

## On some properties of reverses of binary operations

Michal Šabo

Slovak University of Technology  
in Bratislava, Slovakia  
michal.sabo@stuba.sk

### Abstract

The reverses of binary operations defined on the unit interval are also binary operations. Some of them preserve important properties of the original binary operations. A characterization of such binary operations is studied. A special attention is paid to the property of selfreversibility.

**Keywords:** Reverse of binary operation, t-norm, copula

### 1 Introduction

An interesting transformation of t-norm was introduced by C. Kimberling under the name invert [4]. Recently, it is called the reverse of t-norm [1, 5, 10]. The main aim of this paper is to define the reverses of arbitrary binary operations defined on the unit interval and to show some basic properties of them. The binary operations which are invariant under such transformation are called selreversible binary operations. They play an important role in some applications, e.g., selreversible copulas in statistics. Copulas are functions that join two-dimensional distribution function  $H$  to its one-dimensional marginal distribution functions  $F$  and  $G$ , i.e.,

$$H(x,y) = C(F(x), G(y)).$$

For a pair  $(X, Y)$  of continuous random variables with a joint distribution function  $H$  holds: If  $X, Y$  are symmetric about  $a$  and  $b$  respectively, then  $(X, Y)$  is radially symmetric about  $(a, b)$  if and only if the copula  $C$  is selfreversible (see Section 4 or [9]).

The paper is organised as follows. After summarising the known preliminary results related

to reversible t-norms in Section 2, we define the reverses of arbitrary binary operations and show their interesting properties in Section 3. Section 4 is devoted to the selfreversibility of binary operations.

### 2 Reverses of t-norms a t-conorms

The well-known formula of Boolean two-valued logic says that the truth values of propositions  $p, q$ , their disjunction  $p \vee q$  and their conjunction  $p \wedge q$  fulfill the equation:

$$\text{val}(p \vee q) + \text{val}(p \wedge q) = \text{val}(p) + \text{val}(q)$$

In fuzzy set theory, the generalization of the former equation leads to the known Frank functional equation

$$T(x, y) + S(x, y) = x + y$$

where  $x, y \in [0, 1]$ ,  $T$  is a t-norm and  $S$  is a t-conorm. The Frank family of t-norms was introduced as a family of solutions of this functional equation [2].

Let a t-conorm  $S$  be the dual of the t-norm  $H$ , i.e.,

$$S(x, y) = 1 - H(1-x, 1-y).$$

Then the Frank equation can be written in the form

$$T(x, y) = x + y - 1 + H(1-x, 1-y)$$

The t-norm  $T$  can be considered as a transformation of the t-norm  $H$ . It is known [1, 5, 10] that if  $T$  and  $H$  are Frank t-norms, and moreover  $T = H$ , then the former equation holds for all  $x, y \in [0, 1]$ . It is clear that for some t-norms  $T$  the expression

$$x + y - 1 + T(1-x, 1-y)$$

can be negative. For example, we can choose a t-norm  $T \leq W$ ,  $T \neq W$  ( $W$  - Lukasiewicz t-norm). Therefore we define the binary operations  $T^*$  as follows.

**Definition 1** Let  $T$  be a t-norm. Then the binary operation  $T^* : [0, 1]^2 \rightarrow [0, 1]$  defined by

$$T^*(x, y) = \max\{0, x + y - 1 + T(1-x, 1-y)\}$$

is called the reverse of T.

Analogously, we can define the reverse of t-conorm. Because, for some t-conorms S the expression

$$x + y - 1 + S(1-x, 1-y)$$

can exceed one, we define the reverse of t-conorm S by formula:

$$S^*(x, y) = \min\{1, x + y - 1 + S(1-x, 1-y)\}$$

If  $T^*$  is also a t-norm then we say that  $T$  is a reversible t-norm. Analogously, if  $S^*$  is also a t-conorm then we say that  $S$  is a reversible t-conorm.

The problem of reversibility of t-norms has already been intensively studied [1, 5, 10]. For the sake of completeness we briefly list the most relevant results from the mentioned papers.

