

An application of Schur-concave t-norms in pFCSP

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Abstract

Prioritized fuzzy constraint satisfaction problems (pFCSP's) are used to model the concept of priority in real time systems. Schur-concave t-norms are a key factor in the implementation of pFCSP. In this paper there are presented some results on the characterization of Schur-concave t-norms. There are tested two different pFCSP's on a candidate evaluation scenario. Using pFCSP's, different candidates are evaluated based on their education, experience and physical fitness for the jobs of teaching assistant and construction worker.

Keywords: Schur-concavity, t-norms, priority, pFCSP.

1 Introduction

In order to model real time systems sometimes it is necessary to incorporate the concept of priority into them. This can be done using Prioritized fuzzy constraint satisfaction problems (pFCSP's). In a pFCSP, priority of constraints is considered as the global importance of a constraint among other ones. The more important the constraint is, the more impact it has on the aggregated output of the pFCSP system. They were introduced by Dubois et al. [2] (framework 1), an axiomatic framework was given in [7], and an alternative one was given in [10] (framework 2).

The main difference between these two frameworks is the way they treat priority. Framework 2 favors the constraint with the larger priority regardless of its value. On the other hand, frame-

work 1 favors the constraint with the larger priority only when its value is smaller than the value of other less prioritized constraints.

The paper has the following structure. In the section 2 some Schur-concave t-norms are introduced together with some results on their characterization. In the section 3 axiomatic frameworks 1 and 2 are presented. In the section 4 we have tested these two frameworks on a candidate evaluation scenario.

2 Schur-concave t-norms

Let us recall the definition of t-norms. For more details on t-norms see [3].

Definition 1 A mapping $T : [0, 1]^2 \rightarrow [0, 1]$ is called a t-norm if the following conditions are satisfied for all $x, y, z \in [0, 1]$:

$$(T1) \quad T(x, y) = T(y, x)$$

$$(T2) \quad T(x, T(y, z)) = T(T(x, y), z)$$

$$(T3) \quad \text{if } y \leq z \text{ then } T(x, y) \leq T(x, z)$$

$$(T4) \quad T(x, 1) = x.$$

The four basic t-norms are:

$$T_M(x, y) = \min(x, y)$$

$$T_P(x, y) = xy$$

$$T_L(x, y) = \max(x + y - 1, 0)$$

$$T_D(x, y) = \begin{cases} 0, & \text{if } (x, y) \in [0, 1]^2; \\ \min(x, y), & \text{otherwise.} \end{cases}$$

One of the notions related to t-norms are level lines. for a continuous t-norm T they are obtained

for every fixed value $z \in [0, 1]$ such that

$$l_z(x) = \sup\{y \mid T(x, y) = z\}.$$

Since t-norms are symmetric, level lines of continuous t-norms are symmetric to the line $x = y$. Since t-norms are bounded from above by T_M level line $l_z(x)$ are only defined for each x in $[z, 1]$.

In the theory of fuzzy logics, t-norms are generalizations of the conjunction operator. In order to generalize the disjunction, t-conorms or s-norms are used.

Definition 2 A mapping $S : [0, 1]^2 \rightarrow [0, 1]$ is called an s-norm or a t-conorm if the following conditions (T1), (T2), (T3) from the previous definition and the condition (S4) are satisfied for all $x, y, z \in [0, 1]$.

$$(S4) S(x, 0) = x.$$

Since t-conorms are dual to t-norms, every result can be easily transferred from t-norms to t-conorms.

Now, we will introduce Schur-concave t-norms.

Definition 3 A t-norm is called a Schur-concave t-norm if the following condition holds for all $x_1, x_2, y_1, y_2 \in [0, 1]$ such that $x_1 + x_2 = y_1 + y_2 \in [0, 1]$ and $x_1 \leq y_1 \leq y_2 \leq x_2$:

$$T(y_1, y_2) \geq T(x_1, x_2).$$

It can be easily proven that T_M , T_P and T_L are Schur-concave t-norms. However, the fourth basic t-norm which is non-continuous, T_D , is not a Schur-concave t-norm.

