Fuzzy Logics and Substructural Logics without Exchange

Mayuka F. KAWAGUCHI
Division of Computer Science
Hokkaido University
Sapporo 060-0814, JAPAN
mayuka@main.ist.hokudai.ac.jp

Osamu WATARI
Hokkaido Automotive
Engineering College
Sapporo 062-0922, JAPAN
watari@haec.ac.jp

Masaaki MIYAKOSHI
Division of Computer Science
Hokkaido University
Sapporo 060-0814, JAPAN
miyakosi@main.ist.hokudai.ac.jp

Abstract

This report treats the relation between substructural logics and fuzzy logics, especially focuses on the non-commutativity of conjunctive operators i.e. substructural logics without the exchange rule. As the results, the authors show that fuzzy logics based on the left continuous pseudo-t-norms are the extensions of $\mathbf{FL}_{w}$. Also, we introduce the definition of pseudo-uninorms and give some methods to construct them, and show that such functions realize the fuzzy logical systems as the extensions of $\mathbf{FL}_{c}$ and **FL**, which is the weakest substructural logic.

Keywords: Non-commutative fuzzy logics, substructural logics, pseudo-t-norms, pseudo uninorms.

1 Introduction

Substructural logics are logics lacking some or all of the structural rules when they are formalized in sequent systems. It has been known that they cover many of the well-known non-classical logics. According to Ono [17], the purpose of the study of substructural logics is to introduce a uniform framework in which various kinds of non-classical logics that originated from different motivations can be discussed together, and to find common feartures among them, taking structural rules for a clue.

On the other hand, the theoretical aspects of fuzzy logics have been developed mainly from the viewpoint of logical connectives represented by the family of t-norms and their residuals [13]. Recently, the relations of fuzzy logics to substructural logics have attracted the attentions of some researchers in both sides of fuzzy logics [1], [11], [12] and substructural logics [16], [17], i.e. t-norm based fuzzy logics are included in the framework of $\mathbf{FL}_{ew}$ (substructural logic without contraction).

This research treats fuzzy logics from the viewpoint of substructural logics without exchange, because the lack of exchange in sequent systems corresponds to the non-commutativity property in the algebraic systems, non-commutative logics has become relevant especially in the field of computer science, and fuzzy logics can easily provide the various concrete examples of such logical connectives.

2 Substructural Logics and Full Lambek Algebra

The formal systems by sequent calculi corresponding to the classical logic and the intuitionistic logic are called **LK** and **LJ**, respectively. A sequent calculus consists of an initial sequent, structural inference rules, and inference rules on logical connectives. Here, the structural inference rules are weakening, contraction, exchange and cut. Substructural logics are the logical systems in which some of the structural rules except for cut are restricted from **LK** or **LJ**.

The logical system called full Lambek Caluculus **FL** is obtained by removing structural rules: weakening, contraction and exchange from **LJ**, and forms the basis of all substructural logics.

$$\frac{\Gamma,\Gamma' \Rightarrow \Delta}{\Gamma, A,\Gamma' \Rightarrow \Delta}\ \text{(w left)} \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ \text{(w right)}$$

$$\frac{\Gamma, A, A, \Gamma' \Rightarrow \Delta}{\Gamma, A, \Gamma' \Rightarrow \Delta} \text{ (c left)}$$

$$\frac{\Gamma, A, B, \Gamma' \Rightarrow \Delta}{\Gamma, B, A, \Gamma' \Rightarrow \Delta} \text{ (e left)}$$

Here, Δ denotes a sequence of at most one formula. By adding all or some of weakening, contraction and exchange to **FL**, one obtains the various intuitionistic substructural logics $\mathbf{FL}_\mathrm{e}$, $\mathbf{FL}_\mathrm{ec}$, $\mathbf{FL}_\mathrm{ew}$, $\mathbf{FL}_\mathrm{ecw}$ (=**LJ**), $\mathbf{FL}_\mathrm{c}$, $\mathbf{FL}_\mathrm{w}$, and $\mathbf{FL}_\mathrm{cw}$. Here, "w", "c" and "e" denote that the rules: weakening, contraction and exchange are added to **FL**, respectively. On the other hand, by removing weakening and/or contraction from **LK**, one obtains the various classical substructural logics $\mathbf{CFL}_\mathrm{e}$, $\mathbf{CFL}_\mathrm{ec}$, $\mathbf{CFL}_\mathrm{ew}$ and $\mathbf{CFL}_\mathrm{ecw}$ (=**LK**).

