Additive Generators of Discrete Conjunctive Aggregation Operations

G. Mayor
Dept. of Mathematics and C.S.
University of the Balearic Islands
07122 - Palma de Mallorca, Spain
gmayor@uib.es

J. Monreal
Dept. of Mathematics and C.S.
University of the Balearic Islands
07122 - Palma de Mallorca, Spain
jaumemonreal@mallorcaweb.net

Abstract

This paper deals with a class of conjunction-like binary operations defined on a finite chain. The concept of additive generator of a discrete conjunctive aggregation operation is introduced and we obtain a description of those operations having additive generator by means of non-strict Archimedean t-norms. The main goal is to give a procedure for deciding whether a conjunctive aggregation operation is additively generated or not. The problem of the existence of additive generators for discrete t-norms is also discussed.

Keywords: discrete binary operation, conjunction, discrete t-norm, additive generator, system of linear inequalities, polyhedral convex cone, dual cone, Γ-algorithm.

1 Introduction

An old problem is whether there exist constructions involving only a one-place real function and the usual addition (or multiplication) which induce two-place real functions (i.e., binary operations on the real line) having interesting algebraic properties, in particular, the associativity. In [3] the reader can find details about this topic, in particular we point up these facts: "A continuous t-norm is Archimedean if and only if it has a continuous additive generator" and "There exist additive generators for the drastic t-norm (and for other non-continuous t-norms), but none for the minimum t-norm (or for other t-norm with a non-trivial idempotent elements)". In discrete settings things go quite different in some aspects as we have shown in [4]. Now, in this paper we focus our attention on a method to decide whether a discrete conjunctive aggregation operation has an additive generator or not. In this direction, using the minimal and maximal elements of the table of the operation, we reduce the problem of the existence of generator to a problem of consistency of a system of linear inequalities.

While the problem of treating with systems of linear equations implies dealing with linear spaces, main results on systems of (weak or strict) linear inequalities can be derived from convexity theory. Particularly, the key point in studying consistency of that systems is the generation of the dual cone of a given cone. We dedicate section 2 to provide some tools about this theory. In sections 3 and 4 the concepts of discrete conjunctive aggregation operation and additive generator for such an operation are introduced and, finally, in section 5 we deal with the development of a procedure for detecting whether a given operation has an additive generator, and in the affirmative case it gives one of them.

2 Preliminaries

In this section we remind the reader about some elementary concepts and results about convexity and polyhedral convex cones. Basic notation is also introduced. All definitions and results in this preliminary section can be found in [7].

All through this paper R^n is the usual Euclidean vector space of real n-tuples $x = (x_1, ..., x_n)$. The inner product of two vectors x and y in R^n is expressed by $\langle x, y\rangle = x_1y_1 + ... + x_ny_n$.

Definition 1 *A subset C of R^n is said to be convex if $(1-\lambda)x + \lambda y \in C$ whenever $x, y \in C$ and $0 \leq \lambda \leq 1$.*

All affine sets (including $\emptyset$ and R^n) are convex. What makes convex sets more general than affine sets is that they only have to contain, along with two points x, y, the line segment between x and y. Of course, the intersection of an arbitrary collection of convex sets is convex.

Half-spaces are important examples of convex sets. For any non-zero $a \in R^n$ and any $\alpha \in R$, the sets $\{x \in R^n; \langle a, x\rangle \leq \alpha\}$, $\{x \in R^n; \langle a, x\rangle < \alpha\}$ are called closed and open half-spaces respectively. One may speak unambiguously of the open and closed half-spaces corresponding to the hyperplane $\{x \in R^n \; ; \; \langle a, x\rangle = \alpha\}$.

Definition 2 *A polyhedral convex cone (pcc) is the intersection of a finite collection of closed half-spaces whose boundary hyperplanes pass through the origin. A polyhedral convex cone is thus the set of solutions to some finite system of homogeneous weak linear inequalities.*

Proposition 1 *Given a pcc K, $K^o = \{x \in R^n \; ; \; \langle x, y\rangle \leq 0 \; , \; \forall y \in K\}$ is also a polyhedral convex cone that we will call the dual of K.*

Proposition 2 *If K is the pcc of the solutions of the system $\langle a_i, x\rangle \leq 0$, $i = 1, ..., m$, then $K^o = \{\lambda_1 a_1 + ... + \lambda_m a_m \; ; \; \lambda_i \geq 0 \; , \; i = 1, ..., m\}$.*

This set of non-negative linear combinations of $a_1, ..., a_m$ will be denoted by $\langle a_1, ..., a_m\rangle_+$. We say also that $a_1, ..., a_m$ are generators of $\langle a_1, ..., a_m\rangle_+$.