On the other hand, the nilpotent minimum:

$$T^{nM}(x, y) = \begin{cases} 0, & \text{if } x + y \leq 1 \\ \min(x, y), & \text{otherwise,} \end{cases}$$

is a Schur-concave t-norm, but it is not continuous. However, the Schweizer-Sklar t-norms

$$T_\lambda^{SS}(x, y) = \max\left(\left(x^\lambda + y^\lambda - 1\right)^{\frac{1}{\lambda}}, 0\right)$$

for $\lambda \in (1, \infty)$ are not Schur-concave t-norms, although they are continuous. We should note that

Schur-concavity implies continuity on the boundary, see [5].

It has been shown that Schur-concave t-norms are closely related to copulas (see [1, 4]). Therefore we repeat here the definition of copulas (see [3, 8]).

Definition 4 A two-dimensional copula or simply copula is a binary operation C on the unit interval $[0, 1]$, i.e., a function $C : [0, 1]^2 \rightarrow [0, 1]$, such that for all $x \leq x^*$ and $y \leq y^*$ we have:

$$(CO1) C(x, y) + C(x^*, y^*) \geq C(x, y^*) + C(x^*, y),$$

$$(CO2) C(x, 0) = C(0, x) = 0,$$

$$(CO3) C(x, 1) = C(1, x) = x.$$

It has been shown in [1] that if a t-norm is strict, then it is Schur-concave if and only if it is a copula. On the other hand, if a t-norm is a nilpotent copula it is also Schur-concave. Continuous Archimedean t-norms are either strict or nilpotent. It is well known that every continuous t-norm can be represented as an ordinal sum of Archimedean t-norms. From this, we can conclude that every associative copula (1-Lipschitz t-norm) is Schur-concave. Unfortunately, the converse is not true, e.g., a t-norm T generated by $t(x) = \frac{1+\cos(\pi x)}{2}$ yields to

$$T(x, y) = \begin{cases} \frac{1}{\pi} \cos^{-1}(2 \cos \pi x + 2 \cos \pi y + 1), & \text{for } x + y \geq 1 \\ 0, & \text{otherwise,} \end{cases}$$

which is Schur-concave but not a copula.

In the case of level lines, there is an interesting characterization of Schur-concave t-norms concerning them, see [4] (for another characterization see [9]).

Theorem 1 For a continuous Archimedean t-norm the following statements are equivalent:

(i) T is Schur-concave;

(ii) For all $z \in (0, 1)$ such that $T(z, z) = \alpha > 0$, we have that the function $l_\alpha + id_{[\alpha, 1]} : [\alpha, 1] \rightarrow [0, 2]$ is non-increasing on $[\alpha, z]$.

If a continuous t-norm satisfies the conditional cancelation law (CCL), see [3], then it is Schur-

concave if and only if it is a copula, or if and only if it has the 1-Lipschitz property.

The characterization of Schur-concave t-norms by the means of their generators still remains an open problem.

3 Axiomatic framework for pFCSP

The pFCSP have evolved from constraint satisfaction problems (CSP), see [2, 7]. First, fuzzy values of a constrain have been introduced into CSP's and fuzzy constraint satisfaction problems (FCSP) were obtained. Constrains are model as fuzzy sets over a particular domain. The global satisfaction degree is obtained by aggregating the values of each constraint. The operators used for aggregation are Schur-concave t-norms and t-conorms.

In order to incorporate priority into FCSP, we will use axiomatic frameworks 1 and 2. The difference between these two frameworks is in Axiom 3, [2, 7, 10]. Systems that satisfy each of this frameworks are given and compared later in the paper.