The full Lambek algebra (shortly, **FL**-algebra), which is an algebraic interpretation of the full Lambek calculus **FL**, is defined as follows [14], [15].

Definition 1. The algebra $\langle V, \cup, \cap, \circ, \rightarrow, \rightarrow', \mathbf{1}, \mathbf{0}, \top, \bot \rangle$ satisfying the following properties is called full Lambek algebra.

(FL1) $\langle V, \cup, \cap, \top, \bot \rangle$ is a lattice with the largest element $\top$ and the least element $\bot$,

(FL2) $\langle V, \circ, \mathbf{1} \rangle$ is a monoid of which unit element is **1**,

(FL3) $\forall x, y, z, w \in V$:
$z \circ (x \cup y) \circ w = (z \circ x \circ w) \cup (z \circ y \circ w)$,

(FL4) $\forall x, y, z \in V$: $x \circ y \leq z \Leftrightarrow x \leq y \rightarrow z$,

(FL5) $\forall x, y, z \in V$: $x \circ y \leq z \Leftrightarrow y \leq x \rightarrow' z$,

(FL6) $\mathbf{0} \in V$.

It should be noted that an arbitrary element **0** is needed in order to give the interpretation of negation as $\neg x \equiv x \rightarrow \mathbf{0}$ and $\neg' x \equiv x \rightarrow' \mathbf{0}$.

The structural inference rules in **FL** correspond to the following properties in **FL**-algebra, respectively.

(FLe) exchange: $\circ$ is commutative

(FLw) weakening: $\mathbf{0} = \bot$,
$x \circ y \leq x, \quad y \circ x \leq x$,

(FLc) contraction: $x \leq x \circ x$.

With respect to the classical substructural logics, the following property should be added to the algebraic systems.

(CFL) $(x \rightarrow \mathbf{0}) \rightarrow' \mathbf{0} = x = (x \rightarrow' \mathbf{0}) \rightarrow \mathbf{0}$.

3 Algebraic Structures for Substructural Logics without Exchange

In this section, let us investigate the algebraic systems corresponding to the substructural logics without exchange. We shall refer to the algebraic interpretation of the logical system $\mathbf{FL}_\Box$ as $\mathbf{FL}_\Box$-algebra.

$\mathbf{FL}_\mathrm{w}$-algebra: $\langle V, \cup, \cap, \circ, \rightarrow, \rightarrow', \mathbf{1}, \mathbf{0} \rangle$

One obtains $\mathbf{FL}_\mathrm{w}$-algebra by adding the condition (FLw) to **FL**-algebra as follows.

(FLw1) $\langle V, \cup, \cap, \mathbf{1}, \mathbf{0} \rangle$ is a lattice with the largest element **1** and the least element **0**,

(FL2) $\langle V, \circ, \mathbf{1} \rangle$ is a monoid of which unit element is **1**,

(FL3) $\forall x, y, z, w \in V$:
$z \circ (x \cup y) \circ w = (z \circ x \circ w) \cup (z \circ y \circ w)$,

(FL4) $\forall x, y, z \in V$: $x \circ y \leq z \Leftrightarrow x \leq y \rightarrow z$,

(FL5) $\forall x, y, z \in V$: $x \circ y \leq z \Leftrightarrow y \leq x \rightarrow' z$.

It should be noted that the largest element $\top$ and the least element $\bot$ coincide with **1** and **0**, respectively. Also, $\mathbf{0} \circ a = a \circ \mathbf{0} = \mathbf{0}$ holds for any $a \in V$.

$\mathbf{FL}_\mathrm{c}$-algebra: $\langle V, \cup, \cap, \circ, \rightarrow, \rightarrow', \mathbf{1}, \mathbf{0}, \top, \bot \rangle$

One obtains $\mathbf{FL}_\mathrm{c}$-algebra by adding the condition (FLc) to **FL**-algebra.

