On some properties of mixing OWA operators with t-norms and t-conorms

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Abstract

The OWA operators are traditionally used in the context of decision making as the means to aggregate the satisfaction of single criteria into a overall preference index. They belong to the class of averages, entailing compensatory properties. Differently, t-norms and t-conorms entail the reinforcement property, since there is a kind of interaction between the degree of criteria satisfaction. A deeper analysis of the OWA structure shows assumptions that can be highlighted at the linguistic level. This offers the opportunity of extending the definition of ordinary OWA operators in new forms. In this paper we present some of these extensions and we discuss their properties in the context of decision making.

Keywords: Decision making, Aggregation operators, OWA

1 Introduction

In multicriteria decision making, aggregation operators provide the means by which the degree of each alternative in satisfying each single criterion is used to provide an overall preference index. Among the several properties that characterize aggregation operators there are the reinforcement and compensation properties. Let us denote a generic aggregation operator by a mapping $F : [0, 1]^n \to [0, 1]$. The reinforcement property is characteristic of t-norms and t-conorms: it states that $T(a_1, \dots, a_n) \le \min\limits_{i=1..n} a_i$, and $S(a_1, \dots, a_n) \ge \max\limits_{i=1..n} a_i$, for any generic t-norm T and t-conorm S. On the other side, the compensation property is associated to averages, and it states that $M(a_1, \dots, a_n) \in [\min\limits_{i=1..n} a_i, \max\limits_{i=1..n} a_i]$. The min and max are aggregation operators themselves: they respectively represent the upper limit of t-norms and the lower limit of t-conorms. Instead, the lower limit of t-norms is represented by the *drastic t-norm* T_d, defined as

$$T_d(x, y) = \begin{cases} 1 & \text{if } x = 1, y = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

whilst the upper limit to t-conorm is represented by the *drastic t-conorm* S_d, defined as

$$S_d(x, y) = \begin{cases} 0 & \text{if } x = 0, y = 0 \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

Other noticeable examples of t-norms (t-conorms) are the *algebraic product* (*algebraic sum*)

$$T_a(x, y) = xy \tag{3}$$

$$(S_a(x, y) = x + y - xy) \tag{4}$$

and the *bounded difference* (*bounded sum*)

$$T_b(x, y) = \max(0, x + y - 1) \tag{5}$$

$$(S_b(x, y) = \min(1, x + y)) \tag{6}$$

T-norms and t-conorms can be compared with respect to the aggregated value they provide.

Definition 1. *Given two t-norms T_1 and T_2, T_1* dominates *T_2 iff*

$$T_1(x, y) \ge T_2(x, y) \; \forall x, y \in [0, 1] \tag{7}$$

In this case we write $T_1 \ge T_2$.

It can be easily proven that $T_d \leq T_b \leq T_a \leq \min$. The definition of dominance can be applied to t-conorms as well. In this case, it results $\max \leq S_a \leq S_b \leq S_d$. Therefore there is a kind of symmetry between t-norms and t-conorms.

We can cluster aggregation operators as depicted in Fig.1. As measure of dominance, we can adopt

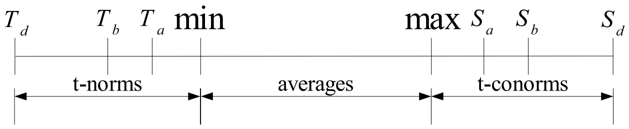

Figure 1: Clusters of aggregation operators

the *power* of a t-norm [8].

Definition 2. *If T is a t-norm, then the power of T is defined as*

$$\pi(T) = 1 - 3 \int_0^1 \int_0^1 T(x,y)dxdy \tag{8}$$

Proposition 1. *For any t-norm $T_1 \leq T_2$, then*

$$\pi(T_1) \geq \pi(T_2) \tag{9}$$

It can be proven that $\pi(T_d) = 1$, $\pi(T_b) = 0.5$, $\pi(T_a) = 0.25$, and $\pi(\min) = 0$. In a similar way we can adopt the *tolerance* as measure of dominance for t-conorms.

Definition 3. *If S is a t-conorm, then the tolerance of S is*

$$\tau(S) = 3 \int_0^1 \int_0^1 S(x,y)dxdy - 2 \tag{10}$$

Proposition 2. *For any t-conorm $S_1 \geq S_2$, then*

$$\tau(S_1) \geq \tau(S_2) \tag{11}$$

In this case $\tau(S_d) = 1$, $\tau(S_b) = 0.5$, $\tau(S_a) = 0.25$, and $\tau(\max) = 0$.