In order to recall these axiomatic frameworks, we need to introduce the label of a variable, which is a well known concept in logic. A label of a variable x is an assignment of a value of the variable, denoted as v_x . A compound label $v_{X'}$ of all variables in the set $X' = \{x'_1, x'_2, \dots, x'_n\}$ is a simultaneous assignment of all variables in the set X' , that is,

$$v_{X'} = (v_{x'_1}, v_{x'_2}, \dots, v_{x'_n}).$$

The membership degree of each constraint indicates the local degree to which the constraint is satisfied with a compound label. In order to obtain the global satisfaction degree, local degrees are aggregated together with the priority of each constraint.

First we recall the definition of FCSP and pFCSP as given in [7].

Definition 5 A fuzzy constraint satisfaction problem (FCSP) is defined as a 3-tuple (X, D, C^f) where:

1. $X = \{x_i \mid i = 1, 2, \dots, n\}$ is a set of variables.

2. $D = \{d_i \mid i = 1, 2, \dots, n\}$ is a finite set of domains. Each domain d_i is a finite set containing the possible values for the corresponding variable x_i in X .

3. C^f is a set of fuzzy constraints. That is,

$$C^f = \{R^f \mid \mu_{R^f_i} : d_i \rightarrow [0, 1], i = 1, 2, \dots, n\}.$$

In a FCSP (X, D, C^f) , given a compound label v_x of all variables X , the global satisfaction degree for the compound label v_x is defined as

$$\alpha(v_x) = \bigoplus_{R^f \in C^f} \mu_{R^f}(v_x),$$

where \bigoplus is an aggregation operator on the unit interval. A solution of FCSP is a compound label v_X such that

$$\alpha(v_X) \geq \alpha_0,$$

where α_0 is called a solution threshold which is usually predetermined.

We will now introduce axiomatic framework 1 avoiding formal definitions, see [2]. The first axiom states that a zero value of the local satisfaction degree of the constraint with the maximum priority implies a zero value of the local satisfaction degree. The second axiom states that, in the case of equal priorities, the pFCSP becomes a FCSP.

The third axiom captures the notion of the priority. If one constraint has a larger priority then, the increase of the value on that constraint should result in a bigger increase of the global satisfaction degree than when the value with the smaller priority has the same increase, but only if the aggregated value of that constraint with the priority function is not bigger than the aggregated value of the constraint with the smaller priority. The last part of the explanation will be removed in order to obtain a different, more strict notion of priority in axiomatic framework 2. The fourth axiom is the monotonicity property, and finally the fifth is the upper boundary condition.

Now, describe how a pFCSP works. Priority of every constraint R_f is evaluated by function $\rho : C_f \rightarrow [0, \infty)$. The larger the value of ρ is the larger the priority. After the normalization of the priority values which is done by dividing each priority by $\rho_{\max} = \max\{\rho(R_f) \mid R_f \in C_f\}$ every priority gets a value in the unit interval. An S-norm aggregates priority of each constraint with its value. This is done in a way that the larger the priority, the more chance it has for the resulting value to stay the same as it was before aggregation. If the priority of constraint is small, then the aggregated value is closer to 1. This leads to greater values for constraints with the smaller priority. It makes sense, since when these aggregated values are again aggregated with a Schur-concave t-norm T , the smaller values have more impact on the result due to properties of Schur-concave t-norms. In [7], it has been proved that a concrete pFCSP the previously given axioms. Now we will describe how the global satisfaction degree of this system is calculated.

Let X, D, C^f be given as in Definition 5, $\rho : [0, \infty) \rightarrow [0, 1]$, $\rho_{\max} = \max\{\rho(R^f) \mid R^f \in C^f\}$ and \oplus is a Schur-concave t-norm T . Then the following system is a pFCSP:

$$\alpha_\rho(v_X) = \bigoplus_{R^f \in C^f} \frac{\rho(R^f)}{\rho_{\max}} \diamond \mu_{R_i^f}(v_X),$$

where $\oplus = T_M$ and $\diamond(x, y) = \max(1 - x, y)$. This system will be called min-max system.

