A note on ordinal sums of t-norms on bounded lattices

Susanne Saminger
Department of Knowledge-Based Mathematical Systems
Johannes Kepler University
Linz, Austria
susanne.saminger@jku.at

Erich Peter Klement
Department of Knowledge-Based Mathematical Systems
Johannes Kepler University
Linz, Austria
ep.klement@jku.at

Radko Mesiar
Department of Mathematics and Descriptive Geometry
Slovak University of Technology
Bratislava, Slovakia
and Institute for Research and Applications of Fuzzy Modeling
University of Ostrava
Czech Republic
mesiar@math.sk

Abstract

Ordinal sums of t-norms on bounded lattices are discussed. It is shown that this construction leads always to a t-norm in the case of horizontal sums of chains. Moreover, we obtain a t-norm also if the underlying lattice is an ordinal sum (in the sense of Birkhoff) of the carriers of the summands.

Keywords: Triangular norm; bounded lattice; ordinal sum

1 Introduction

Many-valued logics are usually based on a bounded lattice $(L, \leq, 0, 1)$ of truth values [6, 7, 10–12]. In such a case, the conjunction is interpreted by some triangular norm on L. The structure of (divisible) t-norms is known in some special cases only (closed real intervals, especially $[0, 1]$, finite chains), see [1, 8].

Recall that from the construction methods of t-norms only the ordinal sum construction can be applied both to t-norms on $[0, 1]$ (closed real intervals) and to t-norms on finite chains. This motivated us to have a deeper look at the construction of t-norms on bounded lattices by means of ordinal sums.

2 On some types of ordinal sums

Ordinal sums have been introduced in quite diverse contexts. One quite general approach is provided in [2] for building the ordinal sum $X \oplus Y$ of two disjoint posets X, Y, which can be generalized as follows:

Definition 2.1 Consider a linearly ordered index set $(I, \preceq_I)$, $I \neq \emptyset$ and a family of pairwise disjoint posets $(X_i, \leq_i)_{i\in I}$. The *ordinal sum* $\oplus_{i\in I} X_i$ (*in the sense of Birkhoff*) is defined as the set $\cup_{i\in I} X_i$ with the following order $\leq$ given by

$$x \leq y :\Longleftrightarrow (\exists i \in I : x, y \in X_i \wedge x \leq_i y) \text{ or } (\exists i, j \in I : x \in X_i \wedge y \in X_j \wedge i \prec_I j).$$

In [3], ordinal sums have been introduced in the context of abstract semigroups in order to construct a new semigroup from a given family of semigroups. The basic idea is to extend an ordinally ordered system of non-overlapping semigroups into a single semigroup whose carrier is equal to the union of the original carriers.

Definition 2.2 [3] Let $(I, \preceq)$, $I \neq \emptyset$ be a linearly ordered index set, $(X_i)_{i\in I}$ a family of pairwise disjoint sets, and $(G_i)_{i\in I}$ with $G_i = (X_i, *_i)$ a family of semigroups. Put $X = \cup_{i\in I} X_i$ and define the binary operation $*$ on X by

$$x * y = \begin{cases} x *_i y & \text{if } (x, y) \in X_i \times X_i, \\ x & \text{if } (x, y) \in X_i \times X_j \text{ and } i \prec j, \\ y & \text{if } (x, y) \in X_i \times X_j \text{ and } i \succ j. \end{cases}$$

Then we say that $(X, *)$ is the *ordinal sum* of all $(X_i, *_i)_{i\in I}$. If necessary, we will refer to this type of ordinal sum as *ordinal sum in the sense of Clifford*.

Proposition 2.3 [3] *With all the assumptions of the previous definition the ordinal sum $(X, *)$ is also a semigroup, i.e., $*$ is an associative operation on X.*

Note that ordinality in the sense of Clifford refers to the linear order of the index set I involved. The elements of some X_i, $i \in I$, need not fulfill some special order relation. On the other hand, taking into account that equality is an order relation on any set we immediately can state the following corollary.