Proposition 3 *If K is a polyhedral convex cone then $K^{oo} = K$.*

From this result we deduce that any pcc K can be expressed in the form $K = \langle a_1, ..., a_m\rangle_+$ for some $a_i \in R^n$, $i = 1, ..., m$.

According to the previous results we can deduce

Proposition 4 *Let x be a vector in R^n. Then $x \in \langle a_1, ..., a_m\rangle_+$ if and only if $\langle b_i, x\rangle \leq 0$, $i = 1, ..., s$, where $b_i \in R^n$ are generators of the cone $(\langle a_1, ..., a_m\rangle_+)^o$.*

Proof If $K = \langle a_1, ..., a_m\rangle_+$ then $K^o = \langle b_1, ..., b_s\rangle_+$ for some $b_i \in R^n$ $i = 1, ..., s$. From Proposition 3 we have $K = K^{oo}$ thus $x \in \langle a_1, ..., a_m\rangle_+$ if and only if $x \in (\langle b_1, ..., b_s\rangle_+)^o = \{x \in R^n; \langle x, b_i\rangle \leq 0, \; i = 1, ..., s\}$ ♢

Consider $n < p < m$ and $a_i \in R^n$, $i = 1, ..., m$. We are interested in systems of linear inequalities of the form

$$\begin{aligned} &\langle a_i, x\rangle < 0, \quad i = 1, ..., p \\ &\langle a_i, x\rangle \leq 0, \quad i = p+1, ..., m \end{aligned} \qquad (*)$$

Proposition 5 *The system $(*)$ is compatible (has a solution) if and only if any solution $(y_1, ..., y_m)$ with $y_i \geq 0$, $i = 1, ..., m$ of the homogeneous linear system $A^T y = 0$ is such that $y_1 = \; ... \; = y_p = 0$, where A is the $m \times n$ real matrix whose rows are $a_1, ..., a_m$.*

Finally, other basic result we will use later is the following

Proposition 6 *Given a_0, $a_i \in R^n$ $i = 1, ..., m$. Then $\langle a_i, x\rangle \leq 0$, $i = 1, ..., m$ implies $\langle a_0, x\rangle \leq 0$ (we say that $\langle a_0, x\rangle \leq 0$ is a consequence of $\langle a_i, x\rangle \leq 0$, $i : 1, ..., m$) if and only if $a_0 \in \langle a_1, ..., a_m\rangle_+$.*

3 Discrete Conjunctive Aggregation Operations

Consider the finite set $L = \{0, 1, ..., n\}$ with $n \geq 1$ equipped with the usual ordering.

Definition 3 *A binary operation $C : L \times L \longrightarrow L$ is a (discrete) conjunctive aggregation operation (coagop, for short) if it verifies for all i, i', $j \in L$:*

1) $C(i, j) = C(j, i)$
2) $i \leq i' \Longrightarrow C(i, j) \leq C(i', j)$
3) $C(i, n) = n \;\; \forall i \in L$

Example 1

1. T-norms on L are associative coagops ([5]). In particular: $C_M(i, j) = min(i, j)$, $C_Ł(i, j) = max(i + j - n, 0)$, $C_D(i, j) = 0$ if $max(i, j) < n$ and $C_D(i, j) = min(i, j)$ otherwise; are coagops that we call Minimum, Łukasiewicz and Drastic respectively.

