Construction of RET operators from aggregation functions

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Abstract

This paper deals with the construction of RET (relevancy transformation) operators for fuzzy systems. The notion of pseudo-duality is introduced to obtain new RET operators from old ones and, t-norms, t-conorms, nullnorms and uninorms are used to construct this kind of operators.

Keywords: RET-operator, fuzzy system, t-norm, t-conorm, nullnorm, uninorm, aggregation function, duality.

1 Introduction.

When fuzzy systems are used to model complex systems, the fuzzy modelling inference process contains two main steps: one consists of determine the contribution of each rule of the model, based on its relevancy to the current input, and the other consists of aggregate all these contributions to obtain the overall system output. Both steps in the inference process are studied in [10], where the minimal requirements of the operators used to perform each step are investigated. From this study, it is deduced that a good operator for the aggregation step is a uninorm whereas the first step requires the use of an operator, related to the implication operator, that is called there a relevancy transformation (RET) operator. Such a RET operator is a binary operator on the unit interval that satisfies several conditions (see Definition 1). All of them are different monotonicities in the first and second variables, except the first one that is required in order to ensure the following behavior:

- if the antecedent of the rule is completely satisfied, the rule relevancy is 1, then the effective output must be the rule consequent, and
- if the rule relevancy is 0, then the effective output must have no influence in the later aggregation process, that is, the effective output must be the neutral element of the aggregation operator.

Thus, the operators used in each step are not independent since the RET operator is dependent upon the neutral element of the uninorm used in the aggregation process. In [10] adequate RET operators for the cases when the uninorm is a t-norm or a t-conorm (that is, the neutral element is 1 or 0, respectively) are stated. For the general case, some constructions of RET operators from t-norms and copulas are presented in [9]. Several examples of RET operators with respect to 1 are also given in [7], called there *priority operators*.

In this paper we want to generalize these results and we give new construction methods generated from t-conorms, from t-operators and from uninorms. We give also a duality method to construct new RET operators from old ones. Note that, in this way, the same uninorm used in the aggregation process can also be used in the construction of the RET operator.

2 Preliminaries.

We begin with some basic definitions and results concerning RET operators that will be used throughout the paper. On the other hand, we will suppose the reader to be familiar with basic results concerning t-norms, t-conorms, uninorms and t-operators or nullnorms. In any case, all these results can be found in [6] for t-norms and t-conorms, in [4] for uninorms and in [8] for t-operators, also called nullnorms in [3].

Definition 1 *Given $c \in [0,1]$, a binary operator $h : [0,1]^2 \to [0,1]$ is called a RET operator with respect to the element c if it satisfies the following conditions:*

(R1) *$h(1,b) = b$ and $h(0,b) = c$ for all $b \in [0,1]$.*

(R2) *h is non-decreasing in the second variable.*

(R3) *If $b \geq c$ then $h(a_1,b) \leq h(a_2,b)$ for all $a_1,a_2 \in [0,1]$ such that $a_1 < a_2$.*

(R4) *If $b \leq c$ then $h(a_1,b) \geq h(a_2,b)$ for all $a_1,a_2 \in [0,1]$ such that $a_1 < a_2$.*

From these axioms it is easily deduced that any RET operator with respect to c always satisfies $h(a,c) = c$ for all $a \in [0,1]$.

For any t-norm T and any element $c \in [0,1]$, consider the operator $h^{T,c} : [0,1]^2 \to [0,1]$ given by

$$h^{T,c}(a,b) = T(a,b) + T(1-a,c).$$

One has the following results that can be found in [9].

Proposition 1 *Let T be a continuous Archimedean t-norm with convex additive generator and take $c \in [0,1]$. Then $h^{T,c}$ is a RET operator with respect to the element c if and only if $T(a,c) + T(1-a,c) = c$ holds for all $a \in [0,1]$.*

Proposition 2 *Let T be a strict t-norm and S a nilpotent t-conorm. Let $g : [0,1] \to [0,1]$ be an increasing bijection such that $g(a) + g(1-a) = 1$ for all $a \in [0,1]$. Then, if g is an additive generator of S and a multiplicative generator of T, the binary operator $h^{S,T}$ defined by*

$$h^{S,T}(a,b) = S(T(a,b), T(1-a,c))$$

for all $a,b \in [0,1]$ is a RET operator with respect to c. Moreover, in this case $h^{S,T}$ is given by

$$h^{S,T}(a,b) = g^{-1}(g(a)g(b) + (1-g(a))g(c)).$$

Let us recall finally that a t-norm T satisfies the Lipschitz condition if

$$T(a,b) - T(a',b) \leq a - a' \quad \text{whenever } a' \leq a.$$

3 RET operators defined from t-norms and t-conorms

Let us begin by recalling some well known examples of RET operators.

