Calculation and reasoning with ordered fuzzy numbers

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Abstract

The fundamental concept of the fuzzy logic has been weakened by requiring a mere membership relation; consequently an ordered fuzzy number (OFN) arises as an ordered pair of continuous real functions defined on the interval $[0,1]$. Four algebraic operations are constructed in a way that renders them an algebra. Further, a norm is introduced which makes them a Banach space. A class of linear defuzzyfication operations on the algebra of OFN's is introduced. New algebraic methods of determining activation levels of multiconditional fuzzy *If–Then* rules are proposed. Examples of compositional rules of inference are given. The proposed operations and methods have been implemented in the form of a *fuzzy calculator* working under Windows and a class of controllers.

Keywords: Fuzzy logic, ordered fuzzy numbers, defuzzyfication, linear functional, inference rules.

1 Introduction

Classical fuzzy sets are convenient as far as a simple interpretation in the set-theoretical language is concerned. However, we could ask: how can we imagine a fuzzy information, say X, in such a way that by adding it to the fuzzy information A the fuzzy information C will be obtained?

In the classical approach for numerical handling of fuzzy quantities the extension principle [1], [2] is of fundamental importance. The commonly accepted theory of fuzzy numbers is that set up by Dubois and Prade [5]. However, if one wants to stay within their class of (L, R)–numbers while following the extension principle, approximations of fuzzy functions (and operations) are needed. They may lead to large computational errors [13] that cannot be further controlled when applying them repeatedly.[1]

In a series of papers (see [19, 20, 21] for reference) we have answered the above question by constructing a more general class of fuzzy numbers called ordered fuzzy numbers. There the concept of the membership function has been weakened by requiring a mere *membership relation*; consequently a fuzzy number arises as an ordered pair of continuous real functions defined on the interval $[0,1]$. Four algebraic operations have been constructed on OFN's in a way that renders them an algebra. Further, a normed topology has been introduced which makes them a Banach space and hence an infinite number of defuzzyfication methods can be defined.

2 Convex fuzzy numbers and invertibility

The concept of an ordered fuzzy number (OFN) has been introduced by the authors in their joint paper [19]. Before the term of the ordered fuzzy number has arrived we discussed an intermediate concept, namely, an oriented fuzzy number in [15, 16, 18, 17]. Such a number possesses properties close to the classical (of Zadeh type) fuzzy number with convex (precisely, strictly quasi–concave) membership functions, and is equipped with an extra feature, namely the orientation of

[1] Overcoming this drawback could be of a great help in constructing fuzzy inference systems, fuzzy controllers, not mentioning an effective fuzzy calculus [3, 14, 25].

its graph. The orientation has introduced a possibility to overcome some drawbacks[2] observed for example in [3, 9, 11, 13, 24] if one wants to stay within the Dubois and Prade's [5] representation of fuzzy numbers (or working on convex fuzzy numbers [4, 11]) while following the Zadeh's extension principle dealing with operations on them.

Classically, in the Zadeh's definition, by a fuzzy real one understands a fuzzy set in which the universe $\mathcal{X}$ is a set of real numbers $\mathbf{R}$. In [15] conditions distinguishing fuzzy membership functions corresponding to fuzzy reals are proposed. They are in some sense similar to those given by Nguyen and Wagenknecht in [11, 13]. The crucial point is the assumption of quasi-invertibility. Let us recall, by [10], the following notion :

Definition 1. Function $f : \mathbf{R} \to \mathbf{R}$ is **strictly quasi-convex**, iff for any $x, y, z \in \mathbf{R}$, $x < y < z$, we have

$$f(x) \le f(z) \Rightarrow f(y) \le f(z) \quad \text{and} \\ f(x) < f(z) \Rightarrow f(y) < f(z)\ . \quad \triangle \tag{1}$$

According to the fundamental theorem for strictly quasi–convex functions [10], fuzzy numbers with quasi-convex membership functions are quasi-invertible:

Proposition 1. (cf. [19]) Let a fuzzy number $A = (\mathbf{R}, \mu_A)$ be given with $\mu_A : \mathbf{R} \to [0,1]$ as its membership function $-\mu_A$ being strictly quasi–convex, and such that $1 \in \mu_A(\mathbf{R})$, and the support of μ_A is an interval (l_A, p_A), $l_A, p_A \in \mathbf{R}$. Then there exist $1_A^-, 1_A^+ \in (l_A, p_A)$ such that μ_A is invertible (increasing) on $(l_A, 1_A^-)$, invertible (decreasing) on $(1_A^+, p_A)$, and constantly equal to 1 on $[1_A^-, 1_A^+]$. $\triangle$

Quasi-invertibility (1) enables to state quite an efficient calculus on fuzzy reals [15]. Given $A = (\mathbf{R}, \mu_A)$, $B = (\mathbf{R}, \mu_B)$, one can construct the sum $C = A + B$ by pairwise adding inverses of the increasing and decreasing parts of functions μ_A and μ_B. In case of trapezoidal fuzzy membership functions this operation can be encoded in terms of equations $l_C = l_A + l_B$, $1_C^- = 1_A^- + 1_B^-$, $1_C^+ = 1_A^+ + 1_B^+$ and $p_C = p_A + p_B$. Analogously, one could define subtraction $A - B$.

