n} (A(\mathbf{x}) \to f_{\mathrm{CNF},k}(\mathbf{x})), \qquad (17)$$

The value B_{DNF} may be viewed as the maximal degree in which A "intersects" $f_{\mathrm{DNF},k}$. Analogously, B_{CNF} represents the minimal degree in which A is a "subset" of $f_{\mathrm{CNF},k}$.

We will consider a *fuzzy control system* consisting of a rule-base (including available knowledge) in the form of fuzzy IF-THEN rules (15) and computing outputs of inference method (applied to the fuzzy rules and to fuzzy input variables) on the basis of (16) or (17), see Figure 1. Restricting those inputs and outputs of fuzzy control system to be crisp, methods of fuzzification and defuzzification are involved. Since the output value B_{DNF} as well as B_{CNF} is already the crisp number belonging to L, therefore, the defuzzification is not needed, but instead of it, the inverse transformation $T^{-1} : L \to [a, b]$ is applied.

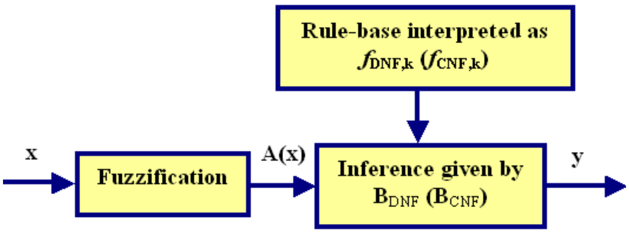

Figure 1: Normal form based fuzzy system.

3.1 Properties of proposed compositions

As the first step in the analysis of proposed compositions based on normal forms, we will take a look at relationship between compositions and their input A. In other words, we want to know whether the composition is implied and the way how the implication works.

Let us assume in the sequel $f, A \underset{\sim}{\subset} M^n$ and $f_{\mathrm{D(C)NF},k}$, $B_{\mathrm{D(C)NF}}$ are defined by (4), (5), (16), (17), respectively.

Proposition 1 *The inequalities*

$$A(\mathbf{x}) * f_{\mathrm{DNF},k}(\mathbf{x}) \le B_{DNF}, \qquad (18)$$
$$B_{CNF} \le A(\mathbf{x}) \to f_{\mathrm{CNF},k}(\mathbf{x}), \qquad (19)$$

are valid for each $\mathbf{x} \in M^n$.

PROOF: Obviously follows from (16) and (17). □

Taking into account the extensionality of f, we can prove the following relationship between compositions.

Theorem 2 *Let f be extensional w.r.t. R. Then*

$$\bigwedge_{\mathbf{x},\mathbf{y} \in M^n} D(\mathbf{x}, \mathbf{y}) \le B_{DNF} \to B_{CNF}, \qquad (20)$$

where $D(\mathbf{x}, \mathbf{y})$ is given by (9).

PROOF: From the extensionality of $f_{\mathrm{DNF},k}$ w.r.t. D given by (9) and using (6), we come to (20). □

Difference between the composition of A with $f_{\mathrm{D(C)NF},k}$ and the original fuzzy relation f can be estimated on the basis of the inequalities formulated below.

Theorem 3 *If f is extensional w.r.t. R then*

$$A(\mathbf{x}) * C(\mathbf{x}) \le f(\mathbf{x}) \leftrightarrow B_{\mathrm{D(C)NF}}, \qquad (21)$$

for each $\mathbf{x} \in M^n$.

PROOF: From (7)

$$M(\mathbf{x}) * A(\mathbf{x}) \le A(\mathbf{x}) * f_{\mathrm{DNF},k}(\mathbf{x}),$$

where $M(\mathbf{x}) = C(\mathbf{x}) * f_{\mathrm{CNF},k}(\mathbf{x})$, which implies

$$M(\mathbf{x}) * A(\mathbf{x}) \le \bigvee_{\mathbf{x} \in M^n} A(\mathbf{x}) * f_{\mathrm{DNF},k}(\mathbf{x}),$$
$$M(\mathbf{x}) \le A(\mathbf{x}) \to B_{\mathrm{DNF}},$$

since f is extensional w.r.t. R, consequently, $f(\mathbf{x}) \le f_{CNF,k}(\mathbf{x})$ that gives

$$C(\mathbf{x}) * f(\mathbf{x}) \le A(\mathbf{x}) \to B_{\mathrm{DNF}},$$
$$C(\mathbf{x}) * A(\mathbf{x}) \le f(\mathbf{x}) \to B_{\mathrm{DNF}}.$$

By the conditional equivalence (7) and (18), we prove the reverse implication, which gives the claim for $f_{\mathrm{DNF},k}$. For $f_{\mathrm{CNF},k}$, the proof is analogous. □

These inequalities provide an idea of how a "closeness" between f and the composition $B_{\mathrm{D(C)NF}}$ can be estimated. Let us remind that $C(\mathbf{x})$ is the lower boundary of equivalences between f and normal forms. If e.g. $A(\mathbf{x}) = \mathbf{1}$ then the value of $f(\mathbf{x}) \leftrightarrow B_{\mathrm{D(C)NF}}$ is influenced only by $C(\mathbf{x})$, i.e. knowing that $C(\mathbf{x}) \ge p$ follows $f(\mathbf{x}) \leftrightarrow B_{\mathrm{D(C)NF}} \ge p$.

