Triangular norms and k-Lipschitz property

A. Mesiarová
Mathematical Institute of SAS
Štefánikova 49
81473 Bratislava, Slovakia
mesiarova@mat.savba.sk

Abstract

Inspired by an open problem of Alsina, Frank and Schweizer, k-Lipschitz t-norms are studied. The k-convexity of continuous monotone functions is introduced. Additive generators of k-Lipschitz t-norms are completely characterized by means of k-convexity. For a given $k \in [1, \infty[$ the pointwise infimum A_{*k} of the class of all k-Lipschitz t-norms is introduced.

Keywords: additive generator, k-Lipschitz property, triangular norm

1 Introduction

Triangular norms are, on the one hand, special semigroups and, on the other hand, solutions of some functional equations [1, 3, 7, 9]. This mixture quite often requires new approaches to answer questions about nature of triangular norms.

A triangular norm (t-norm for short) $T : [0,1]^2 \to [0,1]$ is an associative, commutative, non-decreasing function such that 1 acts as a neutral element [7]. Most important t-norms are the minimum $T_{\mathbf{M}}$, the product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ given by $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$. Observe that each continuous Archimedean t-norm T can be represented by means of a continuous additive generator [3, 4], i.e., a strictly decreasing continuous function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$ such that

$$T(x,y) = t^{(-1)}(t(x) + t(y)),$$

where the pseudo-inverse $t^{(-1)} : [0,\infty] \to [0,1]$ in this special case is given by

$$t^{(-1)}(u) = t^{-1}(\min(u, t(0))).$$

Note that if t is an additive generator of a t-norm T then for any $d \in]0,\infty[$ also $d \cdot t$ is an additive generator of the t-norm T. For continuous t-norms the additive generator is uniquely determined up to a multiplicative constant.

For the sake of completeness recall that each continuous t-norm (see [3, 4]) can be represented as an ordinal sum of continuous Archimedean t-norms (t-norm is called Archimedean if for each $(x,y) \in]0,1[^2$ there is an $n \in \mathbb{N}$ with $x_T^{(n)} < y$, where $x_T^{(n)} = T(x, x_T^{(n-1)})$ and $x_T^{(1)} = x$). More precisely, for each continuous t-norm T there exists a unique (finite or countably infinite) index set A, a family of unique pairwise disjoint open subintervals $(]a_\alpha, e_\alpha[)_{\alpha \in A}$ and a family of unique continuous Archimedean t-norms $(T_\alpha)_{\alpha \in A}$ such that for all $(x,y) \in [0,1]^2$

$$T(x,y) = \begin{cases} a_\alpha + (e_\alpha - a_\alpha) \cdot T_\alpha\left(\frac{x-a_\alpha}{e_\alpha - a_\alpha}, \frac{y-a_\alpha}{e_\alpha - a_\alpha}\right) \\ \qquad \text{if } (x,y) \in [a_\alpha, e_\alpha]^2, \\ \min(x,y) \\ \qquad \text{otherwise.} \end{cases}$$

We shall also write $T = (\langle a_\alpha, e_\alpha, T_\alpha \rangle)_{\alpha \in A}$.

In the center of our interest are t-norms which satisfy k-Lipschitz property.

Definition 1
Let $T : [0,1]^2 \to [0,1]$ be a t-norm and let $k \in]0,\infty[$ be a constant. Then T is *k-Lipschitz* if

$$|T(x_1,y_1) - T(x_2,y_2)| \le k \cdot (|x_1 - x_2| + |y_1 - y_2|) \quad (1)$$

for all $x_1, x_2, y_1, y_2 \in [0,1]$.

Because of the neutral element $e = 1$, a t-norm can be k-Lipschitz only for $k \geq 1$. It is evident that if a t-norm T is k-Lipschitz it is also m-Lipschitz for any $m \in \mathbb{R}$, $k \leq m$. As it was shown in [6, 9], 1-Lipschitz t-norms are exactly those t-norms which are also copulas. By [6, 8], a continuous strictly decreasing function $t : [0,1] \to [0,\infty]$ with $t(1) = 0$ is an additive generator of a 1-Lipschitz Archimedean t-norm if and only if it is convex. The aim of this work is to give an answer to the open problem no. 11 from [2], i.e., to characterize and discuss k-Lipschitz t-norms.