Operating on membership functions satisfying conditions of Proposition defined on $\mathbf{R}$ (which are locally invertible), is similar to the case of convex fuzzy numbers [4, 11, 15].Results of such operations can be treated as inverse parts of resulting fuzzy numbers. However,when the operation of subtraction is done making the inverse of the resulting difference between number A and B, one can get as a result an improper fuzzy number [15, 16, 18], i.e. an object which does not possess a membership functions in the above classical sense.

On the other hand this observation has helped us to make one step more in our development and to define a more general object than the convex fuzzy number, namely an ordered fuzzy number as a pair of functions defined on the interval $[0, 1]$.

3 Ordered fuzzy numbers

In [20], we have referred to one of the very first representations of a fuzzy set defined on a universe $\mathcal{X}$ (the real axis $\mathbf{R}$, in the case of fuzzy numbers) of discourse. In that representation[3] (cf. [1, 7]) a fuzzy set A was defined as a set of ordered pairs $\{(x, \mu_x)\}$, where $x \in \mathcal{X}$ and $\mu_x \in [0,1]$ as the grade (or level) of membership of x in A.

Now the concept of membership functions is weakened by *membership relation.*

Definition 1. An *ordered fuzzy number* $A \in \mathcal{F}$ is an ordered pair of two continuous functions, $A = (x_{up}, x_{down})$, called the up-branch and the down-branch, respectively, both defined on the closed interval $[0, 1]$ with values in $\mathbf{R}$. $\triangle$

From the continuity follows that the images of both functions are bounded intervals, to which the terms *UP* and *DOWN*, can be attached, respectively. If we use the symbols $UP = (l_A, 1_A^-)$ and $DOWN = (1_A^+, p_A)$, and add the third interval $CONST = (1_A^-, 1_A^+)$, then we can see that they are three subintervals that have appeared in the splitting of the support of each convex fuzzy number, discussed above. In general those

[2] Convex fuzzy numbers do not possess neutral elements of addition and multiplication, they lead to the blow-up of the width of supports (i.e. fuzziness) after multiple fuzzy operations, they cannot be equipped with a linear structure and hence any norm.

[3] Later on, one assumed that μ_x is (or must be) a function of x. However, originally, A was just a relation in the product space $\mathcal{X} \times [0, 1]$.

subintervals are directed intervals (in the sense of Kaucher [8]), i.e. they may not satisfied the conditions $l_A \leq 1_A^-$ and $1_A^+ \leq p_A$. However, when both functions are strictly monotonic and moreover $x_{up} \leq x_{down}$, then all above subintervals are proper, with the following relationships:

$$\begin{array}{l} l_A := x_{up}(0) \ , 1_A^- := x_{up}(1) \ , \\ 1_A^+ := x_{down}(1) \ , p_A := x_{down}(0) \ , \end{array} \qquad (2)$$

and the property

$$l_A \leq 1_A^- \leq 1_A^+ \leq p_A.$$

The added interval on which the (membership) function of x variable is constant, i.e. the interval $[1_A^-, 1_A^+]$ gives the whole interval together with the previous *UP* and *DOWN*, namely $UP \cup [1_A^+, 1_A^-] \cup \ DOWN$. A new membership function μ_A can be piecewisely defined on $\mathbf{R}$ by taking the inverse x_{up}^{-1} of the function of $x_{up}(y)$ on *UP* and the inverse x_{down}^{-1} of the function $x_{down}(y)$ on *DOWN*. On the added interval *CONST* we put constant value equal to 1 for the constructed membership μ_A. Notice that now we have an extra feature for that function, namely the orientation of its graph. It is worthwhile to notice that all convex numbers are included in the set of ordered fuzzy numbers as a subset. Curves (x_{up}, x_{down}) and (x_{down}, x_{up}) do not differ[4] graphically on the coordinate system in which x-axis proceeds y-axis.

We have used in our papers [17, 19, 20, 21, 22, 23] the up-arrow and down-arrow to denote the ordered fuzzy number $A = (x_{up}, x_{down}) = (x_A^{\uparrow}, x_A^{\downarrow})$. Now to make it shorter we write two universal letters f and g for any OFN, i.e. for $A = (f_A, g_A)$, and for $B = (f_B, g_B)$. Operations on ordered fuzzy numbers are introduced in the following definition.