- Let  $T \leq W$ . Then  $T^* = W$ .
- Let  $T \geq W$ . Then  $T$  is reversible if and only if  $T$  is a Frank t-norm or an ordinal sum of Frank t-norms.
- If  $T$  is a continuous Archimedean t-norm which is not comparable with  $W$  then  $T$  is not reversible.
- If  $T$  is a reversible t-norm then  $T^*$  is continuous
- If  $T$  is a Frank t-norm then  $T = T^*$ .
- A t-norm  $T$  and t-conorm  $S$  solve the Frank functional equation

$$T(x,y) + S(x,y) = x + y$$

for all  $x, y \in [0, 1]$  if and only if  $T$  is a Frank t-norm or an ordinal sum of Frank t-norms.

Recall that an ordinal sum of t-norms is defined as follows. Let  $\{]a_k, b_k[, k \in K\}$  be a family of pairwise disjoint open subintervals of  $[0, 1]$  and  $\{T_k, k \in K\}$  be a family of t-norms. Then the binary operation  $T$  defined for all  $x, y \in [0, 1]^2$  by

$$T(x, y) = \begin{cases} a_k + (b_k - a_k)T_k\left(\frac{x - a_k}{b_k - a_k}, \frac{y - a_k}{b_k - a_k}\right) & \text{if } x, y \in ]a_k, b_k[ \\ \min\{x, y\} & \text{otherwise} \end{cases}$$

is a t-norm. It is denoted by  $\langle a_k, b_k, T_k \rangle_{k \in K}$  and called the ordinal sum of the summands  $\langle a_k, b_k, T_k \rangle$ ,  $k \in K$ . For more details about t-norms and t-conorms we recommend [7].

### 3 Reverses of binary operations

Now we define the reverse of arbitrary binary operation on the unit interval.

**Definition 2** Let  $B$  be a binary operation on the unit interval. The reverse of a binary operation  $B$  is a binary operation  $B^* : [0,1]^2 \rightarrow [0,1]$  such that

$$B^*(x, y) = /x + y - 1 + B(1-x, 1-y)/$$

where  $/a/ = \min\{1, \max\{0, a\}\}$ . If  $B^* = B$  then the binary operation  $B$  is called selfreversible.

**Example 1** Let  $B : [0; 1]^2 \rightarrow [0;1]$  be a binary operation given by

$$B(x, y) = xy/2,$$

$x, y \in [0; 1]$ . Then

$$\begin{aligned} B^*(x, y) &= \max\{0, x + y - 1 + (1 - x)(1 - y)/2\} = \\ &= \max\{0, (x + y - 1 + xy)/2\} \end{aligned}$$

Recall that the binary operation  $B$  mentioned in the previous example is a binary operation which is called a triangular subnorm [3, 8] and its reverse  $B^*$  is the Sugeno-Weber t-norm [7] and so it is also a triangular subnorm.

The product or arithmetic mean are trivial examples of selfreversible binary operations defined on the unit interval. Recall that the Lukasiewicz t-norm and t-conorm play important roles in this theory,

$$W(x, y) = \max(0, x + y - 1)$$

$$W'(x, y) = \min(1, x + y)$$

$x, y \in [0, 1]$ . The following propositions are direct consequences of Definition 2.

**Proposition 1** Let  $B$  be a binary operation on  $[0, 1]$ . Then

(a)  $W \leq B^* \leq W'$

(b)  $B^* = B^{***}$

Remark that the inequality  $W \leq B^* \leq W'$  is equivalent to the inequality

$$x + y - 1 \leq B^*(x, y) \leq x + y.$$

for all  $x, y \in [0, 1]$ .

**Proposition 2** Let  $B$  be a binary operation on  $[0, 1]$ . If for all  $x, y \in [0, 1]$

$$x + y - 1 \leq B(x, y) \leq x + y$$

then

- (i)  $B^*(x, y) = x + y - 1 + B(1-x, 1-y)$
- (ii)  $B^{**} = (B^*)^* = B$ .

