Copulas with given diagonal section: some new results

Fabrizio Durante
Dipartimento di Matematica
"Ennio De Giorgi"
Università di Lecce
Lecce, Italy 73100
fabrizio.durante@unile.it

Radko Mesiar
STU Bratislava, Radlinskeho 11,
813 68 Bratislava, Slovakia
and UTIA AV CR, P.O. box 18,
18208 Prague, Czech Republic
mesiar@math.sk

Carlo Sempi
Dipartimento di Matematica
"Ennio De Giorgi"
Università di Lecce
Lecce, Italy 73100
carlo.sempi@unile.it

Abstract

We characterize the class of copulas that can be constructed from the diagonal section by means of the functional equation $C(x \vee y, x \vee y) = C(x, y) + |x - y|$, for all (x, y) in the unit square such that $C(x, y) > 0$. A simple construction method for copulas is also given.

Keywords: Copula, quasi–copula, aggregation operator.

1 Introduction

The concept of copula was introduced in 1959 ([17]) and its importance in statistical modeling is described by *Sklar's theorem*, which states that, given two random variables X and Y with joint distribution function H and marginals F and G, respectively, there exists a mapping C from $[0,1]^2$ into $[0,1]$, called *copula*, (which is uniquely determined on $\operatorname{Range} F \times \operatorname{Range} G$) such that $H(x, y) = C(F(x), G(y))$ for all $x, y \in \mathbf{R}$. Since then, copulas have played an important role not only in statistics, but also in many other fields such as probabilistic metric spaces ([16]), generalized measures and integrals ([11]), triangular norms ([10, 16]).

Specifically, $C : [0,1]^2 \to [0,1]$ is a *copula* if, and only if, it has *annihilator element* equal to 0,

$$\forall x \in [0,1] \qquad C(x,0) = C(0,x) = 0, \quad (1)$$

and *neutral element* equal to 1,

$$\forall x \in [0,1] \qquad C(x,1) = C(1,x) = x, \quad (2)$$

and it satisfies the 2–*increasing property*,

$$V_C\left([x,x'] \times [y,y']\right) := C(x',y') - C(x,y') - C(x',y) + C(x,y) \geq 0, \quad (3)$$

for all x, x', y, y' in $[0,1]$, with $x \leq x'$ and $y \leq y'$. A copula is necessarily 1–*Lipschitz*, i.e., for all $x, x', y, y' \in [0,1]$

$$|C(x,y) - C(x',y')| \leq |x - x'| + |y - y'|, \quad (4)$$

and *increasing in each place*,

$$C(x,y) \leq C(x',y') \quad \text{for } x \leq x' \text{ and } y \leq y'. \quad (5)$$

Following [2], we recall that a function A from $[0,1]^2$ into $[0,1]$ is a *binary aggregation operator* (shortly, 2–agop) if, and only if, it satisfies (5) with $A(0,0) = 0$ and $A(1,1) = 1$. In particular, every copula is a 2–agop.

The first generalization of copula is the concept of *quasi–copula*, which appeared in [1] and was studied also in [8]. In detail, a quasi–copula Q is a 2–agop which satisfies (2) and (4). Recently, another interesting generalization of copulas has been introduced in order to study bivariate notion of aging ([3]): it is the notion of *semicopula*, namely a 2–agop that satisfies (2) ([5]).

For every semicopula S, the function $\delta_S : [0,1] \to [0,1]$ given by $\delta_S(t) := S(t,t)$ is called the *diagonal section* (or, briefly, *diagonal*) of S. If S is a semicopula, its diagonal δ_S satisfies the following properties:

(D1) $\delta_S(1) = 1$;

(D2) $\delta_S(t) \leq t$ for all $t \in [0,1]$;

(D3) δ_S is increasing.

Moreover, if S is a (quasi–)copula, then

(D4) $|\delta_S(t) - \delta_S(s)| \le 2|t-s|$ for all $t, s \in [0,1]$.

The set of functions from $[0,1]$ into $[0,1]$ satisfying (D1), (D2) and (D3) will be denoted by $\mathcal{D}$; the set of functions in $\mathcal{D}$ which satisfy also (D4) will be denoted by $\mathcal{D}_2$.