Corollary 2.4 *Any ordinal sum in the sense of Clifford can be expressed as an associative operation on an ordinal sum of a family of sets in the sense of Birkhoff.*

In the case of t-norms on the real unit interval $[0,1]$ a version of the construction of ordinal sums in the sense of Clifford was proposed (see [5, 8, 9, 14]).

Definition 2.5 Let $(]a_i, b_i[)_{i\in I}$ be a family of pairwise disjoint open subintervals of $[0,1]$ and let $(T_i)_{i\in I}$ be a family of t-norms. Then the *ordinal sum* $T = (\langle a_i, b_i, T_i\rangle)_{i\in I}\colon [0,1]^2 \to [0,1]$ is given by

$$T(x,y) = \begin{cases} a_i + (b_i - a_i)T_i(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}) & \\ \qquad\qquad \text{if } (x,y) \in [a_i,b_i]^2, & \\ \min(x,y) \qquad\qquad \text{otherwise.} & \end{cases}$$

This type of ordinal sums is called *ordinal sum of t-norms*.

Note that ordinal sums of t-norms can be viewed as ordinal sums in the sense of Clifford (relaxing the disjointness requirement by allowing intervals overlapping in the boundary elements, and filling the gaps with the minimum). Therefore, associativity, monotonicity, commutativity and the neutral element are preserved by the construction process, and each ordinal sum of t-norms is again a t-norm.

3 Ordinal sums of t-norms on bounded lattices

Consider a bounded lattice $(L, \leq, 0, 1)$ with bottom element 0 and top element 1. Closed and open subintervals of L are defined, as usual, by

$$[a,b] = \{x \in L \mid a \leq x \leq b\},$$
$$]a,b[= [a,b] \setminus \{a,b\}.$$

A binary operation $T\colon L^2 \to L$ which is commutative, associative, non-decreasing in both arguments and has 1 as neutral element is called a *t-norm on L* (compare [4]). Note that the structure of the lattice L heavily influences which and how many t-norms on L can be defined. However, there exist at least two t-norms for arbitrary bounded lattices L with $|L| > 2$, i.e., the minimum ${T_{\mathbf{M}}}^L(x,y) = x \wedge y$ and the drastic product

$${T_{\mathbf{D}}}^L = \begin{cases} x \wedge y & \text{if } 1 \in \{x,y\}, \\ 0 & \text{otherwise,} \end{cases}$$

which are also the greatest and smallest possible t-norms on the lattice L.

Definition 3.1 Consider some bounded lattice $(L, \leq, 0, 1)$ and some linearly ordered index set I. Further, let $(]a_i, b_i[)_{i\in I}$ be a family of pairwise disjoint subintervals of L and $(T_i)_{i\in I}$ a family of t-norms on the corresponding $[a_i, b_i]$. Then the *ordinal sum* $T = (\langle a_i, b_i, T_i\rangle)_{i\in I}\colon L^2 \to L$ is given by

$$T(x,y) = \begin{cases} T_i(x,y) & \text{if } (x,y) \in [a_i,b_i]^2, \\ x \wedge y & \text{otherwise.} \end{cases} \tag{1}$$

Note that if $L = \oplus_{i\in I} [a_i, b_i]$, then an ordinal sum in the sense of Definition 3.1 is an ordinal sum in the sense of Clifford.

In the sequel we will concentrate on under which conditions an operation $T\colon L^2 \to L$ defined by (1) is also a t-norm on the bounded lattice L.

First we have a look at ordinal sums of t-norms on an arbitrary bounded lattice L with one summand only. Further, we will restrict ourselves in the following considerations to just one arbitrary but fixed interval as sublattice and as carrier of the only summand involved.

