Distributive residual implications from uninorms

D. Ruiz-Aguilera and J. Torrens
Departament de Ciències Matemàtiques i Informàtica.
Universitat de les Illes Balears
07122 Palma de Mallorca. Spain
vdmidra4@uib.es, dmijts0@uib.es

Abstract

In this paper the equation

$$I(U_1(x,y),z) = U_2(I(x,z),I(y,z))$$

for all $x,y,z \in [0,1]$ is solved where I is a residual implication defined from a uninorm U, and U_1,U_2 are a conjunctive and a disjunctive uninorm, respectively. Three cases are considered: i) when U is a t-norm, ii) when U_1 is a t-norm and/or U_2 is a t-conorm, and iii) the general case when the neutral elements of U,U_1 and U_2 are in $]0,1[$.

Keywords: t-norm, t-conorm, uninorm, implication operator, S-implication, R-implication, distributivity.

1 Introduction

Uninorms are a special kind of associative aggregation functions that have been extensively studied in the literature. They have proved to be useful for applications in many fields like expert systems, neural networks, aggregation, fuzzy system modelling, etc. It is well known that a uninorm U can be conjunctive (when $U(1,0) = 0$) or disjunctive (when $U(1,0) = 1$). This fact allows to use them also as logical connectives. In this sense, fuzzy implications functions have been defined from uninorms in the following two ways:

- *Strong implications* (or *S-implications*), and
- *Residual implications* (or *R-implications*).

An exhaustive study of both types of implications can be found in [4] and [9]. Uninorms and their derived implications are also used in aggregation applications like mathematical morphology (see [6]).

On the other hand, the equivalence

$$(p \wedge q) \to r \equiv (p \to r) \vee (q \to r), \qquad (1)$$

useful in avoiding combinatorial rule explosion in fuzzy systems (see [2]), was solved in [12] involving t-norms, t-conorms and implications derived from them. Related equations of distributivity between implications and t-norms were solved in [1]. The solutions obtained in both papers, [12] and [1], require that the conjunctions must be given by the minimum and the disjunctions by the maximum.

Moreover, equation (1) was solved in [10] where conjunctions and disjunctions are performed by uninorms and the implications are strong implications derived from disjunctive uninorms. In this case equation (1) becomes

$$I(U_1(x,y),z) = U_2(I(x,z),I(y,z)) \quad x,y,z \in [0,1] \qquad (2)$$

being I defined by $I(x,y) = U(N(x),y)$, where N is a strong negation and U a disjunctive uninorm, and U_1,U_2 are a conjunctive and a disjunctive uninorm, respectively.

In this paper we want to continue the study of equation (2), but now involving residual implications derived from uninorms, that is $I_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \leq y\}$ for any uninorm U such that $U(x,0) = 0$ for all $x < 1$. We solve equation (2) with the mentioned conditions, and we prove that many new solutions appear along our study, different from those already known for strong implications. We will divide it in two cases: i) when U is a t-norm, and ii) the general case when U is a uninorm with neutral element in $]0,1[$.

2 Preliminaries

We suppose the reader to be familiar with basic results concerning t-norms and t-conorms that can be found in [7]. We recall only some notions on uninorms. For more details see [5], [8] and for implications derived from uninorms see [4] and [9].

Definition 1 *A uninorm is a two-place function* $U : [0,1] \times [0,1] \longrightarrow [0,1]$ *which is associative, commutative, increasing in each place and such that there exists some element* $e \in [0,1]$, *called the neutral element, such that* $U(e,x) = x$ *for all* $x \in [0,1]$.

It is clear that the function U becomes a t-norm when $e = 1$ and a t-conorm when $e = 0$. For any uninorm we have $U(0,1) \in \{0,1\}$ and a uninorm U is said conjunctive when $U(1,0) = 0$ and disjunctive when $U(1,0) = 1$.

Definition 2 *A uninorm U with neutral element* $e \in]0,1[$ *is representable if and only if there is a strictly increasing, continuous function* $h : [0,1] \to [-\infty,+\infty]$ *with* $h(0) = -\infty$, $h(e) = 0$ *and* $h(1) = +\infty$ *such that U is given by*

$$U(a,b) = h^{-1}(h(a) + h(b))$$

for all $(a,b) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ *and* $U(0,1) = U(1,0) \in \{0,1\}$. *Function h is usually called an* additive generator *of U*.