2. Commutative copulas on L are also coagops ([6]).

Proposition 7 *Let C be a coagop on L. Then:*

1. *$C(i,j) \leq min(i,j)$ for all $i,j \in L$. Thus, C_M is the strongest coagop.*
2. *$C(i,0) = 0$ for all $i \in L$. Thus, all coagops coincide on the boundary of $L \times L$.*
3. *The only coagop satisfying $C(i,i) = i$ for all $i \in L$ is C_M.*
4. *The only coagop satisfying $C(i,i) = 0$ for all $i \in L - \{n\}$ is C_D.*

Remark 1

It is clear that a coagop C is for $n \geq 2$ determined by its values on $\Delta = \{(i,j) \in L \times L \ ; \ 0 < i \leq j < n\}$. Consider $\Delta_k = \{(i,j) \in \Delta \ ; \ T(i,j) = k\}$, $k = 0, 1, ..., n-1$. We indicate by N_k and M_k the sets of minimal and maximal elements of Δ_k respectively (product order). Of course that some Δ_k can be empty, thus we observe that $\Delta_0 \neq \emptyset$ means $N_0 = \{(1,1)\}$.

Let us supose $N_k = \{(a_\alpha, b_\alpha) \ ; \ \alpha = 1, ..., r\}$ and $M_k = \{(c_\beta, d_\beta) \ ; \ \beta = 1, ..., s\}$ where $0 < a_1 < a_2 < ... < a_r \leq b_r < b_{r-1} < ... < b_1 < n$ and $0 < c_1 < c_2 < ... < c_s \leq d_s < d_{s-1} < ... < d_1 < n$. In these situations we have $C(i,j) = k$ if and only if there exist $(a_\alpha, b_\alpha) \in N_k$, $(c_\beta, d_\beta) \in M_k$ such that $(a_\alpha, b_\alpha) \leq (i,j) \leq (c_\beta, d_\beta)$.

4 Additive Generators of Conjunctive Aggregation Operations

In this section we consider the pseudo-inverse of appropriate monotone functions from L to $[0, +\infty)$, and we introduce a construction similar to the one given in case of ordinary t-norms. Thus, we state a general method to construct conjunctive aggregation operations on L involving only a one-place real function and the usual addition.

Definition 4 *An additive generator $f : L \longrightarrow [0, +\infty)$ of a conjunctive aggregation operation C on L is a strictly decreasing function with $f(n) = 0$ such that*

$$C(i,j) = f^{(-1)}(f(i) + f(j)), \quad i,j \in L \qquad (1)$$

where $f^{(-1)} : [0, +\infty) \longrightarrow L$ is the pseudo-inverse of f, defined by $f^{(-1)}(t) = min\{i \in L \ ; \ f(i) \leq t\}$ $= min \ f^{-1}([0,t])$, $t \in [0, +\infty)$.

If $C : L \times L \longrightarrow L$ is a coagop of the form (1) for some f we say that C is additively generated by f. We indicate $f = (x_1, x_2, ..., x_n, x_{n+1} = 0)$ where $x_{i+1} = f(i)$, $i \in L$. Of course, $x_1 > x_2 > ... > x_n > x_{n+1} = 0$.

Example 2

1. $C_M(i,j)$ is generated by $f = (2^n - 1, 2^{n-1} - 1, ..., 7, 3, 1, 0)$.

2. $C_{\text{Ł}}(i,j)$ is generated by $f = (n, n - 1, ..., 2, 1, 0)$.

3. $C_D(i,j)$ has $f = (2n - 2, 2n - 1, ..., n, n - 1, 0)$ as additive generator.

Remark 2

1. Let C be the coagop defined on $L = \{0, 1, 2, 3, 4\}$ by $C(1,1) = C(1,2) = C(1,3) = 0$, $C(2,2) = C(2,3) = 1$, and $C(3,3) = 3$. This operation has not any additive generator.

2. If a coagop has an additive generator $f : L \longrightarrow [0, +\infty)$ then it also possesses an additive generator $g : L \longrightarrow [0, +\infty)$ with *Ran* $g \subset Z^+$ ([4]). Observe that being $f = (x_1, x_2, ..., x_n, x_{n+1} = 0)$ an additive generator of a coagop C with $x_i \in Z^+$, $i = 1, 2, ..., n+1$, then $x_i \geq n - i + 1$, $i = 1, 2, ..., n+1$.

One of the important problems to be solved is to find out a characterization of the class, say A_n, of coagops on $L = \{0, 1, ..., n\}$ having additive generator. We next propose a description of that class in terms of non-strict Archimedean t-norms.