Example 1 *Let $f(x) = 0.25\sin(20x) + 0.5$ on $M = [0, 1]$, $\mathcal{L}$ is Łukasiewicz algebra with $L = [0, 1]$. Moreover, $R(x, y) = (x \to y)^{k(x,y)} * (y \to x)^{k(y,x)}$, where $k(x, y)$ is determined on the basis of f such that f is extensional w.r.t. R, see [4]. Note that usually $k(x, y) \ne k(y, x)$.*

Nodes for $f_{\mathrm{DNF},8}$ as well as for $f_{\mathrm{CNF},8}$ are as follows

$$C = \{0.02, 0.08, 0.19, 0.4, 0.54, 0.71, 0.87, 0.97\}.$$

As it can be seen on Figure 2, $f_{\mathrm{D(C)NF},8}$ creates the lower (upper) approximation of f and the composition $B_{\mathrm{D(C)NF}}$ lies in boundaries determined as a combination of an error of approximation and a vagueness of the input A, i.e. in our case

$$|f_{\mathrm{D(C)NF},8}(x) - B_{\mathrm{D(C)NF}}| \leq 1 \wedge (2-(A(x)+C(x))).$$

Moreover, $B_{\mathrm{DNF}} \leq B_{\mathrm{CNF}}$ for $A(x)$ depicted on Figure 3. But this inequality is not valid for arbitrary $A(x)$, as it can be seen from (20) saying that $B_{\mathrm{DNF}} \leq B_{\mathrm{CNF}}$ is valid only in some degree, which is based on D given by (9).

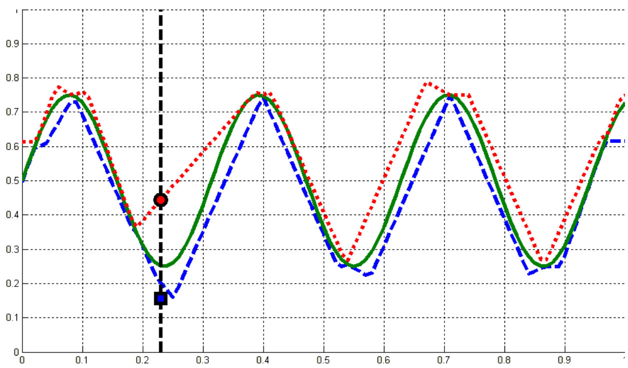

Figure 2: $f_{\mathrm{DNF},8}$ (dashed) and $f_{\mathrm{CNF},8}$ (dotted) for f (solid) from Example 1. Moreover, squared (circled) marker denotes B_{DNF} (B_{CNF}) relating to $A(x)$ depicted on Figure 3 or 4.

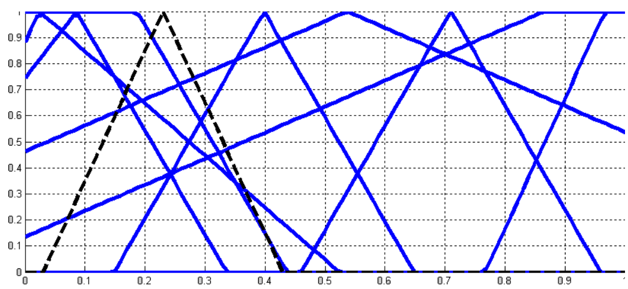

Figure 3: $R(c_i, x)$, $c_i \in C$ specified in Example 1 and input fuzzy set $A(x)$ (dashed).

The results of this subsection show that $B_{\mathrm{D(C)NF}}$ is suitable for deriving conclusions on the basis of inaccurate input and partial knowledge

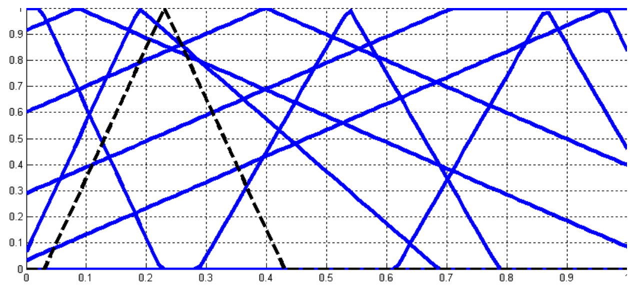

Figure 4: $R(x, c_i)$, $c_i \in C$ specified in Example 1 and input fuzzy set $A(x)$ (dashed).

about some function (represented by fuzzy relation $f_{\mathrm{D(C)NF},k}$). Whenever $f_{\mathrm{D(C)NF},k}$ creates the lower (upper) approximation of f then it is very natural to derive the conclusion as close to f as possible. This requirement is fulfilled by B_{DNF} as well as B_{CNF}. Thus, the fuzzy systems based on fuzzy approximating formulas together with proposed compositions behave properly.