Definition 2. Let $A = (f_A, g_A)$, $B = (f_B, g_B)$ and $C = (f_C, g_C)$ are ordered fuzzy numbers. The sum $C = A + B$, subtraction $C = A - B$, product $C = A \cdot B$, and division $C = A/B$ are defined by formula

$$f_C(s) \ = \ f_A(s) \star f_B(s)$$

$$g_C(s) \ = \ g_A(s) \star g_B(s) \ , \ s \in [0,1]. \qquad (3)$$

where "$\star$" stands for "$+$", "$-$", "$\cdot$", and "$/$", respectively, and where A/B is defined, iff zero does not belong to intervals UP and $DOWN$ of B. △

Notice that the subtraction of B is the same as addition of the opposite of B, i.e. the number $(-1) \cdot B$; and the difference $B - B$ is a crisp (real) zero. On the other hand, if for $A = (f, g)$ we define its complement $\bar{A} = (-g, -f)$ (please note that $\bar{A} \neq (-1) \cdot A$), then the sum $A + \bar{A}$ gives a fuzzy zero $0 = (f - g, -(f - g))$ in the sense of the classical fuzzy number calculus: the *complementary number* plays the role of the opposite number in the sense of the Zadeh's model, since the sum of the both gives a fuzzy zero, non-crisp, in general [15, 16, 18, 20].

3.1 Normed structure of OFN's

The pointwise multiplication by a scalar (crisp) number, together with the operation addition lead to a *linear structure* of $\mathcal{F}$ - the set of all OFN's. Now one can introduce the norm over $\mathcal{F}$ as :

$$||A|| = \max(\sup_{s \in I} |f_A(s)|, \sup_{s \in I} |g_A(s)|) \qquad (4)$$

Hence $\mathcal{F}$ can be identified with $C([0,1]) \times C([0,1])$. Finally, $\mathcal{F}$ is a Banach algebra with the unity $(1^\dagger, 1^\dagger)$– a pair of constant functions $1^\dagger(y) = 1$, for $y \in [0,1]$. Previously, a Banach structure of an extension of convex fuzzy numbers was introduced by Goetschel and Voxman [6]. However, they were only interested in the linear structure of this extension.

On the space $\mathcal{F}$ we can introduce a pre–order [21] by defining a function $W : \mathcal{F} \to C([0,1])$ with the help of the relation

$$W(A) = (f_A + g_A), \qquad (5)$$

its value $W(A)$ is called a variation of the number $A = (f_A, g_A)$. Then we say that the ordered fuzzy number A is not smaller than the number B, and write $A \succ B$, if

$$W(A) \geq W(B) \Leftrightarrow W(A - B) \geq 0 \ . \qquad (6)$$

We say that the number C is non-negative if its variation $W(C) \geq 0$. Notice that there are

[4] However, the corresponding curves determine two different ordered fuzzy numbers: they differ by the orientation: if the first curve has the positive orientation, then the second one has negative.

ordered fuzzy numbers that are not comparable with zero. Thanks to this relation in the Banach algebra $\mathcal{F}$ we may define two ideals: the left and the right ones, which are non-trivial and possess proper divisors of zero [21].

3.2 Defuzzyfication

This is the main operation in fuzzy inference systems and fuzzy controllers [3, 14, 25]. In the case of the product space $\mathcal{F}$, according to the Banach-Kakutami-Riesz representation theorem, each bounded linear functional ϕ is given by a sum of two bounded, linear functionals defined on the factor space $C([0,1])$, i.e.

$$\phi(f_A, g_A) = \int_0^1 f_A(s)\mu_1(ds) + \int_0^1 g_A(s)\mu_2(ds) \tag{7}$$

where the pair of continuous functions $(f_A, g_A) \in \mathcal{F}$ represents an ordered fuzzy number, and μ_1, μ_2 are two Radon measures on $[0,1]$.

From this formula an infinite number of defuzzyfication methods can be defined. In particular, the standard procedure given in terms of the area under membership function (relation) can be generalized. It is realized by the pair of linear combinations of the Lebesgue measure of $[0,1]$. Moreover a number of non-linear defuzzyfication operators can be defined as compositions of multivariant nonlinear functions defined [21] on the Cartesian products of $\mathbf{R}$ and linear continuous functionals on the Banach space $\mathcal{F}$.

4 Reasoning with OFN's

The generalized concept of fuzzy numbers must possess its implementation in the fuzzy reasoning to build a fuzzy inference system in which fuzzy rules appear.

Premise and consequent parts of fuzzy rules contain linguistic variables that can attain values in the form of ordered fuzzy numbers. This makes possible to operate on them in the effective (even symbolic) way while calculating.