Note that any representable uninorm is continuous in $[0,1]^2 \setminus \{(0,1),(1,0)\}$ and all uninorms continuous in this set are in fact representable (see [11]).

Definition 3 *A uninorm U is said to be in* $\mathcal{U}_{\min}$ *when it is given by*

$$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e},\frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2 \\ e + (1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2 \\ \min(x,y) & \text{otherwise.} \end{cases} \quad (3)$$

and is said to be in $\mathcal{U}_{\max}$ *when it is given by*

$$U(x,y) = \begin{cases} eT_U\left(\frac{x}{e},\frac{y}{e}\right) & \text{if } (x,y) \in [0,e]^2 \\ e + (1-e)S_U\left(\frac{x-e}{1-e},\frac{y-e}{1-e}\right) & \text{if } (x,y) \in [e,1]^2 \\ \max(x,y) & \text{otherwise.} \end{cases} \quad (4)$$

In both expressions T_U *denotes a t-norm and* S_U *denotes a t-conorm.*

Remark 1 *In fact any uninorm U has the same structure that in (3) except for the values of* $U(x,y)$ *when* $\min(x,y) < e < \max(x,y)$. *In general it is only known that these values are placed between the minimum and the maximum. Due to this general structure, any uninorm U is usually denoted by* $U = (e, T_U, S_U)$. *Note however that this notation is ambiguous because there are different uninorms with the same* e, T_U *and* S_U.

Theorem 1 ([8]) *U is an idempotent uninorm (that is* $U(x,x) = x$ *for all* $x \in [0,1]$*) with neutral element* $e \in [0,1]$ *if and only if there exists a decreasing function* $g : [0,1] \to [0,1]$ *with* $g(e) = e$, $g(x) = 0$ *for all* $x > g(0)$, $g(x) = 1$ *for all* $x < g(1)$, *satisfying*

$$\inf\{y \mid g(y) = g(x)\} \le g^2(x) \le \sup\{y \mid g(y) = g(x)\}$$

for all $x \in [0,1]$, *such that* $U(x,y) =$

$$\begin{cases} \min(x,y) & \text{if } y < g(x) \text{ or } \left(y = g(x) \text{ and } x < g^2(x)\right) \\ \max(x,y) & \text{if } y > g(x) \text{ or } \left(y = g(x) \text{ and } x > g^2(x)\right) \\ \min(x,y) & \\ \text{or} & \text{if } y = g(x) \text{ and } x = g^2(x) \\ \max(x,y) & \end{cases}$$

being commutative on the set of points (x,y) *such that* $y = g(x)$ *with* $x = g^2(x)$. *Function g is usually called the* associated function *of U*.

Idempotent uninorms will be denoted by $U = (e, g)$ although this notation is again ambiguous because depending on g it is possible to have many idempotent uninorms with the same e and g. Special cases of left-continuous and right-continuous idempotent uninorms are detailed in [3].

Definition 4 *A binary operator* $I : [0,1] \times [0,1] \to [0,1]$ *is said to be an implication operator, or an implication, if it satisfies:*

I1) I is nonincreasing in the first place and nondecreasing in the second one.

I2) $I(0,0) = I(1,1) = 1$ *and* $I(1,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = 1$ and $I(x,1) = 1$ for all $x \in [0,1]$ whereas the symmetrical values $I(x,0)$ and $I(1,x)$ are not determined in general.

Implications from uninorms can be defined in the following two ways:

- Strong implications. That is $I(x,y) = U(N(x),y)$ for all $x,y \in [0,1]$ where N is a strong negation and U a disjunctive uninorm, and
- Residual implications. That is $I_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \le y\}$ where U is a uninorm satisfying $U(x,0) = 0$ for all $x < 1$.

To obtain residual implications any conjunctive uninorm works, but many disjunctive idempotent uninorms and all disjunctive representable uninorms also work (for more details see [4] and [9]). Finally, note that the usual residual implication from a t-norm T, that we will denote by I_T, appears as a special case taking 1 as the neutral element of the uninorm U.

3 Distributive residual implications

As we have commented, equation (1) was solved in [12] when the conjunction is performed by a continuous t-norm, the disjunction by a continuous t-conorm and the implication is a strong implication derived from another continuous t-conorm or a residual implication derived from a continuous t-norm. It was also solved in [10] for the case of strong implications, when t-norms and t-conorms are replaced by conjunctive and disjunctive uninorms respectively.