The question arises whether, for each function $\delta \in \mathcal{D}_2$, there is a copula whose diagonal section coincides with δ. In the literature, this question has already two answers. The first one is the *diagonal copula*

$$K_\delta(x,y) := \min\left\{x, y, \frac{\delta(x)+\delta(y)}{2}\right\}, \qquad (6)$$

introduced in [6]. The second one is the *Bertino copula* ([7])

$$B_\delta(x,y) := \min\{x,y\} - \min_{t\in[x\wedge y, x\vee y]} \{t - \delta(t)\}. \quad (7)$$

¿From a probabilistic point of view, the diagonal section of a copula has interest because, if X and Y are random variables with the same distribution function F and copula C, then the distribution function of $\max(X,Y)$ is $\delta_C(F(t))$. On the other hand, a deep discussion of the problem of copulas and quasi–copulas with given diagonal section and its relationship with aggregation operators can be found in the forthcoming paper [9].

In this note, we analyze under which conditions on $\delta \in \mathcal{D}_2$ the function

$$D_\delta(x,y) := \max\{0, \delta(x\vee y) - |x-y|\} \qquad (8)$$

is a copula. In the class of *t–norms*, viz. associative and commutative semicopulas (see [10]), functions of type (8) were already studied by Mayor and Torrens ([13]), where the following result was obtained:

Theorem 1 *Let T be a continuous t–norm with diagonal section δ. Then T satisfies the functional equation*

$$T(x,y) + |x-y| = \delta(x\vee y),$$

whenever $T(x,y) > 0$, if, and only if, T belongs to the family $\{T_\alpha : \alpha\in[0,1]\}$ defined by

$$T_\alpha(x,y) := \begin{cases} \max\{0, x+y-\alpha\} & (x,y)\in[0,\alpha]^2; \\ \min\{x,y\} & \text{otherwise.}\end{cases}$$

For these reasons, we shall call a function of the type (8) *MT– (quasi–, semi–) copula*, where "MT" stands for "à la Mayor and Torrens".

In Section 2 we characterize the class of MT–copulas and, then, we study its properties (Section 3): for more details and proofs of the statements we refer to the forthcoming paper [4]. In Section 4, we present a simple method to generate copulas, beginning with two copulas having the same diagonal section.

2 Characterization of MT–copulas

In order to characterize MT–copulas, first, we establish an analogous characterization for semicopulas of the same type.

Proposition 1 *The following statements are equivalent:*

(a) $\delta \in \mathcal{D}$ *and there exists* $a \in [0,1[$ *such that* $\delta(x) = 0$ *on* $[0,a]$ *and the function* $x \mapsto (\delta(x) - x)$ *is increasing on* $[a,1]$.

(b) D_δ *is an MT–semicopula;*

Now, we can give the characterization of the *MT*–copulas.

Theorem 2 *The following statements are equivalent:*

(a') $\delta \in \mathcal{D}_2$ *and there exists* $a \in [0,1/2]$ *such that* $\delta(x) = 0$ *on* $[0,a]$ *and the function* $x \mapsto (\delta(x) - x)$ *is increasing on* $[a,1]$*;*

(b') D_δ *is a copula.*

The following result, which can be easily derived from the proof of Theorem 2, states the impossibility of constructing a MT–quasi–copula that is not a copula.

Corollary 1 *The following statements are equivalent:*

(a') $\delta \in \mathcal{D}_2$ *and there exists* $a \in [0, 1/2]$ *such that* $\delta(x) = 0$ *on* $[0, a]$ *and the function* $x \mapsto (\delta(x) - x)$ *is increasing on* $[a, 1]$*;*

(c') D_δ *is a quasi–copula.*

3 Properties of the MT–copulas

In this section, we denote by D an MT–copula and by δ its diagonal, according to Theorem 2.

Proposition 2 *Every MT–copula* D *is a simple Bertino copula, i.e., for all* $(x, y) \in [0, 1]^2$

$$D(x, y) := x \wedge y - \min\left(x - \delta(x), y - \delta(y)\right). \quad (9)$$

In particular, from this result we obtain

Proposition 3 *Every MT–copula* D *is extremal, in the sense that, if there exist two copulas* A *and* B *such that* $D = \alpha A + (1 - \alpha)B$*, with* $\alpha \in]0, 1[$*, then* $D = A = B$*.*

The following statistical characterization of an MT–copula can be formulated

Corollary 2 *Let* X *and* Y *be two random variables uniformly distributed on* $[0, 1]$ *whose joint distribution function is* D*. Then, for each* $(x, y) \in [0, 1]^2$*, either*

$$P(X \le x, Y \le y) = P(\max(X, Y) \le \min(x, y))$$

or

$$P(X > x, Y > y) = P(\min(X, Y) > \max(x, y)).$$

Because of Theorem 1, since T_α is an ordinal sum of W, $T_\alpha = (\langle 0, \alpha, W\rangle)$, and thus a copula for every α in $[0, 1]$, one has

Proposition 4 *The only associative MT–copulas are members of Mayor–Torrens family given in Theorem 1.*

In the following we characterize the MT–copulas that coincide with their corresponding survival copula.