Example 1 *See [9]. (i) Any t-norm is a RET operator with respect to the element $c = 0$.*

(ii) Given any t-conorm S, the operator $h(a,b) = S(1-a,b)$ is a RET operator with respect to the element $c = 1$.

(iii) Let $\varphi : [0,1] \to [0,1]$ be a non-decreasing function with $\varphi(0) = 0$ and $\varphi(1) = 1$, and take $c \in [0,1]$. Then, the operator

$$h^{\varphi,c}(a,b) = \varphi(a)b + (1-\varphi(a))c$$

is a RET operator with respect to the element c.

Note that Example 1 (ii) works in fact for any strong negation N by defining $h(a,b) = S(N(a),b)$, which is the S-implication with respect to N. This operator can be written as

$$h(a,b) = N(S_N(a,N(b))) \quad \text{for all} \quad a,b \in [0,1]$$

where S_N is the N-dual t-norm of S. This fact suggests us the following definition.

Definition 2 *Let N be a strong negation and $h : [0,1]^2 \to [0,1]$ a binary operator. We define the N-pseudo-dual operator of h, denoted h_N, by*

$$h_N(a,b) = N(h(a,N(b))) \quad \textit{for all} \quad a,b \in [0,1].$$

Note that $(h_N)_N = h$ for any strong negation N. We have the following result.

Proposition 3 *A binary operator $h : [0,1]^2 \to [0,1]$ is a RET operator with respect to the element $c \in [0,1]$, if and only if h_N is a RET operator with respect to the element $N(c)$.*

Example 2 *Let N be a strong negation, T a t-norm and S its N-dual t-conorm. Then, cases (i) and, (ii) with N instead of the negation $x \mapsto 1-x$, in Example 1 are N-pseudo-dual one of each other.*

On the other hand, applying pseudo-duality to case (iii) of the mentioned example we get that

$$(h^{\varphi,c})_N(a,b) = N(\varphi(a)N(b) + (1-\varphi(a))c)$$

is a RET operator with respect to $N(c)$. *In particular, taking the usual strong negation* $N(x) = 1-x$, *denoted by* $N = 1-j$, *we get*

$$(h^{\varphi,c})_{1-j} = h^{\varphi,1-c}.$$

Similarly, given any t-norm T and any element $c \in [0,1]$, we get

$$(h^{T,c})_{1-j}(a,b) = 1 - h^{T,c}(a,1-b) =$$

$$1 - [T(a,1-b) + T(1-a,c)] =$$

$$1 - [1 - S(1-a,b) + 1 - S(a,1-c)] =$$

$$S(1-a,b) + S(a,1-c) - 1$$

where S is the $(1-j)$-dual t-conorm of T. Thus, given a t-conorm S and an element $c \in [0,1]$ we can define the following operator:

$$h^{S,c}(a,b) = S(1-a,b) + S(a,c) - 1 \qquad (1)$$

for all $a,b \in [0,1]$ and we have just proved that

$$(h^{T,c})_{1-j} = h^{S,1-c} \qquad (2)$$

whenever T and S are $(1-j)$-dual. As a consequence we obtain the following proposition.

Proposition 4 *Let* S *be a continuous Archimedean t-conorm with convex additive generator and take* $c \in [0,1]$. *Then, the operator* $h^{S,c}$ *is a RET operator with respect to the element* c *if and only if* $S(1-a,c) + S(a,c) = 1+c$.

Let us now deal with a more general class of RET operators given from t-norms and t-conorms. Instead of operators defined by

$$h(a,b) = S(T(a,b), T(1-a,c)) \quad \text{for all} \quad a,b \in [0,1]$$

studied in [9] (see also Proposition 2), we consider the case when N is any strong negation (not necessarily $1-j$) and two possibly different t-norms are involved. That is, we consider operators given by

$$h(a,b) = S(T_1(a,b), T_2(N(a),c)) \quad \text{for all} \quad a,b \in [0,1] \qquad (3)$$

and we want to study when they define RET operators with respect to c. It is clear that a necessary condition will be

$$S(T_1(a,c), T_2(N(a),c)) = c \quad \text{for all} \quad a \in [0,1]$$

which is a weak version of the well known Alsina equation where the same equality is required for all $a,c \in [0,1]$. The solution of Alsina equation can be found in [1] for the case $T_1 = T_2$ and in [2] for the general case. Next proposition shows that all the solutions of Alsina equation give RET operators.