If, for example, two-conditional, fuzzy *If–Then* rule appears

$(\mathcal{R})$ **if** x_1 is A_1 **and** x_2 is A_2 **then** z is C ,

where x_1, x_2 and z are linguistic variables while A_1, A_2 and C are ordered fuzzy numbers, the level of activation $a_{\mathcal{R}}$ of $\mathcal{R}$ or firing strength of the fuzzy rule must be calculated.

Knowing that A_1 and A_2 are two OFN's the activation level of $\mathcal{R}$ at x_1, x_2 can be calculated based on the algebraic multiplication of A_1 and A_2, as follows

$$\begin{array}{l} a_{\mathcal{R}}(x_1, x_2) = \max\arg\{B(y) = x_1 x_2\}, \\ \text{with } \ y \in [0,1] \text{ and } B = A_1 \cdot A_2 \ . \end{array} \tag{8}$$

It should be stressed that before such operations are defined all variables need to be made dimensionless.

In the case when both OFN's A_1 and A_2 and their product $B = A_1 \cdot A_2$ are convex numbers with classical membership functions then the result of calculation in (8) will be

$$a_{\mathcal{R}}(x_1, x_2) = \mu_B(x_1 x_2) \ , \tag{9}$$

with μ_B as the membership function of the convex fuzzy number $B = A_1 \cdot A_2$. If in the fuzzy rule $\mathcal{R}$ the conjunction is replaced by the alternative then the algebraic addition can replace the multiplication in (8), i.e.

$$\begin{array}{l} a_{\mathcal{R}}(x_1, x_2) = \max\arg\{B(y) = x_1 + x_2\}, \\ \text{with } \ y \in [0,1] \text{ and } B = A_1 + A_2 \ . \end{array} \tag{10}$$

Other methods have been proposed by in [26], i.e. arithmetic mean, mean with constrain or with alternation of orientation on the support. The generalization to more than two conditions (linguistic variables) is straightforward.

4.1 Fuzzy inference

In the approximate reasoning the compositional rule of inference says that from a fuzzy fact (e.g. x is A') and the rule (e.g. if x is A then z is B) a consequent (conclusion: z is B') is derived. If the fuzzy rule is denoted by $\mathcal{R}$ then from the fuzzy fact x is A' the deduction of the consequent in the form of a fuzzy set B' made in the course of the fuzzy reasoning, is schematically defined by

$$B' = A' \circ \mathcal{R} \ , \tag{11}$$

where $\circ$ denotes the composition operator; it is just the composition rule of inference in the fuzzy (approximate) reasoning.

For the fuzzy rule $\mathcal{R}$, with $W, B \in \mathcal{F}$, of the form

$$\text{if } x \text{ is } W \text{ then } z \text{ is } B \tag{12}$$

in the first example of compositional rule of inference based on the multiplication operator we put $z_x(\cdot) = B'$ for (11), as

$$\begin{aligned} B' &= y_u(x)(f_B, g_B) \quad \text{if} \quad x \in f_W([0,1]) \\ B' &= y_d(x)(f_B, g_B) \quad \text{if} \quad x \in g_W([0,1]) \end{aligned} \; , \tag{13}$$

if the intervals $g_W([0,1]), f_W([0,1])$ are disjoint, where B and W are represented by the pairs:

$$W = (f_W, g_W), \;\; B = (f_B, g_B) \; , \tag{14}$$

and the numbers y_u and y_d are defined by the relations

$$\begin{aligned} y_u(x) &= \max\arg\{f_W(y) = x\}, \;\; y \in [0,1] \\ y_d(x) &= \max\arg\{g_W(y) = x\}. \end{aligned} \tag{15}$$

If the intervals $g_W([0,1]), f_W([0,1])$ are not disjoint then B' is given by the formula

$$B' = \max\{y_u(x), y_d(x)\}(f_B, g_B) \; . \tag{16}$$

If $f_W(1) \neq g_W(1)$, i.e. there exists (not degenerated to a point) an interval of constancy of W on which the membership relation is equal to one in y variable[5], then both formulae (13) and (16) lead to a unique relationship

$$B' = (f_B, g_B) \tag{17}$$

since then $y_u(x) = y_d(x) = 1$ by (15).

5 Implementation and conclusions

For the practical application of OFN a software in the Object Pascal environment of Delphi has been designed in [26] for implementation of a family of fuzzy controllers that allow to choose different methods of evaluation of the rule activation level, inference and defuzzyfications. It also possesses the possibility to vary between Mamdani and Takagi-Sugeno-Kang controllers. All data are stored as OFN's.

[5]Notice that if the fuzzy number W were convex it could correspond to unity in the value of the membership function of W .

Presented in the paper methods show that convex fuzzy numbers equipped with the extra property–orientation and then regarded as OFN's give more flexibility[6] in applications and lead to new features in dealing with fuzzy rules and fuzzy inference.

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