Proposition 8 *A binary operation C belongs to A_n if and only if there exists a non-strict Archimedean t-norm T on the real interval $[0,n]$ such that $C(i,j) = \lceil T(i,j) \rceil \ \forall i,j \in L = \{0, 1, ..., n\}$, where $\lceil z \rceil$ stands for the ceiling of z (the smallest integer which is greater than or equal to z).*

Proof Let us supose that C is a conjunctive aggregation operation on L with additive generator $f : L \longrightarrow [0, +\infty)$. Consider $\overline{f} : [0,n] \longrightarrow [0, +\infty)$ a continuous strictly decreasing extension of f to the real interval $[0,n]$ and let T be the non-strict Archimedean t-norm on $[0,n]$ generated by $\overline{f}$: $T(x,y) = \overline{f}^{(-1)}(\overline{f}(x) + \overline{f}(y)) \ \forall x,y \in [0,n]$ where $\overline{f}^{(-1)}$ is the pseudo-inverse of $\overline{f}$, defined by $\overline{f}^{(-1)}(t) = min\{x \in [0,n] \; ; \; \overline{f}(x) \le t\}$, $t \in [0,+\infty)$. Obviously, $f^{(-1)}(t) = \left\lceil \overline{f}^{(-1)}(t) \right\rceil$ $\forall t \in [0,+\infty)$. Thus we can write $C(i,j) = f^{(-1)}(f(i)+f(j)) = \left\lceil \overline{f}^{(-1)}(f(i)+f(j)) \right\rceil = \left\lceil \overline{f}^{(-1)}(\overline{f}(i)+\overline{f}(j)) \right\rceil = \lceil T(i,j) \rceil$ $\forall i,j \in L$. Reciprocally, let us consider C given by $C(i,j) = \lceil T(i,j) \rceil$ $\forall i,j \in L$, being T a non-strict Archimedean t-norm on $[0,n]$. It is trivial to show that C verifies all conditions in definition 3, thus C is a conjunctive aggregation operation on L. On the other hand, let $\overline{f} : [0,n] \longrightarrow [0,+\infty)$ be an additive generator of T and $f : L \longrightarrow [0,+\infty)$ the restriction of $\overline{f}$ to L: $f(i) = \overline{f}(i)$, $i \in L$. We have $C(i,j) = \lceil T(i,j) \rceil = \left\lceil \overline{f}^{(-1)}(\overline{f}(i)+\overline{f}(j)) \right\rceil = \left\lceil \overline{f}^{(-1)}(f(i)+f(j)) \right\rceil = f^{(-1)}(f(i)+f(j))$ $\forall i,j \in L$. In other words, f is an additive generator of C and so $C \in A_n$. ◊

At this point, other interesting problem arises: the description of those t-norms on $L = \{0,1,...,n\}$ having additive generator. In other words, describe $T_n \cap A_n$, where T_n stands for the class of t-norms on $L = \{0,1,...,n\}$. In this direction we have proved the following partial results.

Proposition 9

1. *In case $n \le 7$, $T_n \subset A_n$.*
2. *The ordinal sum of additively generated t-norms is an additively generated t-norm. In particular, any divisible (smooth) t-norm has additive generator.*

More details on additive generators and the proof of part 2 in Proposition 9 can be found in [4].

5 Procedure for deciding whether a Conjunctive Aggregation Operation is additively generated or not

Observe that condition $C(i,j) = f^{(-1)}(f(i)+f(j))$, $(i,j) \in L \times L$ can be written in the form $f(C(i,j)) \le f(i)+f(j) < f(C(i,j)-1)$, $(i,j) \in \Delta = \{(i,j) \in L \times L \; ; \; 0 < i \le j < n\}$. In other words, $C(i,j) = k$ if and only if $f(k) \le f(i)+f(j) < f(k-1)$, $k = 0,1,...,n-1$ (in case k = 0 only the left inequality must be considered).