Proposition 5 *Let* $s : [0,1] \to [0,1]$ *be a continuous increasing function with* $s(0) = 0$ *and* $s(1) = 1$ *and let* S *and* N *be given by*

$$S(a,b) = s^{-1}(\min(1, s(a) + s(b)))$$

and $N(a) = s^{-1}(1 - s(a))$. *If* T_1 *and* T_2 *have one of the following forms:*

(i) $T_1(a,b) = \min(a,b)$, *and* $T_2(a,b) = s^{-1}(\max(0, s(a)+s(b)-1))$;

(ii) $T_1(a,b) = s^{-1}(\max(0, s(a)+s(b)-1))$, *and* $T_2(a,b) = \min(a,b)$;

(iii) $T_1(a,b) = T_2(a,b) = s^{-1}(s(a)s(b))$;

(iv) $T_1(a,b) = s^{-1}(T_\alpha(s(a),s(b)))$, *and* $T_2(a,b) = s^{-1}(T_{1/\alpha}(s(a),s(b)))$, *where* $0 < \alpha < \infty$, $\alpha \neq 1$ *and* $T_\alpha, T_{1/\alpha}$ *belong to the Frank's family;*

then, for any $c \in [0,1]$, *the binary operator* h *defined by* (3) *is a RET operator with respect to* c.

The expressions of the RET operators obtained in the proposition above are the following:

(i) $h(a,b) =$

$$\begin{cases} c & \text{if } a \leq b,c \\ a & \text{if } c \leq a \leq b \\ b & \text{if } a \geq b,c \\ s^{-1}(s(b)+s(c)-s(a)) & \text{if } b \leq a \leq c. \end{cases} \qquad (4)$$

(ii) $h(a,b) =$

$$\begin{cases} b & \text{if } a \geq N(b),N(c) \\ N(a) & \text{if } N(c) \leq a \leq N(b) \\ s^{-1}(s(a)+s(b)+s(c)-1) & \text{if } N(b) \leq a \leq N(c) \\ c & \text{otherwise.} \end{cases} \qquad (5)$$

(iii)

$$h(a,b) = s^{-1}(s(a)s(b) + (1-s(a))s(c)). \qquad (6)$$

(iv) $h(a,b) =$

$$s^{-1}\left(\log_\alpha\left(\frac{\alpha^{s(a)+s(b)} - \alpha^{s(a)} - \alpha^{s(b)} + \alpha}{\alpha^{s(a)} + \alpha^{1-s(c)} - \alpha^{s(a)-s(c)} - 1}\right)\right). \tag{7}$$

Remark 1 *When $N = 1 - j$, case (iii) in the previous proposition agrees with the RET operator given in Proposition 2. However, the result proved here shows that the operator h given by (6) is in fact a RET operator for any continuous increasing function s with $s(0) = 0$ and $s(1) = 1$ (and not only for those satisfying $s(a) + s(1 - a) = 1$ for all $a \in [0,1]$, as it is stated in Proposition 2). The only difference is that in these cases h has the form (3) with the strong negation $N(a) = s^{-1}(1 - s(a))$ instead of $1 - j$.*

The structure of cases (i) and (ii) in Proposition 5 can be viewed in Figures 1 and 2 respectively.

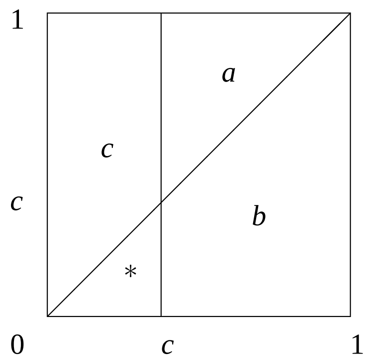

Figure 1: Structure of the RET operator given in Proposition 5 case (i), where $*$ stands for the expression $s^{-1}(s(b) + s(c) - s(a))$.