Note that a partial answer to the problem of Alsina et al. posed in [2] was given by Y.-H. Shyu [10] who has shown that if the additive generator t of a t-norm T is differentiable and $t'(x) < 0$ for $0 < x < 1$, then T is k-Lipschitz if and only if $t'(y) \geq kt'(x)$ whenever $0 < x < y < 1$. This special case, as well as the characterization of additive generators of 1-Lipschitz t-norms, follow from our characterization. For more details and full proofs of the next results see [5].

2 k-Lipschitz t-norms and additive generators

Let T be a k-Lipschitz t-norm. Since it is k-Lipschitz it is evident that it is necessarily also continuous and it can be uniquely expressed as an ordinal sum of continuous Archimedean t-norms (for more details see [4]) which are then necessarily k-Lipschitz Archimedean t-norms. Furthermore, each of these k-Lipschitz Archimedean t-norm has a continuous additive generator.

Definition 2
Let $f : [0,1] \to [0,\infty]$ be a strictly monotone function and let $k > 0$ be a real constant. Then f will be called *k-convex* if

$$f(x + k\varepsilon) - f(x) \leq f(y + \varepsilon) - f(y) \qquad (2)$$

holds for all $x \in [0,1[\,, y \in\,]0,1[\,, \varepsilon \in\,]0,1[$ where $x \leq y$ and $\varepsilon \leq \min(1-y, \frac{1-x}{k})$.

Observe that Shyu's condition mentioned in introduction is a sufficient condition for k-convexity of an additive generator t.

Note also that because of the monotonicity, a continuous strictly decreasing function t can be k-convex only for $k \geq 1$. Moreover, when t is k-convex it is l-convex for all $l \geq k$. In the case of strictly increasing function, a continuous strictly increasing function c can by k-convex only for $k \leq 1$. Moreover, when c is k-convex it is l-convex for all $l \leq k$. Note also that formula (2) immediately implies the continuity of a strictly monotone function.

The following is an equivalent definition of k-convexity.

Lemma 1
Let $t : [0,1] \to [0,\infty]$ be a continuous strictly monotone function then the following are equivalent.

(i) t is k-convex.

(ii) For all $x \in [0,1[\,, y \in\,]0,1[\,, \varepsilon \in\,]0,1[$ where $x \leq y$ and $\varepsilon \leq 1-y$ it holds

$$t(\min(x+k\varepsilon, 1)) - t(x) \leq t(y+\varepsilon) - t(y). \quad (3)$$

Theorem 1
Let $T : [0,1]^2 \to [0,1]$ be an Archimedean t-norm and let $t : [0,1] \to [0,\infty]$ be an additive generator of T. Then T is k-Lipschitz if and only if t is k-convex.

Note that for $k = 1$ we get $t(y+\varepsilon) - t(y) \geq t(x+\varepsilon) - t(x)$ whenever $x \leq y$, $0 < \varepsilon \leq 1-y$, i.e., the function t is convex.

Corollary 1
Let $t : [0,1] \to [0,\infty]$ be an additive generator of a k-Lipschitz Archimedean t-norm. Let $x_0, y_0 \in\,]0,1]\,, x_0 \leq y_0$ ($x_0, y_0 \in [0,1[\,, x_0 \leq y_0$). Then if there exist left (right) derivatives $t'_-(x_0)$ and $t'_-(y_0)$ ($t'_+(x_0)$ and $t'_+(y_0)$) we have

$$t'_-(x_0) \leq \frac{1}{k} t'_-(y_0)$$

$$(t'_+(x_0) \leq \frac{1}{k} t'_+(y_0)).$$

Moreover, let $z_0 \in\,]0,1[$ be such that both left and right derivatives $t'_-(z_0)$ and $t'_+(z_0)$ exist. Then we have

$$t'_-(z_0) \leq \frac{1}{k} t'_+(z_0).$$

From Corollary 1 and Theorem 1 follows the necessity of the result of Y.-H.Shyu [10]

Corollary 2 (Y.-H. Shyu)
Let $t : [0,1] \to [0,\infty]$ be an additive generator of a t-norm T, differentiable on $]0,1[$ and let $t'(x) < 0$ for $0 < x < 1$. Then T is k-Lipschitz if and only if $t'(y) \geq kt'(x)$ whenever $0 < x < y < 1$.

Corollary 3
Let $T : [0,1]^2 \to [0,1]$ be a continuous Archimedean t-norm and let $t : [0,1] \to [0,\infty]$ be an additive generator of T such that t is differentiable on $]0,1[\setminus \mathcal{R}$, where $\mathcal{R} \subset [0,1]$ is a discrete set. Then T is k-Lipschitz if and only if $kt'(x) \leq t'(y)$ for all $x, y \in [0,1]$, $x \leq y$ such that $t'(x)$ and $t'(y)$ exist.