We are interested in solving the same equation when the involved operators are conjunctive and disjunctive uninorms respectively, but now using residual implications derived from uninorms. Namely, let $U = (e, T_U, S_U)$ be a uninorm such that $U(x,0) = 0$ for all $x < 1$ and I the residual implication defined by

$$I_U(x,y) = \sup\{z \in [0,1] \mid U(x,z) \le y\} \quad x,y \in [0,1].$$

Let $U_c = (e_c, T_{U_c}, S_{U_c})$ be a conjunctive uninorm and $U_d = (e_d, T_{U_d}, S_{U_d})$ a disjunctive one. We want to solve the equation

$$I_U(U_c(x,y),z) = U_d(I_U(x,z), I_U(y,z)) \qquad (5)$$

for all $x,y,z \in [0,1]$ where the uninorm $U = (e, T_U, S_U)$ is in one of the three classes stated in the preliminaries and T_U and S_U are continuous.

We do this by distinguishing four different cases:

i) When U is a t-norm T,

ii) When U is in $\mathcal{U}_{\min}$,

iii) When $U = (e,g)$ is idempotent with $g(0) = 1$, and

iv) When U is representable (conjunctive or disjunctive).

However, we begin by proving that the fourth case is not possible. Effectively, suppose that U is any representable uninorm with additive generator h. If we denote by U^* the disjunctive representable uninorm with the same additive generator h, we know (see [4]) that

$$I_U(x,y) = U^*(N(x),y) \quad \text{for all } x,y \in [0,1]$$

where N is the strong negation given by $N(x) = h^{-1}(-h(x))$. That is, I_U is in fact a strong implication and it is proved in [10] that no strong implications from representable uninorms satisfy equation (5).

Thus, let us divide our reasoning in three subsections devoted to each one of the remaining three cases. Due to the restricted space we do not include the proofs of the results, but we will give all necessary lemmata in order to establish the complete process that leads to the solutions. Although the first steps always consist in proving that U_d and U_c must be idempotent, the proof is different in each case and so we need to handle them separately.

3.1 When U is a t-norm T

The following lemma can be easily proved in general without any assumption on continuity.

Lemma 1 *Let T be a t-norm, U_c a conjunctive uninorm and U_d a disjunctive one. If I_T, U_c and U_d satisfy (5), then U_d and U_c are idempotent. Moreover, U_c must be a t-norm and consequently $U_c = \min$.*

Now, the result for continuous t-norms is as follows.

Theorem 2 *Let T be a continuous t-norm, I_T its residual implication, U_c a conjunctive uninorm and U_d a disjunctive one. Then, I_T, U_c and U_d satisfy (5) if and only if $U_c = \min$, U_d is idempotent and $T(x,x) = x$ for all $x \le e_d$.*

Note that the condition on T in the Theorem above is equivalent to say that T must be an ordinal sum of the form $(\langle e_d, 1, T_1\rangle)$ where T_1 is any continuous t-norm.

Such family of t-norms jointly with their residual implications can be viewed in Figure 1. Note also that the border case $e_d = 0$ gives $U_c = \min$, $U_d = \max$ and then any t-norm works.

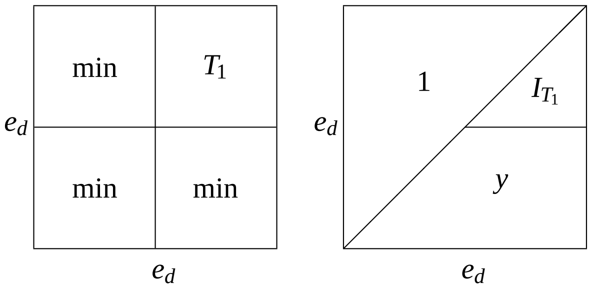

Figure 1: A t-norm T (left) and its residual implication I_T (right) such that I_T, min and U_d satisfy (5), where U_d is an idempotent uninorm with neutral element e_d.

3.2 When U is in $\mathcal{U}_{\min}$

In this case we take $U = (e, T_U, S_U)$ a uninorm in $\mathcal{U}_{\min}$ with neutral element e such that $0 < e < 1$.