Proposition 5 *Let* D *be an MT–copula and* $\widehat{D}$ *be its corresponding survival copula, defined on* $[0, 1]^2$ *by* $\widehat{D}(x, y) := x + y - 1 - D(1 - x, 1 - y)$*. Then* $D = \widehat{D}$*, if, and only if, there exists* $a \in [0, 1/2]$ *such that* D *is a member of a family of copulas given by*

$$C_a(x, y) = \max\{W(x, y), M(x, y) - a\}. \quad (10)$$

4 A construction method

Let Δ_+ and Δ_- be the two subsets of the unit square defined by

$$\Delta_+ := \{(x, y) \in [0, 1]^2 : x \ge y\},$$
$$\Delta_- := \{(x, y) \in [0, 1]^2 : x < y\}.$$

For every subset S of $[0, 1]^2$, 1_S denotes the *indicator function* of S, i.e., $1_S(x, y)$ is equal to 1, if $(x, y) \in S$, and 0, otherwise.

Given the 2–agops A and B, we introduce the function $F_{A,B} : [0, 1]^2 \to [0, 1]$ given, for all x, y in $[0, 1]$, by

$$F_{A,B}(x, y) := A(x, y)\, 1_{\Delta_+}(x, y) + B(x, y)\, 1_{\Delta_-}(x, y).$$

In other words, if we divide the unit square by means of the diagonal $y = x$, F is equal to A in the lower triangle and equal to B in the upper one.

The following result is easily proved.

Proposition 6 *If A and B are 2–agops with the same diagonal section, then $F_{A,B}$ is a 2–agop. Moreover, if A and B have neutral element 1, so has $F_{A,B}$.*

Remark 1 *If A and B are 2–agops such that $\delta_A \neq \delta_B$, then $F_{A,B}$ need not be increasing. For example, if $A(x,y) = \min(x,y)$ and $B(x,y) = xy$, then*

$$F_{A,B}(0.5, 0.4) = 0.4 > 0.3 = F_{A,B}(0.5, 0.6).$$

In the following two theorems we investigate the cases in which A and B are copulas or quasi–copulas.

Theorem 3 *If A and B are quasi–copulas with the same diagonal section, then $F_{A,B}$ is a quasi–copula.*

Corollary 3 *If A and B are 1–Lipschitz 2–agops with the same diagonal section, then $F_{A,B}$ is a 1–Lipschitz 2–agop.*

Theorem 4 *Let A and B be copulas with the same diagonal section. If A and B are symmetric, then $F_{A,B}$ is a copula.*

Proof. Notice that inequality (3) follows directly from the 2–increasing property of A and B on rectangles entirely contained in either Δ_+ or Δ_-. Therefore, it suffices to show that, for all $s, t \in [0,1]$ with $s < t$,

$$V_F([s,t]^2) := A(s,s) + A(t,t) - B(s,t) - A(t,s) \geq 0.$$

But, if $B(s,t) \leq A(s,t)$, then $V_F\left([s,t]^2\right) \geq V_A\left([s,t]^2\right) \geq 0$; if $B(s,t) > A(s,t)$, then $B(t,s) > A(t,s)$, so that

$$\begin{aligned} &V_F([s,t]^2) = B(s,s) \\ &+ B(t,t) - B(s,t) - A(t,s) \geq V_B([s,t]^2) \geq 0, \end{aligned}$$

which concludes the proof.

Remark 2 *In Theorem 4, the assumption of the symmetry of copulas is essential. If, for example, A is a non–symmetric copula, then $F_{A,B}$ need not be a copula. We consider, for example, the copula A given by*

$$A(x,y) = \begin{cases} \max\left(x + \dfrac{1}{2}(y-1), 0\right), & x \in \left[0, \dfrac{1}{2}\right]; \\[2ex] \min\left(x + \dfrac{1}{2}(y-1), y\right), & x \in \left]\dfrac{1}{2}, 1\right], \end{cases}$$

and the copula B given by

$$B(x,y) := \min\left(x, y, \frac{\delta_A(x) + \delta_A(y)}{2}\right).$$

If $R := [1/3, 2/3]^2$, one has

$$V_{F_{A,B}}(R) = -1/12 < 0,$$

viz. $F_{A,B}$ is not a copula. Specifically, because of Theorem 3, $F_{A,B}$ is a (proper) quasi–copula.