Between t-norms and t-conorms there is the class of *averages*. In particular, the *Ordered Weighted Averaging* (OWA) [6] operator is defined as

$$M_{[w]}(a_1, \ldots, a_n) = \sum_{i=1}^{n} w_i a_{(i)} \tag{12}$$

where $a_{(1)}, \ldots, a_{(n)}$ is a non-increasing permutation of arguments $a_1, \ldots, a_n$. The OWA operators are parametric with respect to weights $w_i \in [0,1]$, such that $\sum_{i=1}^{n} w_i = 1$. When $w_1 = 1$ ($w_{i \neq 1} = 0$) we get $M_{[(1,\ldots,0)]}(a_1, \ldots, a_n) = \max_{i=1..n} a_i$; whilst $w_n = 1$ ($w_{i \neq n} = 0$) entails $M_{[(0,\ldots,1)]}(a_1, \ldots, a_n) = \min_{i=1..n} a_i$. When $w_i = 1/n$, then $M_{[(1/n,\ldots,1/n)]}(a_1, \ldots, a_n) = AM(a_1, \ldots, a_n)$, that is the arithmetic mean of arguments. Therefore the OWA operators are compensatory: they build an ideal bridge between the boundary of t-norms (min) and the boundary of t-conorms (max), made of a continuum of operators, depending on the aggregation weights. A measure of distance from t-norms is provided by the *attitudinal character*, that is a measure of weights defined as

$$AC(w) = \sum_{i=1}^{n} w_i \frac{n-i}{n-1} \in [0,1] \tag{13}$$

It results

$$AC(w) = \begin{cases} 1 & w_1 = 1, w_{i \neq 1} = 0 \\ 0.5 & w_i = 1/n \\ 0 & w_n = 1, w_{i \neq n} = 0 \end{cases} \tag{14}$$

Therefore the attitudinal character allows to locate the OWA operators between the t-norms and t-conorms, with respect to the weight vector. In reason of that the attitudinal character is also known as *orness*. The attitudinal character can be itself computed by OWA operators.

Proposition 3.

$$AC(w) = M_{[w]}(1, \tfrac{n-2}{n-1}, \ldots, \tfrac{1}{n-1}, 0) \tag{15}$$

2 The multicriteria decision making setting

In multicriteria decision making problems, we have a set of alternatives X which must satisfy a collection of criteria $\mathcal{A}$. Each alternative $x_j \in X$ satisfies the criterion $\mathcal{A}_i \in \mathcal{A}$ by a degree s_{ij}. Given two alternatives x_j and x_k, we will prefer x_j to x_k if

$$s_{ij} \geq s_{ik} \forall i \tag{16}$$

as x_j *dominates* x_k with respect to each criterion $\mathcal{A}_i$. If condition (16) is not verified, the decision maker is in face of deciding which alternative to choose. A method consists in defining a decision function $D : X \to [0, 1]$ by which to rank alternatives, so that x_j is preferred to x_k if

$$D(x_j) \geq D(x_k) \tag{17}$$

There are countless ways of building the decision function $D(\cdot)$. In this paper we follow the approach proposed in [9, 4, 5] and inspired to the pioneering work of Bellman and Zadeh [1], that describes a situation in which the decision maker chooses an alternative if *it satisfies at least h criteria*. Let us consider the alternative x_j in isolation, and let be $a_i = s_{ij}$. If the number of criteria h is precisely defined and the alternative x_j fully satisfies or does not satisfy at all each criterion ($a_i \in \{0, 1\}$), we can assume

$$D_h(x_j) = \begin{cases} 1 & \text{if } \mathrm{count}(a_i = 1) \geq h \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