Lemma 2 *Let $U = (e, T_U, S_U)$ be a uninorm in $\mathcal{U}_{\min}$ with $e < 1$, I_U its residual implication, U_c a conjunctive uninorm and U_d a disjunctive one. If I_U, U_c and U_d satisfy (1) then U_d is idempotent and $e \le e_c$.*

Lemma 3 *Let $U = (e, T_U, S_U)$ be a uninorm in $\mathcal{U}_{\min}$ with $e < 1$ and S_U continuous, and I_U its residual implication. Let U_c be a conjunctive uninorm and U_d a disjunctive one. If I_U, U_c and U_d satisfy (5), then U_c is idempotent.*

Next we distinguish two cases: when U_c is a t-norm, that is $e_c = 1$, and when $e_c < 1$. We first solve the case when U_c is a t-norm.

Lemma 4 *Let $U = (e, T_U, S_U)$ be a uninorm in $\mathcal{U}_{\min}$ with $e < 1$ and S_U continuous, and I_U its residual implication. Let $U_c = \min$ and U_d be a disjunctive idempotent uninorm. If I_U, U_c and U_d satisfy (5) then $e_d \le e$ and $U(x,x) = x$ for all $x \le e_d$.*

Note again that the condition on U given in the lemma above is equivalent to say that $U(x,y) = \min(x,y)$ for all x,y such that $\min(x,y) \le e_d$ and $\max(x,y) \le e$. That is, U must be given by $U(x,y) =$

$$\begin{cases} e + (1-e)S_U(\frac{x-e}{1-e}, \frac{y-e}{1-e}) & \text{if } x,y \in [e,1] \\ e_d + (e-e_d)T_1(\frac{x-e_d}{e-e_d}, \frac{y-e_d}{e-e_d}) & \text{if } x,y \in [e_d,e] \\ \min(x,y) & \text{otherwise.} \end{cases} \quad (6)$$

for any continuous t-norm T_1. It is understood that in the special case $e_d = e$, the second expression in equation (6) disappears and then the uninorm U becomes a uninorm in $\mathcal{U}_{\min}$ with associated t-norm $T_U = \min$.

Theorem 3 *Let $U = (e, T_U, S_U)$ be a uninorm in $\mathcal{U}_{\min}$ with $e < 1$ and S_U continuous, and I_U its residual implication. Let U_c be a t-norm and $U_d = (e_d, T_d, S_d)$ a disjunctive uninorm. Then, I_U, U_c and U_d satisfy (5) if and only if $U_c = \min$, U_d is idempotent with $e_d \le e$ and U is given by equation (6).*

The family of uninorms U stated in the previous theorem, jointly with their residual implications can be viewed in Figure 2, where R_S means the residual operator associated to the t-conorm S, that is, $R_S(x,y) = \sup\{z \in [0,1] \mid S(x,z) \le y\}$.

Now, we study the case when U_c is not a t-norm, that is, when $0 < e_c < 1$.

Lemma 5 *Let $U = (e, T_U, S_U)$ be a uninorm in $\mathcal{U}_{\min}$ with $e < 1$ and S_U continuous, and I_U its residual implication. Let $U_c = (e_c, g_c)$ be a conjunctive idempotent uninorm with $0 < e_c < 1$ and $U_d = (e_d, g_d)$ a disjunctive one. If I_U, U_c and U_d satisfy (5), then $e = e_c < e_d$.*

Lemma 6 *Let $U = (e, T_U, S_U)$ be a uninorm in $\mathcal{U}_{\min}$ with $e < 1$ and S_U continuous, and I_U its residual implication. Let $U_c = (e_c, g_c)$ be a conjunctive idempotent uninorm with $0 < e_c < 1$ and $U_d = (e_d, g_d)$ a disjunctive one. If I_U, U_c and U_d satisfy (5), then $T_U = \min$ and $U(x,x) = x$ for all $x \ge e_d$.*

Again the conditions in the previous lemma ensures that U must be given by $U(x,y) =$

$$\begin{cases} \min(x,y) & \text{if } \min(x,y) \le e \\ e + (e_d - e)S_1\left(\frac{x-e}{e_d-e}, \frac{y-e}{e_d-e}\right) & \text{if } x,y \in [e,e_d] \\ \max(x,y) & \text{otherwise} \end{cases} \quad (7)$$

for any continuous t-conorm S_1.