The function ρ represents the priority of each constraint. Operator \diamond aggregates priority of each constraint with the value of that constraint. These are then aggregated by the operator \oplus , which results in the satisfaction degree of an evaluation.

In the axiomatic framework 2, constraints with a larger priority are favored regardless of their values. This is done by slightly modifying Axiom 3, see [10]. The changed Axiom 3 captures the concept of priority in a linear sense. For example take two investments where one of them results in a bigger profit (larger priority) then it is expected that it is better to invest in a more profitable investment then in a less profitable one, if the profit increase is linear to the investment sum. One of

the interesting facts is the min-max system which is used most often in engineering does not satisfy the changed Axiom 3. On the other hand when the t-norm T_L is used together with the s-norm S_P we obtain the system that satisfies axiomatic framework 2. The global satisfaction degree in this system is calculated using the following formula:

$$\alpha_\rho(v_X) = \bigoplus_{R^f \in C^f} \frac{\rho(R^f)}{\rho_{\max}} \diamond \mu_{R_i^f}(v_X),$$

where $\oplus(x, y) = T_L(x, y)$, and $\diamond(x, y) = S_P(1 - x, y)$. We will call this system T_L-S_P .

Beside min-max and T_L-S_P many more systems satisfy axiomatic frameworks 1 and 2. It has been proven in [7] that if we take any Schur-concave t-norm instead of T_M , then we obtain a system that satisfies axiomatic framework 1. On the other hand, if we take a Frank t-conorm for some $\lambda \in (1, \infty)$

$$S_\lambda^F(x, y) = 1 - \log_\lambda \left(1 + \frac{(\lambda^{1-x} - 1)(\lambda^{1-y} - 1)}{\lambda - 1} \right)$$

instead of S_P , then we obtain a system that satisfies axiomatic framework 2 (see [10]).

In the following section we will test the two systems min-max and T_L-S_P on a candidate evaluation scenario.

4 Testing systems on a candidate evaluation scenario

Now we will describe the scenario. Suppose we have job openings for a teaching assistant and construction worker. We evaluate candidates based on their education, experience and physical fitness. Depending on the job, each constrain will have a certain priority. We will suppose that for the job of teaching assistant education is most important, experience is mildly important and physical fitness is the least important. For the construction worker things will be vise versa. The min-max and T_L-S_P systems will be used for evaluation.

The results for five candidates are given in the following tables. Table 1 represents evaluations for the teaching assistant job. We assume that the

priority of *Education* constraint is 1, *Experience* has priority 0.6 and finally *Physical Fitness* has priority 0.1. Similarly, Table 2 represents evaluations for the construction worker job where priority *Education* constraint is 0.2, *Experience* has priority 0.7 and finally *Physical Fitness* has priority 1.

no.	Edu	Exp	PhF	min-max	T_L-S_P
1	1	0.6	0.2	0.6	0.68
2	0.55	0.65	0.7	0.55	0.31
3	0.1	0.6	1	0.1	0
4	0.8	0.7	0.7	0.7	0.59
5	0.9	0.6	0.3	0.6	0.59

Table 1. Evaluations for the teaching assistant job

no.	Edu	Exp	PhF	min-max	T_L-S_P
1	1	0.6	0.2	0.2	0
2	0.55	0.65	0.7	0.65	0.445
3	0.1	0.6	1	0.6	0.67
4	0.8	0.7	0.7	0.7	0.5
5	0.9	0.6	0.3	0.3	0.05

Table 2. Evaluations for the construction worker job

We see that candidate 1 who is the most educated was picked by T_L-S_P system for the teaching assistant, and candidate 3 with the best physical fitness was picked for the construction worker. It is interesting that the min-max has not picked these candidates but it has picked the ones with the more averaging values of constraints.

These experimental results confirm the theoretical background for each pFCSP. We see that if we want a system that favors highest values of large priority constraints regardless of the values of smaller priority constraints we should pick T_L-S_P . On the other hand, if we want a system that also values the smaller priority values to a certain degree we should pick min-max.

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