Alternative ways of writing (18) are

$$D_h(x_j) = \min\{a_{(1)}, \ldots, a_{(h)}\} \tag{19}$$

$$D_h(x_j) = \max\{a_{(h)}, \ldots, a_{(n)}\} \tag{20}$$

where $\{a_{(1)}, \ldots, a_{(h)}\}$ are the first h highest arguments, and $\{a_{(h)}, \ldots, a_{(n)}\}$ the $n - h + 1$ lowest arguments in $\{a_1, \ldots, a_n\}$. In both cases, it is $D_h(x_j) = a_{(h)}$, that is the overall satisfaction with h required criteria. Now, assume that the number of required criteria h is not precisely defined within the set of possible values $\{1, \ldots, n\}$. In particular let us suppose that h is a fuzzy number whose degree of truth at $i \in \{1, \ldots, n\}$ is $h(i)$. Since there are n possible values for h, at the linguistic level we can state that

$$\begin{aligned} D(x_j) =& (D_1(x_j) \textbf{ given } h = 1) \textbf{ or} \\ & (D_2(x_j) \textbf{ given } h = 2) \textbf{ or} \\ & \ldots \\ & (D_n(x_j) \textbf{ given } h = n) \end{aligned} \tag{21}$$

If we adopt the Sugeno-Takagi's inference calculus [3], that is based on the interpolation of rule consequenses, we get

$$D(x_j) = \sum_{i=1}^{n} h(i) D_i(x_j) \Big/ \sum_{i=1}^{n} h(i) \tag{22}$$

Since $D_i(x_j) = a_{(i)}$, if we put

$$w_i = h(i) \Big/ \sum_{i=1}^{n} h(i) \tag{23}$$

then we get

$$D(x_j) = M_{[w]}(a_1, ..., a_n) = \sum_{i=1}^{n} w_i a_{(i)} \tag{24}$$

There are other ways of computing $D(x_j)$; for instance

$$D(x_j) = \max_{i=1..n} (\min\{h(i), D_i(x_j)\}) \tag{25}$$

3 The satisfaction of criteria

If we require h criteria to be satisfied, it means there exists at least a subset of cardinality h in which each criterion is satisfied. Let $\mathcal{H}_h \subseteq 2^{|\mathcal{A}|}$ be the collection of subsets with cardinality h. We get

$$D_h(x_j) = Sat(H_1) \textbf{ or } \ldots \textbf{ or } Sat(H_{m_h}) \tag{26}$$

where $H_v \in \mathcal{H}_h, v = 1..m_h$. In other terms, we require that among the m_h possible combinations of h over n criteria, there is at least one that satisfies all criteria, that is

$$Sat(H_v) = b_1 \textbf{ and } b_2 \textbf{ and } \ldots \textbf{ and } b_h \tag{27}$$

where $H_v = \{b_1, \ldots, b_h\} \subseteq \{a_1, \ldots, a_n\}$. In particular, $J_h = \{a_{(1)}, \ldots, a_{(h)}\} \in \mathcal{H}_h$.

Computation of (26) and (27) can be respectively performed by a t-conorm S_D and t-norm T_H, then

$$D_h(x_j) = S_D(Sat(H_1), \ldots, Sat(H_{m_h})) \tag{28}$$

and

$$Sat(H_v) = T_H(b_1, \ldots, b_h) \tag{29}$$

In particular, we can adopt $S_D \equiv \max$ and $T_H \equiv \min$, then we get (19). More in general, the monotonicity property of t-norms assures that

$$T_H(a_{(1)}, \ldots, a_{(h)}) \geq T_H(b_1, \ldots, b_h) \tag{30}$$

since $a_{(j)} \geq b_i, \forall i, j = 1..h$. Moreover

$$\begin{aligned} &\max(Sat(H_1), \ldots, Sat(H_{m_h})) \leq \\ &\quad S_D(Sat(H_1), \ldots, Sat(H_{m_h})) \ \forall S_D \end{aligned} \tag{31}$$

Therefore,

$$D_h(x_j) = T_H(a_{(1)}, \ldots, a_{(h)}) \quad (32)$$

is the minimum satisfaction value that D_h can assume. This looks as a rational choice in the context of decision making. However other choices are possible.

We can extend (24) as

$$D_T(x_j) = M_{T_H[w]}(a_1, \ldots, a_n) = \\ = \sum_{i=1}^{n} w_i T_H(a_{(1)}, \ldots, a_{(i)}) \quad (33)$$

where $M_{T_H[w]}(a_1, \ldots, a_n)$ is a T-OWA operator as discussed in [8].