Corollary 4
Let $t : [0,1] \to [0,\infty]$ be a strictly decreasing function differentiable on $]0,1[$ and let $t(0) = 1$. If $k \cdot \sup_{x\in]0,1[} t'(x) \leq \inf_{x\in]0,1[} t'(x)$ then t is an additive generator of some k-Lipschitz t-norm.

Example 1

(i) Let $t : [0,1] \to [0,\infty]$ be given by $t(x) = \frac{sin(\frac{\pi}{3}(1-x))}{\frac{\pi}{3}}$. Then $\sup_{x\in]0,1[} t'(x) = -\frac{1}{2}$ and $\inf_{x\in]0,1[} t'(x) = -1$, and hence $2 \cdot \sup_{x\in]0,1[} t'(x) \leq \inf_{x\in]0,1[} t'(x)$, i.e., t is an additive generator of some 2-Lipschitz t-norm.

(ii) Let $t : [0,1] \to [0,\infty]$ be given by $t(x) = (1-x) + \frac{(1-x)^2}{4}$. Then $\sup_{x\in]0,1[} t'(x) = -1$ and $\inf_{x\in]0,1[} t'(x) = -\frac{3}{2}$, and we have $\frac{3}{2} \cdot \sup_{x\in]0,1[} t'(x) \leq \inf_{x\in]0,1[} t'(x)$, i.e., t is an additive generator of some $\frac{3}{2}$-Lipschitz t-norm.

Although in the case of 1-Lipschitz t-norms their additive generators have left (right) derivative everywhere on $]0,1[$ (since $\frac{t(x)-t(x-\varepsilon)}{\varepsilon}$ is increasing when ε is decreasing), in the case of k-Lipschitz t-norms with $k > 1$ the situation is different. The following is an example of an additive generator of a 2-Lipschitz t-norm with no right derivative in $\frac{1}{2}$.

Example 2
Let $(a_n)_{n\in\mathbb{N}_0}$ be a sequence, where $a_n = \frac{1}{2} + (\frac{1}{2})^{n+1}$. Let $t : [0,1] \to [0,\infty]$ be given by

$$t(x) = \begin{cases} -x + \frac{11}{12} + \frac{1}{6}\frac{1}{2^{2n+1}} \\ \qquad \text{if } x \in [a_{2n+1}, a_{2n}], n \in \mathbb{N}_0 \\ -\frac{x}{2} + \frac{2}{3} - \frac{1}{3}\frac{1}{2^{2n+3}} \\ \qquad \text{if } x \in]a_{2n+2}, a_{2n+1}[\,, n \in \mathbb{N}_0 \\ -x + \frac{11}{12} \\ \qquad \text{if } x \in \left[0, \frac{1}{2}\right[. \end{cases}$$

Then t has no right derivative in point $\frac{1}{2}$. Moreover, since for all $x \in [0,1]$ and all $\varepsilon \in]0, 1-x[$ it is $t(x+\varepsilon) - t(x) \in [-\varepsilon, -\frac{\varepsilon}{2}]$ we have $t(x + 2\varepsilon) - t(x) \leq -\varepsilon \leq t(y+\varepsilon) - t(y)$ for all $x, y, 0 < \varepsilon \leq \min(1-y, \frac{1-x}{2})$, i.e., t is 2-convex and due to Theorem 1 it is an additive generator of some 2-Lipschitz t-norm.

Note only that each continuous monotone function has derivative almost everywhere, i.e., the Lebesgue measure of the set S of all points from $[0,1]$ where derivative does not exist is equal to zero.

The following example shows that the requirement in Corollary 3 for set $\mathcal{R}$ to be discrete is substantial.

Example 3
Let $t : [0,1] \to [0,\infty]$ be given by $t(x) = 1 - x + f(1-x)$ for all $x \in [0,1]$, where $f : [0,1] \to [0,1]$ is the Cantor function, i.e., $f(\frac{1}{3}) = f(\frac{2}{3}) = \frac{1}{2}$, etc. Then $t'(x) = -1$ for all $x \in [0,1]$ where $t'(x)$ exist. Since t is continuous and strictly decreasing with $t(1) = 0$ we know that t is an additive generator of some continuous t-norm. But t is not k-Lipschitz for any $k \in [1,\infty[$. For example