According to Remark 1, condition $f(k) \le f(i)+f(j) < f(k-1)$, $k = 0,1,...,n-1$ can be written as follows:

$$\begin{array}{ll} f(a_\alpha)+f(b_\alpha) < f(k-1) & for\ all\ (a_\alpha,b_\alpha) \in N_k, \\ & \alpha = 1,...,r \\ f(k) \le f(c_\beta)+f(d_\beta) & for\ all\ (c_\beta,d_\beta) \in M_k, \\ & \beta = 1,...,s \end{array}$$

Thus, given a conjunctive aggregation operation C on L, the problem of the existence of an additive generator f of C is equivalent to the problem of the compatibility (consistency) of a system of (weak and strict) linear inequalities. In concrete terms, and denoting $f = (x_1,x_2,...,x_n,x_{n+1}=0)$ where $x_{k+1} = f(k)$, $k = 0,1,...,n$, this system is of the following form:

$$\begin{array}{l} -x_1+x_2 < 0 \\ -x_2+x_3 < 0 \\ ... \\ -x_{n-1}+x_n < 0 \\ -x_n < 0 \\ \left.\begin{array}{l} -x_k+f(a_1)+f(b_1) < 0 \\ -x_k+f(a_2)+f(b_2) < 0 \\ ... \\ -x_k+f(a_r)+f(b_r) < 0 \end{array}\right\} \quad k = 1,2,...,n-1 \\ \left.\begin{array}{l} x_{k+1}-f(c_1)-f(d_1) \le 0 \\ x_{k+1}-f(c_2)-f(d_2) \le 0 \\ ... \\ x_{k+1}-f(c_s)-f(d_s) \le 0 \end{array}\right\} \quad k = 0,1,...,n-1 \end{array} \qquad (2)$$

Note that in system (2), $-x_k+x_i+x_j < 0$ means that $C(i-1,j-1) = k$ being $(i-1,j-1)$ a minimal element of Δ_k. Analogously, if $(i-1,j-1)$ is a maximal element of Δ_k then $x_{k+1}-x_i-x_j \le 0$ means that $C(i-1,j-1) = k$.

Example 3

Consider $L = \{0,1,...,n\}$ and let C be the conjunctive aggregation operation on L determined by its values on $\Delta = \{(i,j) \in L \times L \; ; \; 0 < i \le j < 4\}$: $C(1,1) = 0$, $C(1,2) = C(1,3) = C(2,2) = 1$, $C(2,3) = 2$, $C(3,3) = 2$.
In this case, we have:
$N_0 = M_0 = \{(1,1)\}$
$N_1 = \{(1,2)\}$, $M_1 = \{(1,3),(2,2)\}$
$N_2 = \{(2,3)\}$, $M_2 = \{(3,3)\}$
$N_3 = M_3 = \emptyset$

and the corresponding system is (once eliminated redundant inequalities):

$$
\begin{array}{l}
-x_1 + x_2 < 0 \\
-x_2 + x_3 < 0 \\
-x_3 + x_4 < 0 \\
-x_4 < 0 \\
-x_1 + x_2 + x_3 < 0 \\
-x_2 + x_3 + x_4 < 0 \\
x_1 - 2x_2 \leq 0 \\
x_2 - 2x_3 \leq 0 \\
x_3 - 2x_4 \leq 0
\end{array}
$$

where $f = (x_1, x_2, x_3, x_4, x_5 = 0)$ with $x_{k+1} = f(k)$, $k \in \{0, 1, 2, 3, 4\}$.
Observe that $f = (7, 4, 2, 1, 0)$ is a solution. Thus the coagop C has $f = (7, 4, 2, 1, 0)$ as additive generator. Observe also that C is not associative (it is not a t-norm): $C(C(1, 2), 2) = C(1, 2) = 1$ and $C(1, C(2, 2) = C(1, 1) = 0$.

Let us go back to system (2) that for short we denote by

$$
\begin{array}{ll}
\langle a_i, x\rangle < 0, & i = 1, ..., p \\
\langle a_i, x\rangle \leq 0, & i = p + 1, ..., m
\end{array}
\tag{3}
$$

where p is the number of strict inequalities in the system. It is $n < p < m$.
Let A be the $m \times n$ real matrix whose rows are the vectors $a_1, ..., a_m$.