If we require at least h criteria to be satisfied, this means that any subset $H'_{\overline{h}}$ of $\overline{h} = n-h+1$ criteria should have at least one criterion to be satisfied. Among them there is $K_h = \{a(h), \ldots, a(n)\}$.

$$D'_h(x_j) = Zat(H'_1) \textbf{ and } \ldots \textbf{ and } Zat(H'_{m_{\overline{h}}}) \quad (34)$$

where $H'_v = \{b'_1, \ldots, b'_{\overline{h}}\} \in \mathcal{H}_{\overline{h}}, v = 1..m_{\overline{h}}$ and

$$Zat(H'_v) = b'_1 \textbf{ or } \ldots \textbf{ or } b'_{\overline{h}} \quad (35)$$

For computing (34) and (35), we can respectively use a t-norm T_D and a t-conorm S_H, then

$$D'_h(x_j) = T_D(Zat(H'_1), \ldots, Zat(H'_{m_{\overline{h}}})) \quad (36)$$

and

$$Zat(H'_v) = S_H(b'_1, \ldots, b'_{\overline{h}}) \quad (37)$$

If we adopt $T_D \equiv \min$ and $S_H \equiv \max$, then we get (20). The monotonicity property of t-conorms entails

$$S_H(a_{(h)}, \ldots, a_{(n)}) \leq S_H(b'_1, \ldots, b'_{\overline{h}}) \quad (38)$$

Furthermore,

$$\min(Zat(H'_1), \ldots, Zat(H'_{m_{\overline{h}}})) \geq \\ T_D(Zat(H'_1), \ldots, Zat(H'_{m_{\overline{h}}})) \ \forall T_D \quad (39)$$

$$D'_h(x_j) = S_H(a_{(h)}, \ldots, a_{(n)}) \quad (40)$$

is the maximum satisfaction value that D'_h can assume. This choice leads us to extend (24) as

$$D_S(x_j) = M_{S_H[w]}(a_1, \ldots, a_n) = \\ \sum_{i=1}^{n} w_i S_H(a_{(i)}, \ldots, a_{(n)}) \quad (41)$$

where $M_{S_H[w]}(a_1, \ldots, a_n)$ is a S-OWA operator [8]. Also in this case, other choices are possible.

Proposition 4. *For any t-norm T, t-conorm S, and weight vector w*

$$M_{T[w]}(\cdot) \leq M_{[w]}(\cdot) \leq M_{S[w]}(\cdot) \quad (42)$$

Proof. The boundary condition of t-norms and t-conorms implies

$$T_H(J_i) = T_H(a_{(1)}, \ldots, a_{(i)}) \leq a_{(i)}$$

and

$$S_H(K_i) = S_H(a_{(i)}, \ldots, a_{(n)}) \geq a_{(i)}$$

Then

$$\sum_{i=1}^{n} w_i T_H(J_i) \leq \sum_{i=1}^{n} w_i a_{(i)} \leq \sum_{i=1}^{n} w_i S_H(K_i)$$

□

As discussed in [8], Prop.3 suggests a way for computing the attitudinal character of T-OWA operators as

$$AC(w, T) = \sum_{i=1}^{n} w_i T(1, \ldots, \tfrac{n-i}{n-1}) \quad (43)$$

In a similar way we can compute the attitudinal character of S-OWA operators as

$$AC(w, S) = \sum_{i=1}^{n} w_i S(\tfrac{n-i}{n-1}, \ldots, 0) \quad (44)$$

Proposition 5. *For any, weight vector w, if $T_1 \leq T_2$ ($S_1 \geq S_2$) then*

$$\begin{gathered} AC(w, T_1) \leq AC(w, T_2) \\ (AC(w, S_1) \geq AC(w, S_2)) \end{gathered} \quad (45)$$

Proposition 6. *For any t-norm T and t-conorm S,*

$$AC(w, T) \leq AC(w) \leq AC(w, S) \quad (46)$$

Prop.5 and 6 suggest that, given a weight vector w, T-OWA operators are more restrictive than usual OWA operators, while S-OWA operators are more tolerant. The degree by which they are more restrictive (tolerant) depends on the power of T (tolerance of S). Moreover, in accordance with usual OWA operators, we get