4 Conclusion

The main goal of this work lies in formalization of basic properties of the composition w.r.t. the respective normal form. Observe that to maintain the character of fuzzy approximation, special compositional rule has to be used. It is the $\{\vee, *\}$-composition for the case of disjunctive normal form. The conjunctive normal form requires different treating and thus $\{\wedge, \rightarrow\}$-composition has been introduced to preserve analogous properties as composition used for disjunctive normal form.

Both compositions have a meaningful interpretation. They are used to draw a value on the basis of inaccurate input and partial knowledge about some fuzzy relation forming its lower or upper approximation. Hence, it is very natural to derive this value as close to the original fuzzy relation as possible. This attempt can be successfully implemented into the fuzzy system and used in fuzzy control.

Acknowledgments

The hereby material was and is deeply influenced by the important, instructive and inspiring work in this field done by my supervisor I. Perfilieva, to whom all my gratefulness for her support and

engagement goes.

This investigation has been partially supported by projects MSM6198898701 of the MŠMT ČR and IAA 1187301 GAAV ČR.

References

[1] K. E. Årzén, M. Johansson, R. Babuška: "Fuzzy control versus conventional control." In: H. Verbruggen, H. J. Zimmermann, R. Babuška (Eds.) *Fuzzy Algorithms for Control*, Boston, Kluwer, 1999, pp. 111–125.

[2] U. Bodenhofer: "Applications of Fuzzy Orderings: An Overview." In: K.T. Atanassov, O. Hryniewicz, J. Kacprzyk (Eds.) *Soft Computing - Foundations and Theoretical Aspects*, EXIT, Warsaw, 2004.

[3] M. Daňková: "Representation of logic formulas by normal forms," Kybernetika, Nr. 6, Vol. 38, pp. 717-728, 2002.

[4] M. Daňková: "Generalized extensionality of fuzzy relations." *Fuzzy Sets and Systems*, 2004, Vol. 148, pp. 291–304.

[5] M. Daňková and I. Perfilieva: "Logical approximation II." Soft Computing, Vol. 7, Nr. 4, pp. 228-233, 2003.

[6] D. Dubois, H. Prade: "Fuzzy sets in approximate reasoning, Part 1: Inference with possibility distributions." *Fuzzy Sets and Systems*, 1999, Vol. 100, pp. 73–132.

[7] P. Hájek: "Metamathematics of fuzzy logic," Kluwer, Dordrecht, 1998.

[8] V. Novák, I. Perfilieva, J. Močkoř: *Mathematical Principles of Fuzzy Logic.* Kluwer, Boston Dordrecht, 1999.

[9] V. Novák: "On the logical basis of approximate reasoning." In: V. Novák, J. Ramík, M. Mareš, M. Černý, and J. Nekola (Eds.) *Fuzzy Approach to Reasoning and Decision Making*, Kluwer, Dordrecht, 1992.

[10] F. Klawonn, V. Novák: "The relation between inference and interpolation in the framework of fuzzy systems." *Fuzzy sets and systems*, 1996, Vol. 81, pp. 331–354.

[11] F. Klawonn and R. Kruse: "Equality relations as a basis for fuzzy control." *Fuzzy sets and systems*, Nr. 2, Vol. 54, pp. 147 - 156, 1993.

[12] E.P. Klement, R. Mesiar and E. Pap: "Triangular norms." Kluwer, Boston, Dordrecht, 2000.

[13] R. Mesiar and V. Novák: "Operations fitting triangular-norm-based biresiduation." *Fuzzy Sets and Systems*, Vol. 104, pp. 77-84, 1999.

[14] B. Moser, M. Navara: "Fuzzy Controllers With Conditionally Firing Rules." *IEEE Trans. Fuzzy Systems*, 2002, Vol. 10, Nr. 3, 340-348.

[15] I. Perfilieva: "Normal forms for fuzzy logic functions and their approximation ability." *Fuzzy Sets and Systems*, Vol. 124, pp. 371-384, 2001.

[16] I. Perfilieva: "Logical Approximation." *Soft Computing*, 2001, Vol. 2, 73–78.

[17] I. Perfilieva: "Functions represented by BL-algebra formulas: characterization and approximate representation." *Soft Computing*, to appear

[18] T. Sudkamp: "Similarity, interpolation, and fuzzy rule construction." *Fuzzy Sets and Systems*, 1993, Vol. 58, Iss. 1, 73–86.