$\mathbf{FL}_\mathrm{cw}$-algebra: $\langle V, \cup, \cap, \circ, \rightarrow, \rightarrow', \mathbf{1}, \mathbf{0} \rangle$

One obtains $\mathbf{FL}_\mathrm{cw}$-algebra by adding the condition (FLc) to $\mathbf{FL}_\mathrm{w}$ -algebra.

4 Non-Commutative Fuzzy Logics and Substructural Logics without Exchange

4.1 Pseudo-t-Norms and Pseudo-Uninorms

Flondor et al. [6] have introduced pseudo-t-norms as conjunctive operators in order to construct a non-commutative fuzzy logic.

Definition 2. A two-place function $\hat{T}:[0,1]^2 \to [0,1]$ satisfying the following properties is called a pseudo-t-norm.

(pT1) $\hat{T}(a,1)=\hat{T}(1,a)=a$

(pT2) $a \le b \Rightarrow \hat{T}(a,c) \le \hat{T}(b,c), \hat{T}(c,a) \le \hat{T}(c,b)$

(pT3) $\hat{T}(a,\hat{T}(b,c)) = \hat{T}(\hat{T}(a,b),c)$

Now, we need to consider the left-hand continuity of conjunctive operators in logical systems because the axioms (FL4) and (FL5) are required in **FL**-algebra. Flondor et al. [6] have given the following family of functions as an example of left-hand continuous pseudo-t-norms:

$$\hat{T}(x,y) = \begin{cases} a_i & \text{if } a_i \le x \le b_i, a_i \le y \le c_i \\ \min(x,y) & \text{otherwise} \end{cases}, \quad (1)$$

where $0 \le a_1 < b_1 < c_1 < a_2 < b_2 < c_2 < \cdots < a_n < b_n < c_n \le 1$. When $b_i = c_i\ (i=1,\ldots,n)$, this function becomes commutative, i.e. a t-norm. The simplest example of this family is given as follows:

$$\hat{T}_{Mesiar}(x,y) = \begin{cases} 0 & \text{if } x \le a, y \le b \\ \min(x,y) & \text{otherwise} \end{cases}, \quad (2)$$

where $0 < a < b < 1$.

Now, the authors introduce the definition of a non-commutative extension of uninorms [20], according to the way by Flondor et al. to introduce a pseudo-t-norm.

Definition 3. A two-place function $\hat{U}:[0,1]^2 \to [0,1]$ satisfying the following properties is called a pseudo-uninorm.

(pU1) $\hat{U}(a,e)=\hat{U}(e,a)=a$

(pU2) $a \le b \Rightarrow$

$\hat{U}(a,c) \le \hat{U}(b,c),\ \hat{U}(c,a) \le \hat{U}(c,b)$

(pU3) $\hat{U}(a,\hat{U}(b,c)) = \hat{U}(\hat{U}(a,b),c)$

As the same way in the case of uninorms [7], we have to distinguish the conjunctive pseudo-uninorms satisfying $\hat{U}(0,1)=\hat{U}(1,0)=0$ from others in order to give the role as conjunctive operators to the pseudo-uninorms.

We can consider the following functions as examples of the conjunctive pseudo-uninoms.

$$\hat{U}_1(x,y) = \begin{cases} a & \text{if } a \le x < b \text{ and } a \le y < c \\ \min(x,y) & \text{if } x < e \text{ and } y < e \\ \max(x,y) & \text{otherwise} \end{cases}, (3)$$

Here, $b < c < e$ or $c < b < e$.

$$\hat{U}_2(x,y) = \begin{cases} a & \text{if } a \le x \le b \text{ and } a \le y \le c \\ \min(x,y) & \text{if } y \le 2e-x \\ \max(x,y) & \text{otherwise} \end{cases}, (4)$$

Here, $e > 0.5$; and $b < c < 2e-1$ or $c < b < 2e-1$. Clearly, $\hat{U}_1$ is right continuous, and $\hat{U}_2$ is left continuous. As another example of the left continuous and conjunctive pseudo-uninorms, an extension of the idempotent uninorms by De Baets [3] should be considered.

$$\hat{U}_c^g(x,y) = \begin{cases} \min(x,y) & \text{if } y \le g(x) \\ \max(x,y) & \text{otherwise} \end{cases}, \quad (5)$$

Here, $g:[0,1] \to [0,1]$ is a decreasing and continuous function satisfying $g(1)=0$, $g(e)=e$, and $g(x) < e$ (for $\forall x > e$).