Proposition 7. *For, any choice of T and S, the maximum aggregated value is reached when $w_1 = 1$, so that*

$$AC(w,T) = AC(w,S) = AC(w) = 1 \tag{47}$$

$$M_{T[(0,\dots,1)]}(a_1,\dots,a_n) = \max_{i=1..n} a_i \tag{48}$$

$$M_{S[(0,\dots,1)]}(a_1,\dots,a_n) = S(a_1,\dots,a_n) \tag{49}$$

Vice versa, the minimum aggregated value is obtained when $w_n = 1$, so that

$$AC(w,T) = AC(w,S) = AC(w) = 0 \tag{50}$$

$$M_{T[(0,\dots,1)]}(a_1,\dots,a_n) = T(a_1,\dots,a_n) \tag{51}$$

$$M_{S[(0,\dots,1)]}(a_1,\dots,a_n) = \min_{i=1..n} a_i \tag{52}$$

At the middle point $w_i = 1/n$, so that

$$AC(w,T) \leq AC(w) = 1/2 \leq AC(w,S) \tag{53}$$

$$M_{T[(\frac{1}{n},\dots,\frac{1}{n})]}(a_1,\dots,a_n) \leq AM(a_1,\dots,a_n) \tag{54}$$

$$M_{S[(\frac{1}{n},\dots,\frac{1}{n})]}(a_1,\dots,a_n) \geq AM(a_1,\dots,a_n) \tag{55}$$

Proposition 8. *If a_i is boolean ($a_i \in \{0,1\}$), then*

$$M_{T[w]}(\cdot) = M_{[w]}(\cdot) = M_{S[w]}(\cdot) \tag{56}$$

for any choice of T and S.

Proof. If a_i is boolean then

$$T_H(J_h), S_H(K_h) = 0, 1$$

In particular,

$$T_H(a_{(1)},\dots,a_{(h)}) = 1 \Leftrightarrow a_{(h)} = 1 \Rightarrow S_H(K_h) = 1$$

In analogy,

$$S_H(a_{(1)},\dots,a_{(h)}) = 0 \Leftrightarrow a_{(h)} = 0 \Rightarrow T_H(J_h) = 0$$

Thus,

$$T_H(J_h) = S_H(K_h) = a_{(h)}$$

and

$$\sum_{i=1}^{n} w_i T_H(J_i) = \sum_{i=1}^{n} w_i a_{(i)} = \sum_{i=1}^{n} w_i S_H(K_i)$$

□

4 The dissatisfaction of criteria: duality

If a_i is a measure of satisfaction of one criterion, $1 - a_i$ is a measure of dissatisfaction. More in general, given a complement operator $c : [0,1] \to [0,1]$, $c(a_i)$ provides a degree of dissatisfaction. The problem of choosing the best alternative, can be regarded as the problem of choosing the alternative that less dissatisfies the criteria. This symmetry imposes to verify the *duality* of T-OWA and S-OWA operators, that is

$$\overline{D}_S(x_j) = c(D_T(x_j)) \tag{57}$$

$$\overline{D}_T(x_j) = c(D_S(x_j)) \tag{58}$$

where $\overline{D}_S$ ($\overline{D}_T$) is an aggregated measure of criteria dissatisfaction for the alternative x_j. The duality of OWA operators is defined as follows [7]

Definition 4. *An OWA operator $M_{[\hat{w}]}$ is dual of $M_{[w]}$ if weights are such that,*

$$\hat{w}_i = w_{n-i+1} \tag{59}$$

In this case, weight vectors $\hat{w}$ and w are also said to be dual.

We can easily see [7] that if $M_{[\hat{w}]}$ and $M_{[w]}$ are dual, then

$$AC(\hat{w}) = 1 - AC(w) \tag{60}$$

Moreover,

Proposition 9. *$M_{[\hat{w}]}$ is dual of $M_{[w]}$, iff*

$$M_{[\hat{w}]}(1-a_1,\dots,1-a_n) = 1 - M_{[w]}(a_1,\dots,a_n) \tag{61}$$

Proof. For any $a_1,\dots,a_n$,

$$\sum_{i=1}^{n} \hat{w}_i(1 - a_{(n-i+1)}) = 1 - \sum_{i=1}^{n} \hat{w}_i a_{(n-i+1)} =$$

$$\underset{j=n-i+1}{=} 1 - \sum_{j=1}^{n} \hat{w}_{n-j+1} a_{(j)} = 1 - \sum_{j=1}^{n} w_j a_{(j)}$$

if and only if $w_j = \hat{w}_{n-j+1}$. □

Therefore, the OWA operators $M_{[\hat{w}]}$ and $M_{[w]}$ are dual with respect to De Morgan's Laws when standard complement is chosen.