$$\begin{aligned} T(\tfrac{55}{81}, \tfrac{74}{81}) &= t^{(-1)}(t(\tfrac{55}{81}) + t(\tfrac{74}{81})) \\ &= t^{-1}(\tfrac{26}{81} + \tfrac{7}{16} + \tfrac{7}{81} + \tfrac{3}{16}) \\ &= \tfrac{16}{27} + \tfrac{7}{8} - 1 \end{aligned}$$

and

$$\begin{aligned} T(\tfrac{2}{3}, \tfrac{74}{81}) &= t^{(-1)}(t(\tfrac{2}{3}) + t(\tfrac{74}{81})) \\ &= t^{-1}(\tfrac{1}{3} + \tfrac{1}{2} + \tfrac{7}{81} + \tfrac{3}{16}) \\ &= \tfrac{47}{81} + \tfrac{13}{16} - 1. \end{aligned}$$

We get that

$T(\frac{55}{81},\frac{74}{81}) - T(\frac{2}{3},\frac{74}{81}) = \frac{1}{81} + \frac{1}{16} = \frac{97}{1296} = \frac{6.0625}{81}$

and

$$\frac{55}{81} - \frac{2}{3} = \frac{1}{81}.$$

We have

$$|T(\tfrac{55}{81},\tfrac{74}{81}) - T(\tfrac{2}{3},\tfrac{74}{81})| > 6|\tfrac{55}{81} - \tfrac{2}{3}|,$$

i.e., T is not 6-Lipschitz. Similarly we can show for any $k \in [1,\infty[$ that T is not k-Lipschitz.

We will now continue in the investigation of additive generators of k-Lipschitz t-norms.

Proposition 1
Let $t : [0,1] \to [0,\infty]$ be an additive generator of a k-Lipschitz t-norm T. Then for any $x, y \in [0,1], x < y$ and any $z \in [x,y]$ we have

$$t(z) \le \frac{z(kt(x) - t(y)) + xt(y) - kyt(x)}{(k-1)z + x - ky}. \qquad (4)$$

Remark 1
Supposing the differentiability of t on $[0,1]$ we get the following easy proof of Proposition 1: for $z \in \{x,y\}$ the inequality 4 trivially holds. Assume $\alpha \in]0,1[$ then from Lagrange formula we get that $t(y) - t(\alpha x + (1-\alpha)y) = t'(\theta)(\alpha y - \alpha x)$ for some $\theta \in [\alpha x + (1-\alpha)y, y]$ and that $t(\alpha x + (1-\alpha)y) - t(x) = t'(\varphi)((1-\alpha)y - (1-\alpha)x)$ for some $\varphi \in [x, \alpha x + (1-\alpha)y]$. Since $\varphi \le \theta$ from Corollary 2 we have $t'(\theta) \ge kt'(\varphi)$. We get

$$t(y) - t(\alpha x + (1-\alpha)y) \ge \frac{\alpha k}{1-\alpha}\left(t(\alpha x + (1-\alpha)y) - t(x)\right),$$

i.e.,

$$(1-\alpha)t(y) + \alpha k t(x) \ge (\alpha k + 1 - \alpha)t(\alpha x + (1-\alpha)y). \qquad (5)$$

Note that the inequality (5) with $\alpha = \frac{z-y}{x-y}$ is just the inequality (4).

Recall the classical definition of convexity of a function t, in which for all $x, y \in \mathrm{Dom}(t)$ and $\alpha \in [0,1]$ it holds

$$t(\alpha x + (1-\alpha)y) \le \alpha t(x) + (1-\alpha)t(y).$$

However, the last inequality is just the inequality (5) for $k = 1$.

Since the left derivative of the function $t(z) = \frac{z(kt(x)-t(y))+xt(y)-kyt(x)}{(k-1)z+x-ky}, z \in [x,y]$ in the point x is $t'_-(x) = \frac{t(x)-t(y)}{k(x-y)}$ and the right derivative in the point y is $t'_+(y) = \frac{k(t(x)-t(y))}{x-y}$, from Corollary 4 it follows that this function is not itself an additive generator of some k-Lipschitz t-norm, but it is an additive generator of some k^2-Lipschitz t-norm. This also means that the set of all normed additive generators of nilpotent k-Lipschitz t-norms has no strongest element and its supremum is the function $\frac{k(z-1)}{z(k-1)-k}$.