4.2 Fuzzy Logics as $\mathbf{FL_w}$ and $\mathbf{FL_{cw}}$

Hajék [9], [10] has formulated the algebraic structures for non-commutative fuzzy logics as follows.

Definition 4. Consider an algebraic system $\langle L, \vee, \wedge, *, \to, \to', \mathbf{1}, \mathbf{0} \rangle$ and the following properties on it:

(i) $\langle L, \vee, \wedge, \mathbf{1}, \mathbf{0} \rangle$ is a lattice with the largest element $\mathbf{1}$ and the least element $\mathbf{0}$,

(ii) $\langle L, *, \mathbf{1} \rangle$ is a monoid of which unit element is $\mathbf{1}$,

(iii) $\forall x,y,z \in L$:

$x * y \le z \Leftrightarrow x \le y \to z \Leftrightarrow y \le x \to' z$

(iv) $(x \to y) \vee (y \to x) = \mathbf{1}$

$(x \to' y) \vee (y \to' x) = \mathbf{1}$

(v) $x \wedge y = (x \to y) * x = x * (x \to' y)$

When L satisfies (i), (ii) and (iii), L is called an integral residuated lattice-ordered monoid (shortly, an integral residuated l-monoid). Also, L satisfying (i), (ii), (iii), (iv) and (v) is called a pseudo-**BL** algebra (shortly, **psBL**-algebra), and L satisfying (i), (ii), (iii) and (iv) is called a pseudo-**MTL** algebra (shortly, **psMTL**-algebra).

It is clear that an integral residuated l-monoid is equivalent to $\mathbf{FL}_\mathrm{w}$-algebra. Therefore, **psBL**-algebra and **psMTL**-algebra are included in the framework of $\mathbf{FL}_\mathrm{w}$-algebra.

The algebraic structures: **psBL**-algebra and **psMTL**-algebra correspond to the logical systems: pseudo-**BL** (pseudo basic logic) [4], [6], [10] and pseudo-**MTL** (pseudo monoidal t-norm based logic)[6], [10], respectively, which have been introduced as non-commutative versions of **BL** [2], [8] and **MTL** [5], respectively.

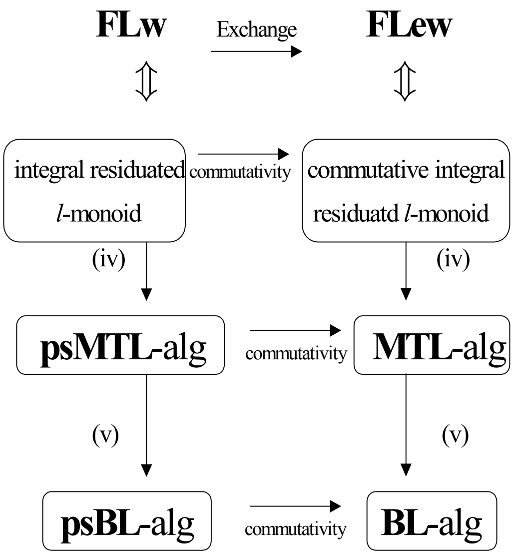

Figure 1: Connections between various fuzzy logics in the framework of $\mathbf{FL}_\mathrm{w}$ and $\mathbf{FL}_\mathrm{ew}$.

Hajék [10] has also given the definition of **psMTL** as follows.