Consider a bounded lattice $(L, \leq, 0, 1)$ and a subinterval $[a,b]$. Further assume a t-norm $T^{[a,b]}$ on the bounded sublattice $[a,b]$. Then $T\colon L^2 \to L$ is defined by

$$T(x,y) = \begin{cases} T^{[a,b]}(x,y) & \text{if } (x,y) \in [a,b]^2, \\ x \wedge y & \text{otherwise,} \end{cases} \tag{2}$$

and is an *ordinal sum $(\langle a, b, T^{[a,b]}\rangle)$ of $T^{[a,b]}$ on L* with one summand.

T	0	x	a	u	1
0	0	0	0	0	0
x	0	x	0	0	x
a	0	0	a	a	a
u	0	0	a	u	u
1	0	x	a	u	1

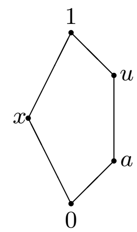

Figure 1: T-norm on some special bounded lattice

Proposition 3.2 *Consider some bounded lattice $(L, \leq, 0, 1)$ and a subinterval $[a, b]$ of L. If L is an ordinal sum of intervals, i.e., $L = [0, a] \oplus [a, b] \oplus [b, 1]$, then $T: L^2 \to L$ defined by (2) is a t-norm for arbitrary t-norm $T^{[a,b]}$ on $[a, b]$.*

The previous proposition only provides a sufficient and not a necessary condition such that T is a t-norm as the next example will illustrate.

Example 3.3 Consider the lattice $(L, \leq, 0, 1)$ with $L = \{0, x, a, u, 1\}$, further the subinterval $[a, 1] = \{a, u, 1\}$ and the operation T as shown in Figure 1. It can be easily checked that T is an operation defined by (2). Moreover, it is a t-norm although L is not an ordinal sum of intervals (note that the T is not the only t-norm on L since $T^{[a,1]} = T_{\mathbf{M}}{}^{[a,1]}$).

However, in some special cases the lattice L must have some special form:

Proposition 3.4 *Consider some bounded lattice $(L, \leq, 0, 1)$ and a subinterval $[a, b]$. Assume that T defined by (2) is a t-norm for arbitrary t-norms $T^{[a,b]}$ on $[a, b]$. If for all $x \in L$ there exists some $u \in \,]a, b[$ such that x can be compared with u, then $L = [0, a] \oplus [a, b] \oplus [b, 1]$.*

Example 3.3 shows that there exist ordinal sum t-norms on bounded lattices L with some subinterval $[a, b]$ where L is not an ordinal sums of intervals. Those $x \in L$ which do not belong to any of the subintervals $[0, a]$, $[a, b]$ or $[b, 1]$ are incomparable to all $u \in \,]a, b[$. They are further at most comparable to either a or b, but not to both at the same time (see also Figure 2).

A necessary and sufficient condition for (2) yielding a t-norm is the following:

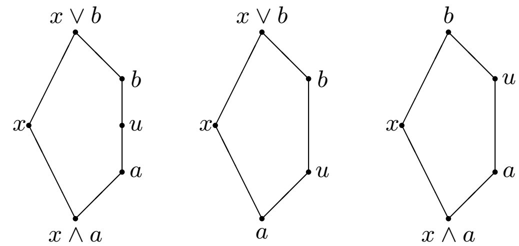

Figure 2: Lattice cases

Theorem 3.5 *Consider some bounded lattice $(L, \leq, 0, 1)$ and a subinterval $[a, b]$ of L. Then the following are equivalent:*

(i) *The ordinal sum $T: L^2 \to L$ defined by (2) is a t-norm for an arbitrary t-norm $T^{[a,b]}$ on $[a, b]$.*

(ii) *For all $x \in L$ it holds that*

  (a) *if x is incomparable to a, then it is incomparable to all $u \in [a, b[$ and*

  (b) *if x is incomparable to b, then it is incomparable to all $u \in \,]a, b]$.*