Corollary 5
Let $t : [0,1] \to [0,\infty]$ be an additive generator of a k-Lipschitz t-norm T. Then for any $x, y \in [0,1], x \le y$ and any $z \in [x,y]$ we have

$$t(z) \le t(x) + \frac{1}{k}\frac{t(x) - t(y)}{x - y}(z - x)$$

and

$$t(z) \le t(y) + k\frac{t(x) - t(y)}{x - y}(z - y)$$

3 Approximation of k-Lipschitz t-norms

It is easy to prove that for a given $k \in [1,\infty[$ the limit of the Cauchy sequence of k-Lipschitz t-norms is again a k-Lipschitz t-norm. Moreover, we have the following result:

Theorem 2
The set of all k-Lipschitz t-norms $\mathcal{K}$ is the closure of both the set of all strict k-Lipschitz t-norms and the set of all nilpotent k-Lipschitz t-norms.

This means that each k-Lipschitz t-norm can be approximated with an arbitrary precision by strict as well as by nilpotent k-Lipschitz t-norms.

The minimum t-norm $T_{\mathbf{M}}$ is k-Lipschitz for all $k \in [1,\infty[$, i.e., for all $k \in [1,\infty[$ the minimum t-norm $T_{\mathbf{M}}$ is the maximum of the class of all k-Lipschitz t-norms. However, though there are several minimal k-Lipschitz t-norms there is no weakest k-Lipschitz t-norm for $k > 1$.

Proposition 2
Let $A_{*k} : [0,1]^2 \to [0,1]$ be given by

$$A_{*k}(x,y) = \inf\{T(x,y) \mid T \text{ is a } k\text{-Lipschitz t-norm}\},$$

i.e., A_{*k} is the pointwise infimum of all k-Lipschitz t-norms. Then A_{*k} is the weakest k-Lipschitz aggregation operator with neutral element 1, i.e., $A_{*k}(x,y) = \max(x + ky - k, y + kx - k, 0)$.

The aggregation operator from the above proposition is a t-norm only for $k = 1$.

Theorem 3
Let $T : [0,1]^2 \to [0,1]$ be a k-Lipschitz t-norm such that A_{*k} and T are ε-close for some $\varepsilon \geq 0$, i.e., $||A_{*k} - T|| \leq \varepsilon$. Then $\varepsilon \geq \frac{k^2-k}{(k+1)(3k+1)}$. Moreover, there exists a t-norm T_{*k} such that A_{*k} and T_{*k} are $\frac{k^2-k}{(k+1)(3k+1)}$-close.

Example 4
Let $t : [0,1] \to [0,\infty]$ be given by

$$t(x)=\begin{cases} 1-x & \text{if } x \in [\frac{3k}{3k+1}, 1], \\ \frac{k+1}{2k}(1-x) + \frac{k-1}{2k(3k+1)} & \text{if } x \in \left[\frac{k}{k+1}, \frac{3k}{3k+1}\right[, \\ \frac{1-x}{k} + \frac{2k^2-k-1}{k(k+1)(3k+1)} & \text{otherwise.} \end{cases}$$

Then the t-norm T generated by the additive generator t is $\frac{k^2-k}{(k+1)(3k+1)}$-close to A_{*k}.

This means that the t-norm from the previous example is the best approximation of A_{*k}.

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References

[1] J. Aczél, "Lectures on Functional Equations and their Applications," Academic Press, New York, 1966.

[2] C. Alsina, M.J. Frank and B. Schweizer "Problems on associative functions," Aequationes Math, vol. 66, 2003, pp. 128–140.

[3] E.P. Klement, R. Mesiar and E. Pap, "Triangular Norms," vol. 8 of Trends in Logic, Studia Logica Library, Kluwer Acad. Publishers, Dordrecht, 2000.

[4] C.M. Ling, "Representation of associative functions," Publ. Math. Debrecen, vol. 12, 1965, pp. 189–212.

[5] A. Mesiarová (2005). *Special classes of triangular norms.* Ph.D. thesis, Mathematical Institute of Slovak Academy of Sciences.

[6] R. Moynihan, "On τ_T semigroups of probability distribution functions II," Aequationes Math., vol. 17, 1978, pp. 19–40.

[7] B. Schweizer and A. Sklar, "Statistical metric spaces," Pacific J. Math., vol. 10, 1960, pp. 313–334.

[8] B. Schweizer and A. Sklar, "Associative functions and abstract semigroups," Publ. Math. Debrecen, vol. 10, 1963, pp. 69–81.

[9] B. Schweizer and A. Sklar, "Probabilistic Metric Spaces," North-Holland, New York, 1983.

[10] Y.-H. Shyu, "Absolute continuity in the τ_T-operations," PhD Thesis, Illinois Institute of Technology, Chicago, 1984.