Definition 5. **psMTL** is the logic with primitive connectives $\to, \to', \&, \wedge, \vee$, the constant $\overline{0}$ and the following axioms and deduction rules:

Axioms:

(A1) $(\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
$(\psi \to' \chi) \to' ((\varphi \to' \psi) \to' (\varphi \to' \chi))$

(A2) $(\varphi \,\&\, \psi) \to \varphi$

(A3) $(\varphi \,\&\, \psi) \to \psi$

(A4a) $((\varphi \to \psi) \,\&\, \varphi) \to (\varphi \wedge \psi)$
$(\varphi \,\&\, (\varphi \to' \psi)) \to (\varphi \wedge \psi)$

(A4b) $(\varphi \wedge \psi) \to \varphi$

(A4c) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

(A5) $(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \,\&\, \psi) \to \chi)$
$(\varphi \to' (\psi \to' \chi)) \leftrightarrow ((\psi \,\&\, \varphi) \to' \chi)$

(A6) $((\varphi \to \psi) \to \chi)) \to (((\psi \to \varphi) \to \chi) \to \chi)$
$((\varphi \to' \psi) \to' \chi)) \to' (((\psi \to' \varphi) \to' \chi) \to' \chi)$

(A7) $\overline{0} \to \varphi$

(A8) $(\varphi \vee \psi)$
$\leftrightarrow \left[((\varphi \to' \psi) \to \psi) \wedge ((\psi \to' \varphi) \to \varphi) \right]$
$\leftrightarrow \left[((\varphi \to \psi) \to' \psi) \wedge ((\psi \to \varphi) \to' \varphi) \right]$

Deduction rules:
Modus ponens: from $\varphi, \varphi \to \psi$ infer ψ;
from $\varphi, \varphi \to' \psi$ infer ψ.
Implications (IMP): from $\varphi \to \psi$ infer $\varphi \to' \psi$ and vise versa.

The pre-linearity property (iv) is expressed in logical axiom form as follows:

$$\begin{aligned} &(\varphi \to \psi) \vee (\psi \to \varphi) \\ &(\varphi \to' \psi) \vee (\psi \to' \varphi). \end{aligned} \tag{6}$$

Theorem 1. The following three statements are equivalent:

(a) A formula φ is provable in **psMTL**;

(b) A sequent $\Rightarrow \varphi$ is provable in $\mathbf{FL}_\mathrm{w}$ satisfying (6);

(c) For **psMTL**-algebra (i.e. $\mathbf{FL}_\mathrm{w}$-algebra satisfying (iv)), $v(\varphi) = \mathbf{1}$.

Let us focus our attention on the framework of fuzzy logics i.e. the case that $L = [0,1]$. The divisibility property (v) means that the monoid $*$ is continuous when L is a continuum. According to Definition 2, mentioned at the previous subsection, it is clear that a left continuous pseudo-t-norm based fuzzy logic $\left\langle [0,1], \max, \min, \hat{T}, \to, \to', 1, 0 \right\rangle$ forms a **psMTL**-algebra.

With respect to $\mathbf{FL}_\mathrm{cw}$, it has been already known that $\mathbf{FL}_\mathrm{cw}$ coincides with $\mathbf{FL}_\mathrm{ecw}$ i.e. **LJ** (see [17] for the details.). Thus, a pseudo-t-norm $\hat{T}$ reduces to minimum operator in the framework of $\mathbf{FL}_\mathrm{cw}$, that is, as Takeuti & Titani have already shown in [18], the fuzzy intuitionistic logic $\langle [0,1], \max, \min, \to, 1, 0 \rangle$ forms Heyting algebra which is well-known as the algebra of **LJ**.

4.3 Fuzzy Logics as FL and $\mathbf{FL_c}$

Let us consider in this subsection, a method to construct the algebraic systems corresponding to fuzzy logics as an extension of **FL**. On the analogy of the case of $\mathbf{FL_w}$, it should be adequate to add the pre-linearily property to **FL**-algebra (Definition 1).

Definition 6. A pseudo-**UL**-algebra (shortly, **psUL**-algebra) is **FL**-algebra equipped with the following pre-linearily condition:

(iv)' $(x \to y) \vee (y \to x) \geq \mathbf{1}$

$(x \to' y) \vee (y \to' x) \geq \mathbf{1}.$

Here, we introduce the definition of pseudo-**UL** (shortly, **psUL**).