Thus, the previous proposition proves that all solutions of Alsina equation provide RET operators via equation (3). But this is not a necessary condition, that is, there are t-norms, t-conorms and strong negations, not satisfying Alsina equation, such that they define RET operators through equation (3). Some easy examples of this fact for the trivial cases $c = 0$ and $c = 1$ are given in [9], whereas for the case $c \in (0,1)$ a lot of them are furnished by the following proposition.

Proposition 6 *Let us fix $c \in (0,1)$. Let N be a strong negation with fixed point c (i.e. $N(c) = c$) and let T, S be a t-norm and a t-conorm respectively, such that*

$$T(a,c) = \min(a,c) \quad \textit{and} \quad S(a,c) = \max(a,c) \tag{8}$$

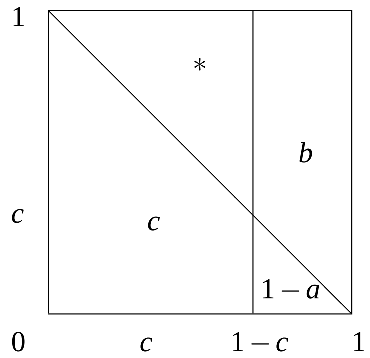

Figure 2: Structure of the RET operator given in Proposition 5 case (ii) taking the negation $N = 1 - j$, where $*$ stands for the expression $s^{-1}(s(a) + s(b) + s(c) - 1)$.

for all $a \in [0,1]$. Then the function h defined by

$$h(a,b) = S(T(a,b), T(N(a),c)) \tag{9}$$

for all $a, b \in [0,1]$ is a RET operator with respect to the element c.

Remark 2 *Note that T and S in the previous proposition need not to be continuous. In fact, whenever they are, equation (8) is equivalent to require simply that c is an idempotent element for T and S, that is $T(c,c) = S(c,c) = c$ and consequently both, T and S, are ordinal sums on the intervals $[0,c]$ and $[c,1]$.*

The structure of RET operators obtained in the previous proposition can be viewed in Figure 3.

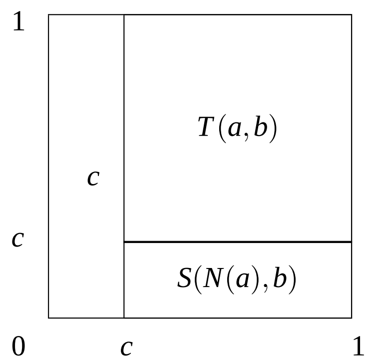

Figure 3: Structure of the RET operator given in Proposition 6.

Note that these operators are constantly equal to c whenever $a \leq c$. This behavior can be read as follows.

If the rule relevancy is less than or equal to a concrete value c, then the effective rule output must have no influence in the later aggregation process. Thus, using this kind of RET operators, only the output of rules with a "sufficient" rule relevancy have influence in the aggregation process.

Example 3 *For any RET operator h with expression as in equation (3), we can obtain a new one by considering its N-pseudo-dual, h_N, for any strong negation N. In particular, if we consider the same strong negation N used in the definition of h, a simple calculation shows that in this case h_N is given by*

$$h_N(a,b) = T(S_1(N(a),b), S_2(a,N(c))) \qquad (10)$$

for all $a,b \in [0,1]$ where T, S_1, S_2 are the N-dual operators of S, T_1, T_2 respectively. The same considerations work for RET operators h with expression as in equation (9), obtaining RET operators of the form

$$h_N(a,b) = T(S(N(a),b), S(a,c))$$

for all $a,b \in [0,1]$, where N is a strong negation with fixed point c and T, S are ordinal sums on the intervals $[0,c]$ and $[c,1]$.

4 RET operators defined from nullnorms and uninorms

We want to use in this section another two special kinds of aggregation functions to construct RET operators: nullnorms and uninorms.

Let us begin with nullnorms or t-operators (the structure as well as a detailed study of this kind of operators can be found in [8]).

Proposition 7 *Let N be any strong negation and F a nullnorm with annihilator element k. Then the operator $h : [0,1]^2 \to [0,1]$ defined by*

$$h(a,b) = \begin{cases} F(a,b) & \text{if } b \geq k \\ F(N(a),b) & \text{if } b \leq k \end{cases} \qquad (11)$$

is a RET operator with respect to k.

The structure of RET operators constructed in the previous proposition is studied in next proposition and it is similar to the structure of RET operators given in Proposition 6, specially when $N(k) = k$. This special case can be viewed in Figure 4.