Lemma 7 *Let $U = (e, T_U, S_U)$ be a uninorm in $\mathcal{U}_{\min}$ with $e < 1$ and S_U continuous, and I_U its residual implication. Let $U_c = (e_c, g_c)$ be a conjunctive idempotent uninorm with $0 < e_c < 1$ and $U_d = (e_d, g_d)$*

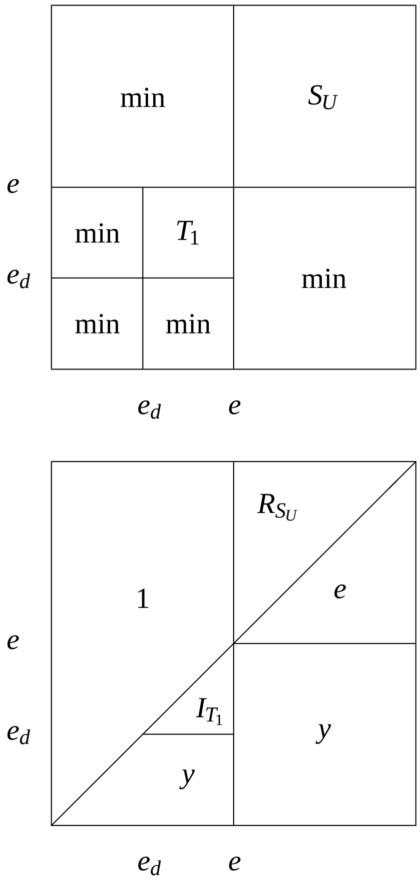

Figure 2: A uninorm U (top) in $\mathcal{U}_{\min}$ and its residual implication I_U (bottom) such that I_U, min and U_d satisfy (5), where U_d is any idempotent uninorm with neutral element e_d.

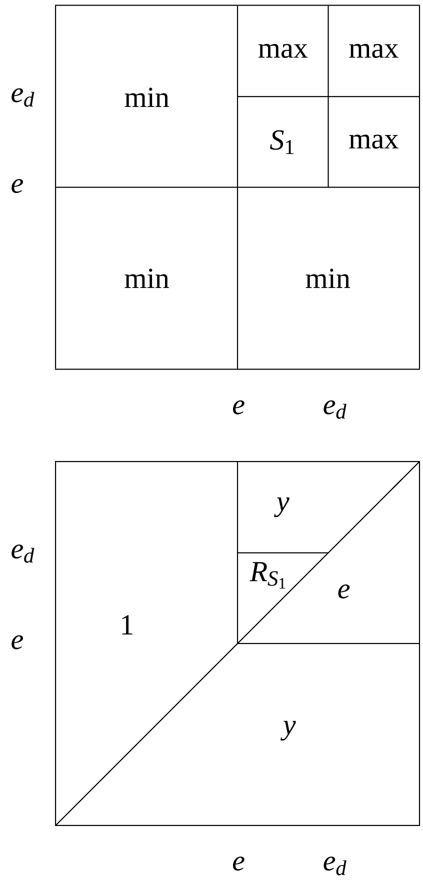

Figure 3: A uninorm U (top) in $\mathcal{U}_{\min}$ and its residual implication I_U (bottom) such that I_U, U_c and U_d satisfy (5), where U_c is idempotent in $\mathcal{U}_{\min}$ and $U_d = (e_d, g_d)$ is idempotent with $e_d > e$ and $g_d(e) = 1$.

a disjunctive one. If I_U, U_c and U_d satisfy (5), then $g_d(e) = 1$ and U_c must be in $\mathcal{U}_{\min}$.

Theorem 4 *Let $U = (e, T_U, S_U)$ be a uninorm in $\mathcal{U}_{\min}$ with $e < 1$ and S_U continuous, and I_U its residual implication. Let $U_c = (e_c, T_c, S_c)$ be a conjunctive uninorm with $0 < e_c < 1$ and $U_d = (e_d, T_d, S_d)$ a disjunctive one. Then, I_U, U_c and U_d satisfy (5), if and only if U_c is idempotent in $\mathcal{U}_{\min}$, U_d is idempotent with associated function g_d such that $g_d(e) = 1$, $e = e_c < e_d$, and U is given by equation (7).*

The family of uninorms U stated in the previous theorem, jointly with their residual implications can be viewed in Figure 3.

3.3 When U is idempotent

In this case we take $U = (e, g)$ an idempotent uninorm with neutral element e such that $0 < e < 1$.