**Proposition 3** Let  $B$  be a binary operation on  $[0, 1]$ . Then

- (a) If  $B \leq W$  then  $B^* = W$
- (b) If  $B \geq W$  then  $B^* = W$

There is no relation between the monotonicity of a binary operation and the monotonicity of its reverse. For example, the reverse of fuzzy implication

$$I(x, y) = \min\{1, 1-x+y\},$$

$x, y \in [0, 1]$  is nondecreasing though  $I$  isn't. Conversely, the reverse of the ordinal sum

$$T = \{<0, \frac{1}{2}, D>\},$$

where  $D$  is the weakest t-norm given by

$$D(x, y) = \begin{cases} xy & \text{if } \max\{x, y\} = 1 \\ 0 & \text{otherwise} \end{cases}$$

is not nondecreasing.

**Proposition 4** Let  $A : [0, 1]^2 \rightarrow [0, 1]$  be a binary operation nondecreasing in both arguments such that for all  $x, y \in [0, 1]$

$$x + y - 1 \leq A(x, y) \leq x + y.$$

Then  $A^*$  is nondecreasing if and only if the binary operation  $A$  is 1-Lipschitz.

Recall that a binary operation  $A: [0, 1]^2 \rightarrow [0, 1]$  is 1-Lipschitz if

$$|A(x, y) - A(z, t)| \leq |x - z| + |y - t|$$

for all  $x, y, z, t \in [0, 1]$

**Proposition 5** Let  $A: [0, 1]^2 \rightarrow [0, 1]$  be nondecreasing in both arguments and 1-Lipschitz. Then  $A^*$  is also nondecreasing.

Denote

$$A_L(x, y) = \max\{W(x, y), \min\{A(x, y), W'(x, y)\}\}$$

**Proposition 6** Let  $A$  be a binary operation on the unit interval. Then

$$A^{**} = A_L.$$

The next theorem gives a necessary and sufficient condition for preserving the property of monotonicity.

**Proposition 7** Let  $A: [0, 1]^2 \rightarrow [0, 1]$  be a binary operation which is non-decreasing in both

arguments. Then  $A^*$  is non-decreasing if and only if  $A_L$  is 1-Lipschitz.

The following proposition also brings an interesting information concerning reverses of binary operations.

**Proposition 8** Let  $A$  and  $A^*$  be nondecreasing binary operations defined on the unit interval. Then  $A^*$  is continuous.

#### 4 Selfreversibility

The characterization of selfreversible binary operations defined on the unit interval is also an interesting problem. We have already mentioned that the Frank t-norms are selfreversible. Also, the symmetrical ordinal sum of the Frank t-norms, i.e.,

$$T = \{<a_k, b_k, T_k>\}_{k \in K}$$

such that for any  $i \in K$  there is  $j \in K$  such that

$$[a_i, b_i] = [1-b_j, 1-a_j] \text{ and } T_i = T_j$$

is selfreversible.

The problem of selfreversibility of copulas was studied by Klement, Mesiar and Pap in [6]. Recall that the copula  $C : [0, 1]^2 \rightarrow [0, 1]$  is a binary operation satisfying :

- (1)  $C(x, 0) = C(0, x) = 0, C(x, 1) = C(1, x) = x$
- (2)  $C$  is 2-increasing, i.e.,

$$C(x_1, y_1) + C(x_2, y_2) \geq C(x_1, y_2) + C(x_2, y_1)$$

for all  $x_1, x_2, y_1, y_2 \in [0, 1]$ , with  $x_1 \leq x_2, y_1 \leq y_2$

For more details about copulas see [9]. We only emphasize that every copula  $C$  is 1-Lipschitz,  $C^*$  is also copula and

$$W \leq C \leq W'.$$

Therefore

$$C^{**} = C.$$

As we have already mentioned, selfreversible copulas play an important role in statistics. Copulas are functions that join two-dimensional distribution function to its one-dimensional marginal distribution functions. If  $H$  is a joint distribution function with margins  $F, G$ , then there exist a copula  $C$  such that for all  $x, y \in R \cup \{-\infty, \infty\}$