Proposition 8 *Let N be any strong negation and F a nullnorm with annihilator element $k \in (0,1)$ and with associated t-norm and t-conorm T_F and S_F respectively. Then the RET operator h defined by (11) is given by*

$$h(a,b) =$$

$$\begin{cases} k + (1-k)T_F\left(\frac{a-k}{1-k}, \frac{b-k}{1-k}\right) & \text{if } a,b \geq k \\ kS_F\left(\frac{N(a)}{k}, \frac{b}{k}\right) & \text{if } a \geq N(k) \text{ and } b \leq k \\ k & \text{otherwise.} \end{cases}$$

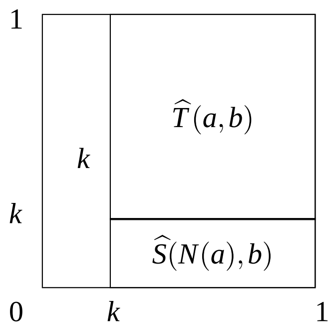

Figure 4: Structure of the RET operator given in Proposition 8 in the special case $N(k) = k$, where $\widehat{T}(a,b) = k + (1-k)T_F(\frac{a-k}{1-k}, \frac{b-k}{1-k})$ and $\widehat{S}(N(a),b) = kS_F(\frac{N(a)}{k}, \frac{b}{k})$.

We can construct as usually the pseudo-dual RET operators of the above ones. In this case we have the following result concerning auto-duality. Recall that a strong negation N (see [5]) is a k-negation if it satisfies

$$N(ka) = k + (1-k)N(a) \quad \text{for all } a \in [0,1]$$

and in this case k is the fixed point of N.

Proposition 9 *Take $k \in (0,1)$ and let N be any k-negation. Let F be a nullnorm with annihilator k and h the RET operator defined by (11). Let T_F and S_F the t-norm and the t-conorm associated to F. Then $h_N = h$ if and only if T_F and S_F are N-dual one of each other.*

Let us now deal with uninorms. Given any strong negation N and any uninorm U with neutral element $e \in (0,1)$, we consider the operator given by

$$h(a,b) = U(ea,b) + eN(a) \quad \text{for all } a,b \in [0,1]. \quad (12)$$

It is clear that, taking U conjunctive, the operator h defined by (12) satisfies conditions $R1)$ and $R2)$ of definition (1) with respect to e. However, to obtain a RET operator we have a necessary condition:

Proposition 10 *Let N be any strong negation, U a conjunctive uninorm with neutral element $e \in (0,1)$ and h the operator defined by* (12). *If h is a RET operator with respect to e then $N(a) = 1 - a$ for all $a \in [0,1]$.*

Thus, if we want to obtain RET operators through equation (12) we need to define

$$h(a,b) = U(ea,b) + e(1-a) \quad \text{for all } a,b \in [0,1] \tag{13}$$

for a conjunctive uninorm U.

Proposition 11 *Let U be a uninorm in $\mathcal{U}_{\min}$ with neutral element $e \in (0,1)$ and h the operator defined by* (13). *Let T_U be the t-norm associated to U. Then, h is a RET operator with respect to e if and only if T_U satisfies the Lipschitz condition. In this case, h is given by*

$$h(a,b) = \begin{cases} e & \text{if } b \geq e \text{ and } a < 1 \\ e\left(T_U(a,\frac{b}{e}) + 1 - a\right) & \text{otherwise.} \end{cases}$$

The structure of RET operators obtained in the last proposition can be viewed in Figure 5

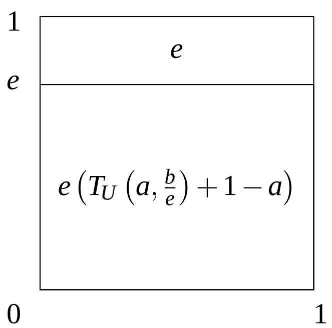

Figure 5: Structure of the RET operator given in Proposition 11.

Conclusions. In this work several constructions of RET operators are studied using t-norms and t-conorms (especially those satisfying the Alsina's equation), nullnorms, and uninorms. Moreover, the notion of pseudo-duality is introduced allowing us to construct new RET operators from old ones. We think that the results obtained here could inspire other possible methods of constructing new RET operators from old ones in a future work.

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