When U is idempotent, we obtain the following lemmata.

Lemma 8 *Let $U = (e, g)$ be an idempotent uninorm with $e < 1$ and I_U its residual implication. Let U_c be a conjunctive uninorm and U_d a disjunctive one. If I_U, U_c and U_d satisfy (5) then U_c and U_d are idempotent.*

Lemma 9 *Let $U = (e, g)$ be an idempotent uninorm with $e < 1$ and I_U its residual implication. Let U_c and U_d be idempotent uninorms with U_c conjunctive and U_d disjunctive. If I_U, U_c and U_d satisfy (5) then U_c and U_d are g-dual, that is, $g(U_c(x, y)) = U_d(g(x), g(y))$.*

Currently, we are working in the general solution of distributivity in the case when U is idempotent. In fact, we already know several solutions in this case that can be derived from the previous cases. Namely:

- When g is a strong negation N, it is known that the residual implication I_U coincides with the strong implication derived from the right continuous idempotent uninorm $U_r = (e, N)$ (see [9]) and, from results in [10], we obtain that I_U, U_c and U_d satisfy equation (5) when $U_d = U_r$ and U_c is the N-dual of U_r (that necessarily is the left continuous idempotent uninorm $U_l = (e, N)$).
- When U is idempotent in $\mathcal{U}_{\min}$ and $U_c = \min$ we have from Theorem 3 that $I_U, \min$ and U_d satisfy equation (5) when U_d is idempotent with $e_d \leq e$.
- When U is idempotent in $\mathcal{U}_{\min}$ and U_c is idempotent in $\mathcal{U}_{\min}$ with $e_c < 1$, we have, from Theorem 4, that I_U, U_c and U_d satisfy equation (5) when $U_d = (e_d, g_d)$ is idempotent with $e = e_c < e_d$ and $g_d(e) = 1$.

However, in the case when U is idempotent, are there any other solution of equation (5) than the ones stated before?

Acknowledgements: This work has been partially supported by the Government of the Balearic Islands projects PRDIB-2002GC3-19 and PRIB-2004-9250 and the Spanish grant TIC2002-11942-E.

References

[1] J. Balasubramaniam and C.J.M. Rao (2004), On the distributivity of implication operators over T and S norms, *IEEE Transactions on Fuzzy Systems*, **12**, pp. 194-198.

[2] W.E. Combs and J.E. Andrews (1998), Combinatorial rule explosion eliminated by a fuzzy rule configuration, *IEEE Transactions on Fuzzy Systems*, **6**, pp. 1-11.

[3] B. De Baets (1999), Idempotent uninorms, *European J. Oper. Res.* **118**(3), pp. 631-642.

[4] B. De Baets and J.C. Fodor (1999), Residual operators of uninorms. Soft Computing **3**, pp. 89-100.

[5] J. C. Fodor, R. R. Yager and A. Rybalov (1997), Structure of Uninorms, *Int. J. of Uncertainty, Fuzziness and Knowledge-based Systems*, **5** N.4, pp. 411-427.

[6] M. González, D. Ruiz and J. Torrens (2003), Algebraic properties of fuzzy morphological operators based on uninorms, in I. Aguiló, L. Valverde and M. Escrig editors, *Artificial Intelligence Research and Development*, IOS Press, pp. 27-38.

[7] E.P. Klement, R. Mesiar and E. Pap, *Triangular norms*, Kluwer Academic Publishers, Dordrecht (2000).

[8] J. Martín, G. Mayor and J. Torrens (2003), On locally internal monotonic operations, *Fuzzy Sets and Systems*, **137** (1), pp. 27-42.

[9] D. Ruiz and J. Torrens (2004), Residual implications and co-implications from idempotent uninorms, *Kybernetika*, **40**, pp. 21-38.

[10] D. Ruiz-Aguilera and J. Torrens (2005), Distributive strong implications from uninorms, *submitted*.

[11] D. Ruiz and J. Torrens (2005), Distributivity and conditional distributivity of a uninorm and a continuous t-conorm, *submitted*.

[12] E. Trillas and C. Alsina (2002), On the law $[p \wedge q \to r] \equiv [(p \to r) \vee (q \to r)]$ in fuzzy logic, *IEEE Transactions on Fuzzy Systems*, **10**, pp. 84-88.