Now we look for conditions on the lattice L only ensuring that, for arbitrary subintervals, the construction (2) leads to a t-norm. Recall that a bounded poset $(X, \leq, 0, 1)$ is called a *horizontal sum* of the bounded posets $((X_i, \leq_i, 0, 1))_{i \in I}$ if $X = \bigcup_{i \in I} X_i$ with $X_i \cap X_j = \{0, 1\}$ whenever $i \neq j$, and $x \leq y$ if and only if there is an $i \in I$ such that $\{x, y\} \subseteq X_i$ and $x \leq_i y$ (compare, e.g., horizontal sums of effect algebras [13]). A nontrivial example of a horizontal sum is the set

$$\{(-1, -1), (1, 1), (-x, 1-x), (x, x-1) \mid x \in \,]0, 1[\}$$

equipped with the product order on $\mathbb{R}^2$.

Theorem 3.6 *Let $(L, \leq, 0, 1)$ be a bounded lattice. Then the following are equivalent:*

(i) *For any $[a, b] \subseteq L$ and any t-norm $T^{[a,b]}$ the ordinal sum operation T on L given by (2) is a t-norm on L.*

(ii) *For all $x, y \in L$: $\{x \wedge y, x \vee y\} \subseteq \{0, 1, x, y\}$.*

(iii) *$(L, \leq, 0, 1)$ is a horizontal sum of chains.*

This result can be generalized to suitable families of intervals.

Proposition 3.7 *Consider some bounded lattice* $(L, \leq, 0, 1)$. *Then the following are equivalent:*

(i) *The ordinal sum* T *as defined by* (1) *is a t-norm for arbitrary families of subintervals* $([a_i, b_i])_{i\in I}$ *and for arbitrary t-norms* $(T_i)_{i\in I}$ *on the corresponding* $[a_i, b_i]$.

(ii) $(L, \leq, 0, 1)$ *is a horizontal sum of chains.*

4 Conclusion

We have seen that the ordinal sum of t-norms leads always to a t-norm only in some special cases, namely, for horizontal sums of chains. As a consequence, this construction is rather limited in the case of product lattices.

Acknowledgments

The support of COST Action 274 TARSKI is acknowledged. The third author was also supported by grant MSM6198898701.

References

[1] C. Alsina, M. J. Frank, and B. Schweizer. *Associative Functions on Intervals: A Primer on Triangular Norms.* In press.

[2] G. Birkhoff. *Lattice Theory.* American Mathematical Society, Providence, 1973.

[3] A. H. Clifford. Naturally totally ordered commutative semigroups. *Amer. J. Math.*, 76:631–646, 1954.

[4] G. De Cooman and E. E. Kerre. Order norms on bounded partially ordered sets. *J. Fuzzy Math.*, 2:281–310, 1994.

[5] S. Gottwald. *Fuzzy Sets and Fuzzy Logic. Foundations of Application—from a Mathematical Point of View.* Vieweg, Braunschweig/Wiesbaden, 1993.

[6] S. Gottwald. *A Treatise on Many-Valued Logic.* Studies in Logic and Computation. Research Studies Press, Baldock, 2001.

[7] P. Hájek. *Metamathematics of Fuzzy Logic.* Kluwer Academic Publishers, Dordrecht, 1998.

[8] E. P. Klement, R. Mesiar, and E. Pap. *Triangular Norms.* Kluwer Academic Publishers, Dordrecht, 2000.

[9] C. M. Ling. Representation of associative functions. *Publ. Math. Debrecen*, 12:189–212, 1965.

[10] J. Łukasiewicz. Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls. *Comptes Rendus Séances Société des Sciences et Lettres Varsovie cl. III*, 23:51–77, 1930.

[11] V. Novák. *Fuzzy Sets and their Applications.* Adam Hilger, Bristol, 1989.

[12] E. L. Post. Introduction to a general theory of elementary propositions. *Amer. J. Math.*, 43:163–185, 1921.

[13] Z. Riečanová. Pastings of MV-effect algebras. *Internat. J. Theoret. Phys.*, 43:1875–1883, 2004.

[14] B. Schweizer and A. Sklar. *Probabilistic Metric Spaces.* North-Holland, New York, 1983.