Definition 7. **psUL** is the logic with primitive connectives $\to, \to', \&, \wedge, \vee$, the constant $\overline{0}$ and the following axioms and deduction rules:

Axioms:

$(\tilde{A}1)$ $(\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$

$(\psi \to' \chi) \to' ((\varphi \to' \psi) \to' (\varphi \to' \chi))$

$(\tilde{A}2a)$ $(\varphi \wedge \psi) \to \varphi$

$(\tilde{A}2b)$ $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$

$(\tilde{A}3)$ $(\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \& \psi) \to \chi)$

$(\varphi \to' (\psi \to' \chi)) \leftrightarrow ((\psi \& \varphi) \to' \chi)$

$(\tilde{A}4)$ $((\varphi \to \psi) \to \chi)) \to (((\psi \to \varphi) \to \chi) \to \chi)$

$((\varphi \to' \psi) \to' \chi)) \to' (((\psi \to' \varphi) \to' \chi) \to' \chi)$

$(\tilde{A}5)$ $\overline{0} \to \varphi$

$(\tilde{A}6)$ $(\varphi \vee \psi)$

$\leftrightarrow [((\varphi \to' \psi) \to \psi) \wedge ((\psi \to' \varphi) \to \varphi)]$

$\leftrightarrow [((\varphi \to \psi) \to' \psi) \wedge ((\psi \to \varphi) \to' \varphi)]$

Deduction rules:

Modus ponens: from $\varphi, \varphi \to \psi$ infer ψ;

from $\varphi, \varphi \to' \psi$ infer ψ.

Implications (IMP): from $\varphi \to \psi$ infer $\varphi \to' \psi$ and vise versa.

Theorem 2. The following three statements are equivalent:

(a) A formula φ is provable in **psUL**;

(b) A sequent $\Rightarrow \varphi$ is provable in **FL** satisfying (6);

(c) For **psUL**-algebra (i.e. **FL**-algebra satisfying (iv)'), $v(\varphi) \geq \mathbf{1}$.

Considering the left continuous and conjunctive pseudo-uninorms $\hat{U}$ which the authors introduced in §3.1, it is clear that an algebra $\langle [0,1], \max, \min, \hat{U}, \to, \to', e, f, 1, 0 \rangle$ forms a **psUL**-algebra. Also, it can be taken as the non-commutative version of 'uninorm based logics (**UL**)', which has been introduced by the authors in [19].

With respect to $\mathbf{FL_c}$, as a similar way to the case of **FL**, it is expected that **FL**-algebra equipped with the properties (iv) gives the algebraic structure for fuzzy logics on $[0,1]$. As a concrete example, left continuous, conjunctive and idempotent pseudo-uninorm $\hat{U}_c^g$ expressed as (5) can play the role of the conjunctive operators as $\langle [0,1], \max, \min, \hat{U}_c^g, \to, \to', e, f, 1, 0 \rangle$. Here, f is an arbitrary element of $[0,1]$.

5 Concluding Remarks

Through this research work, our interests have been always on the logical structures without the exchange rule (i.e. commutativity property). The authors have formulated the algebraic structures: $\mathbf{FL_w}$-algebra, $\mathbf{FL_c}$-algebra and $\mathbf{FL_{cw}}$-algebra, for substractural logics: $\mathbf{FL_w}$, $\mathbf{FL_c}$ and $\mathbf{FL_{cw}}$, respectively. Then, we have shown that fuzzy logics based on the left continuous pseudo-t-norms are the extensions of $\mathbf{FL_w}$. As the main results of this work, we have introduced the definition of pseudo-uninorms and give some methods to construct them, and show that such functions realize the fuzzy logical systems on [0,1] as the extdnsions of $\mathbf{FL_c}$ and **FL**.

It should be noted that the notion of uninorms was originally introduced in [20] as a family of aggregation operators in the field of fuzzy modeling, not as logical connectives. Nevertheless, the authors have shown in this report that uninorm (and pseudo-uninorm) based fuzzy logics can be justified from the viewpoint of substructural logics without the weakening rule. Our forthcoming research paper should be closely related to substructural logics without the weakening rule.

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