Remark 3 *In* [15], *a general method was described to symmetrize a 2–agop. Specifically, let A be a 2–agop (generally, non–symmetric), for every $x, y \in [0,1]$, the symmetrized version of A is defined by*

$$\widetilde{A}(x,y) = \begin{cases} A(x,y), & \text{if } x \geq y; \\ A(y,x), & \text{if } x \leq y. \end{cases} \tag{11}$$

Since it is a clear that, if A is a 2–agop (quasi–copula), then the function $A'(x,y) = A(y,x)$ for every (x,y) in $[0,1]^2$, is a 2–agop (quasi–copula), it follows from Proposition 6 (Theorem 3) that $\widetilde{A}$ is a 2–agop (quasi–copula). Notice that, given a copula C, $\widetilde{C}$ is not a copula. We consider, for example, the copula C_λ ($\lambda \in [0,1]$) defined by

$$C_\lambda(x,y) = \begin{cases} y, & y \leq \lambda x; \\ \lambda x, & \lambda x < y \leq 1 - (1-\lambda)x; \\ x + y - 1, & 1 - (1-\lambda)x < y \leq 1. \end{cases}$$

For a fixed $\epsilon \in \left]0, \frac{1}{2}\right[$, one has

$$\begin{aligned} &V_{\widetilde{C_\lambda}}\left(\left[\frac{1}{2}, \frac{1}{2} + \epsilon\right] \times \left[\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon\right]\right) \\ &= \frac{\lambda}{2} - \lambda\left(\frac{1}{2} + \epsilon\right) < 0. \end{aligned}$$

A similar construction method can also be introduced for 2–agops that have the same values in some fixed linear section [9, 12], for example, with the same opposite diagonal sections, i.e., for 2–agops A and B $A(x,1-x) = B(x,1-x)$ for every x in $[0,1]$. In this case, let Γ_+ and Γ_- be the two subsets of the unit square defined by

$$\Gamma_+ := \{(x,y) \in [0,1]^2 : x+y \leq 1\},$$
$$\Gamma_- := \{(x,y) \in [0,1]^2 : x+y > 1\}.$$

Given the 2–agops A and B, we introduce the function $F^{A,B} : [0,1]^2 \to [0,1]$ given, for all $x,y \in [0,1]$, by

$$F^{A,B}(x,y) := A(x,y)\, 1_{\Gamma_+}(x,y) + B(x,y)\, 1_{\Gamma_-}(x,y).$$

As above, we have

Proposition 7 *If A and B are 2–agops with the same opposite diagonal section, then $F^{A,B}$ is a 2–agop. Moreover, if A and B are quasi–copulas, so is $F^{A,B}$.*

Theorem 5 *Let A and B be copulas with the same opposite diagonal section. If $B(x,y) \geq A(x,y)$ for every $(x,y) \in \Gamma_-$, then $F^{A,B}$ is a copula.*

5 Conclusion

We have found a characterization of the copulas which satisfy the functional equation $C(x,y) + |x-y| = C(x \vee y, x \vee y)$ whenever $C(x,y) > 0$, already studied in the class of triangular norms in the paper [13]. Then, we have introduced a method to generate a new copula, beginning from two copulas with given diagonal section. However, this method can be generalized and studied in a more general form:

Problem 1.

Let Ω be a connected subset of the unit square. Let A and B be two copulas which coincide on the border of Ω. Under which conditions

$$F(x,y) = A(x,y)\, 1_{\Omega}(x,y) + B(x,y)\, 1_{[0,1]^2 \setminus \Omega}(x,y)$$

is a copula?

On the other hand, this suggests also an investigation on the sets of uniqueness for a copula.

Problem 2.

Let f and g be distinct continuous functions from $[0,1]$ into $[0,1]$ and let A and B be copulas such that $A(x,f(x)) = B(x,f(x))$ and $A(g(y),y) = B(g(y),y)$ for all $x,y \in [0,1]$. Under which conditions on f and g does it follow that $A = B$?

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