According to the Proposition 5, we have to analyse solutions $(y_1, ..., y_m)$ with $y_i \geq 0$, $i = 1, ..., m$, of the homogeneous linear system $A^T y = 0$:

$$
\begin{pmatrix}
-1 & 0 & 0 & ... & 0 & 0 & a_{1,n+1} & ... & a_{1,m} \\
1 & -1 & 0 & ... & 0 & 0 & a_{2,n+1} & ... & a_{2,m} \\
0 & 1 & -1 & ... & 0 & 0 & a_{3,n+1} & ... & a_{3,m} \\
.. & .. & .. & ... & .. & .. & .. & ... & .. \\
.. & .. & .. & ... & .. & .. & .. & ... & .. \\
.. & .. & .. & ... & .. & .. & .. & ... & .. \\
0 & 0 & 0 & ... & -1 & 0 & a_{n-1,n+1} & ... & a_{n-1,m} \\
0 & 0 & 0 & ... & 1 & -1 & a_{n,n+1} & ... & a_{n,m}
\end{pmatrix}
\begin{pmatrix} y_1 \\ y_2 \\ .. \\ .. \\ y_n \\ .. \\ .. \\ y_p \\ .. \\ .. \\ y_m \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ .. \\ .. \\ 0 \\ .. \\ .. \\ 0 \\ .. \\ .. \\ 0 \end{pmatrix}
$$

This is a system of rank n that can be easily solved

$$
\begin{array}{rcl}
y_1 & = & a_{1,n+1} y_{n+1} \ + \ ... \ + \ a_{1,m} y_m \\
y_2 & = & (a_{1,n+1} + a_{2,n+1}) y_{n+1} \ + \ ... \ + \\
 & & + \ (a_{1,m} + a_{2,m}) y_m \\
... & & \\
y_n & = & (a_{1,n+1} + ... + a_{n,n+1}) y_{n+1} \ + \ ... \ + \\
 & & + \ (a_{1,m} + ... + a_{n,m}) y_m
\end{array}
$$

Now, let us indicate $s_{i,j} = a_{1,j} + ... + a_{i,j}$, $i = 1, ..., n$, $j = n + 1, ..., m$; and consider the vectors $s_1, ..., s_m \in R^{m-n}$ defined by:

$$
\begin{array}{rcl}
s_1 & = & (s_{1,n+1}, ..., s_{1,m}) \\
s_2 & = & (s_{2,n+1}, ..., s_{2,m}) \\
... & & \\
s_n & = & (s_{n,n+1}, ..., s_{n,m}) \\
s_{n+1} & = & (1, 0, 0, ..., 0) \\
s_{n+2} & = & (0, 1, 0, ..., 0) \\
... & & \\
s_p & = & (0, ..., 1, ..., 0) \\
... & & \\
s_m & = & (0, 0, ..., 0, 1)
\end{array}
$$

Again from Proposition 5, we know that the system (3) is compatible if and only if all solutions $y^* = (y_{n+1}, ..., y_m) \in R^{m-n}$ of the system $\langle s_i, y^*\rangle \geq 0$, $i = 1, ..., m$

$$
\begin{array}{lcl}
s_{1,n+1} y_{n+1} + ... + s_{1,m} y_m & \geq & 0 \\
s_{2,n+1} y_{n+1} + ... + s_{2,m} y_m & \geq & 0 \\
... & & \\
s_{n,n+1} y_{n+1} + ... + s_{n,m} y_m & \geq & 0 \\
y_{n+1} & \geq & 0 \\
... & & \\
y_p & \geq & 0 \\
... & & \\
y_m & \geq & 0
\end{array}
$$

are in the intersection of the hyperplanes $\langle s_i, y^*\rangle = 0$, $i = 1, ..., p$. In other words, (3) is compatible if and only if:

$\langle -s_i, y^*\rangle \leq 0$, $i = 1, ..., m$, implies $\langle s_i, y^*\rangle \leq 0$, $i = 1, ..., p$.