$$H(x, y) = C(F(x), G(y))$$

([9], Sklar's Theorem). For a pair  $(X, Y)$  of continuous random variables with joint distribution function  $H$ , the joint survival function is given by

$$H_s(x, y) = Prob(X > x, Y > y)$$

It can be shown [9] that

$$H_s(x, y) = C^*(1-F(x), 1-G(y))$$

If  $X, Y$  are symmetric about  $a$  and  $b$  respectively, i.e.

$$F(a+x) = 1 - F(a-x), \quad G(b+y) = 1 - G(b-y)$$

then  $(X, Y)$  is radially symmetric about  $(a, b)$  if and only if  $C = C^*$ , i.e.,  $C$  is selfreversible

Recall that  $(X, Y)$  is radially symmetric about  $(a, b)$  if

$$(X-a, Y-b) \text{ and } (a-X, b-Y)$$

have the same joint distribution function. It means that the three-dimensional graph of  $H$  is symmetric with respect to the point  $[a, b, 1/2]$ .

Of course, there exist selreversible binary operations on the unit interval which are neither copulas nor t-norms, e.g., the Frank t-conorms or the arithmetic mean. It can be easily shown that the family of selfreversible binary operations is convex. It means that if  $A$  and  $B$  are selreversible binary operations on the unit interval and  $k \in [0, 1]$  then the binary operation given by

$$V(x, y) = kA(x, y) + (1-k)B(x, y),$$

$x, y \in [0, 1]$ , is also a selfreversible binary operation on the unit interval. The following theorem gives some possibility to construct selfreversible binary operations. It is a generalization of the result reached in [6].

**Proposition 9** Let  $A$  be a binary operation on  $[0, 1]$  such that for all  $x, y \in [0, 1]$

$$x + y - 1 \leq A(x, y) \leq x + y.$$

Then the binary operation given by

$$B(x, y) = \frac{A(x, y) + A^*(x, y)}{2}$$

$x, y \in [0, 1]$ , is selfreversible.

Proposition 9 can be reformulated using Proposition 1. For arbitrary binary operation  $A$  defined on the unit interval, the following binary operation is selfreversible:

$$B(x, y) = \frac{A^*(x, y) + A^{**}(x, y)}{2}.$$

It can be shown that composition of selfreversible binary operations on the unit interval need not be selreversible. For example, the product  $P$  is a

selfreversible binary operations, but the composition  $Q = P(P, P)$  given by

$$Q(x, y) = P(P(x, y), P(x, y)) = (xy)^2$$

$x, y \in [0, 1]$ , is not selfreversible. We are looking for a binary operation  $L$  having property: the binary operation  $L(A, B)$  is selfreversible for any selreversible binary operations  $A$  and  $B$ . The next proposition solves this problem.

**Proposition 10** Let  $L$  be a selfreversible binary operations defined on the unit interval. Then the binary operations  $Q$  given by

$$Q(x, y) = L(A(x, y), B(x, y)),$$

$x, y \in [0, 1]$ , is selfreversible for all selfreversible binary operations  $A$  and  $B$  defined on the unit interval if and only if  $L$  is shift-stable, i.e.,

$$L(k+u, k+v) = k + L(u, v)$$

for all  $k \in [-\min(u, v), 1 - \max(u, v)]$ .

The mentioned convexity of the family of all selfreversible binary operations is a direct consequence of Proposition 10.

## 5 Conclusion

This paper can be considered as an introduction to the theory of reverses of arbitrary binary operations defined on the unit interval. We have also derived their basic properties. We have attempted to find out the condition under which the reverses inherit the monotonicity from original binary operations. The similar problems concerning associativity, bisymmetry, etc., are still open. We have also investigated the family of selreversible binary operations. The full characterization of selfreversible binary operations on the unit interval is also an open field for investigation.

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