Finally, from Proposition 6 we have the following main result

Proposition 10 *Let C be a conjunctive aggregation operation on $L = \{0, 1, ..., n\}$. C admits an additive generator if and only if $s_i \in \langle -s_1, ..., -s_m\rangle_+$, $i = 1, ..., p$.*

We know that $K = \langle -s_1, ..., -s_m\rangle_+$ is the dual of the cone determined by the solutions of the system $\langle s_i, y^*\rangle \geq 0$, $i = 1, ..., m$. On the other hand, from Proposition 3, we have $K^{oo} = K$, thus condition in Proposition 10 can be written $s_i \in K^{oo}$, $i = 1, ..., p$. Finally, if $K^o = \langle v_1, ..., v_t\rangle_+$ then our condition can be rewritten as follows

Proposition 11 *Let C be a conjunctive aggregation operation on $L = \{0, 1, ..., n\}$. C admits an additive generator if and only if $\langle s_i, v_r\rangle \leq 0$, $i = 1, ..., p$, $r = 1, ..., t$, where $v_1, ..., v_t$ generate* *the dual of the cone* $\langle -s_1, ..., -s_m \rangle_+$.

Thus, if we have a method for obtaining the generators of the dual of the cone $\langle -s_1,\ \dots\ , -s_m \rangle_+$, we also have a method for deciding whether a conjunctive aggregation operation is additively generated or not. To this purpose we will next use an algorithm, the Γ-algorithm, that sequentially obtains the duals of the cone generated by $\{-s_1,\ \dots\ , -s_h\}$, $h = 1, 2, \dots$. It starts with the dual R^{m-n} of the empty set cone and, in iteration h, a new vector $-s_h$ is incorporated to get the generators of the dual cone $(\langle -s_1,\ \dots\ , -s_m \rangle_+)^o$. All vectors that are not minimal generators are removed from the tableau. Starting with the unit matrix that generates R^{m-n}, the dual of the empty set, we end with a minimal set of generators for the dual $(\langle -s_1,\ \dots\ , -s_m \rangle_+)^o$. All details on the Γ-algorithm can be foud in [1, 2].

Thus, given a coagop C we can know whether it can be additively generated and, in this case, we can obtain an integer additive generator ($Ran\ f \subset Z^+$) of it. The sequence of this procedure is:

1. From C, we can determine the sets N_k and M_k of minimal and maximals elements, respectively. After, we calculate the vectors $\{s_1, ..., s_p, ..., s_m\}$ in the form described in this section.

2. Applying the Γ-algorithm to the vectors $\{-s_1, ..., -s_m\}$ we obtain the generators of the dual cone $(\langle -s_1,\ \dots\ , -s_m \rangle_+)^o$. Denote them as $\{v_1, ..., v_t\}$.

3. According to Proposition 11, we check whether $\langle s_i, v_r \rangle \leq 0$, $i = 1, ..., m$, $r = 1, ..., t$. In the affirmative case, we continue with step 4. Otherwise, we show a message "C has not any additive generator." and end the procedure.

4. Starting with $(n, n-1, n-2, ..., 2, 1, 0)$, we do $x_1 = n$ and we try all possibilities $(x_1, x_2, ..., x_n, x_{n+1} = 0)$ with $x_1 > x_2 > \dots > x_n > x_{n+1} = 0$. For any case, we check if it is an additive generator for C according to Definition 4. When all cases are computed, x_1 increments one unit and repeat again the process from this new situation. The program ends showing an additive generator of C.

Note that step 4 always will be ended, because from step 3 we know that C has an additive generator and from Remark 2, any coagop having an additive generator has an integer additive generator.

As an application of this method, given the coagop

C_1	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	1	1	1	1
2	0	0	0	1	2	2	2	2	2
3	0	1	1	1	2	2	2	2	3
4	0	1	2	2	2	2	2	2	4
5	0	1	2	2	2	2	2	2	5
6	0	1	2	2	2	2	5	5	6
7	0	1	2	2	2	2	5	5	7
8	0	1	2	3	4	5	6	7	8

we obtain

C has not any additive generator.

And given the coagop

C_2	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	1
2	0	0	0	0	0	2	2	2
3	0	0	0	0	0	3	3	3
4	0	0	0	0	3	4	4	4
5	0	0	2	3	4	5	5	5
6	0	0	2	3	4	5	5	6
7	0	1	2	3	4	5	6	7

we obtain

$$f = (19, 18, 15, 12, 7, 2, 1, 0)$$

Observe that C_1 is not a t-norm (it is not associative): $T(T(3,3),1) = 0$ and $T(3, T(3,1)) = 1$.

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