The definition of duality with respect to De Morgan's Laws is also applicable to t-norms and t-conorms [2].

Definition 5. *A t-norm $T(\cdot,\cdot)$ is dual to a t-conorm $S(\cdot,\cdot)$ by the complement $c(\cdot)$, iff*

$$T(c(x), c(y)) = c(S(x, y)) \tag{62}$$

$$S(c(x), c(y)) = c(T(x, y)) \tag{63}$$

Proposition 10. *For any dual pairs $< T_1, S_1 >$ and $< T_2, S_2 >$, such that $T_1 \geq T_2$ ($S_1 \geq S_2$), then $S_1 \leq S_2$ ($T_1 \leq T_2$).*

Proposition 11. *For any dual pair $< T, S >$ then $\pi(T) = \tau(S)$.*

The standard complement is $c(x) = 1 - x$. Well known pairs of dual t-norms and t-conorms by the standard complement are

$$\begin{array}{c} \langle \min, \max \rangle \\ \langle T_a, S_a \rangle \\ \langle T_b, S_b \rangle \\ \langle T_d, S_d \rangle \end{array} \tag{64}$$

Proposition 12. *A T-OWA operator $M_{T[w]}$ is dual to an S-OWA operator $M_{S[\hat{w}]}$, iff $\hat{w}$ and w are dual, and $< T, S >$ are dual by the standard complement.*

Proof. For any $a_1, \ldots, a_n$,

$$\sum_{i=1}^{n} w_i T(1 - a_{(n-i+1)}, ..., 1 - a_{(n)}) =$$

$$= 1 - \sum_{i=1}^{n} w_i S(a_{(n-i+1)}, ..., a_{(n)}) =$$

$$\underset{j=n-i+1}{=} 1 - \sum_{j=1}^{n} w_{n-j+1} S(a_{(1)}, ..., a_{(j)}) =$$

$$= 1 - \sum_{j=1}^{n} \hat{w}_j S(a_{(1)}, ..., a_{(j)})$$

□

Proposition 13. *For any T-OWA and S-OWA operators, such that T and S are dual, then*

$$AC(w, T) = 1 - AC(w, S) \tag{65}$$

Therefore, for any T-OWA (S-OWA) operator there is a dual S-OWA (T-OWA) operator. This is in accordance to the symmetry of satisfaction and dissatisfaction of criteria.

5 ST-OWA

Proposition 4 highlights that there are two ways of looking at satisfaction of h criteria, and they are not equivalent, as they denote a different semantics.

Using a t-conorm S_D (28) of t-norms T_H (29), we prefer alternatives that satisfy as much as possible the minimum number of criteria h, ignoring by which degree it satisfies more criteria. In particular, choosing $S_D \equiv \max$ we consider the most conservative measure, as the maximum operator is the strictest among t-conorms. Vice versa, using a t-norm T_D (36) of t-conorms S_H (37), we prefer alternatives that satisfy as many as possible criteria.

The choice depends on the number of minimum criteria h, thus on the attitudinal character. The higher h is, the less additional criteria there are; vice versa the lower h is, the more additional criteria there are. Moreover, it has been noticed [7] that the attitudinal character is a measure of the decision maker risk-adversion or severity. Both observations lead to consider a convex combination, as

$$M_{ST[w]}(a_1, ..., a_n) =$$

$$= \sum_{i=1}^{n} w_i((1 - \sigma)T(a_{(1)}, ..., a_{(i)}) + \sigma S(a_{(i)}, ..., a_{(n)})) \tag{66}$$

where $\sigma = AC(w)$ is the OWA attitudinal character provided by (13). When $\sigma = 0$, then

$$M_{ST[w]}(a_1, ..., a_n) = T(a_{(1)}, ..., a_{(n)})$$

that is the minimum aggregated value, whilst when $\sigma = 1$,

$$M_{ST[w]}(a_1, ..., a_n) = S(a_{(1)}, ..., a_{(n)})$$

that is the maximum aggregated value. Therefore, an ST-OWA operator varies in a wider range of values, as depicted by Fig.2. In analogy to T-OWA and S-OWA operators, the attitudinal character can be computed as

$$AC(w, S, T) = M_{ST[w]}(1, \tfrac{n-2}{n-1}, \ldots, 0) \tag{67}$$

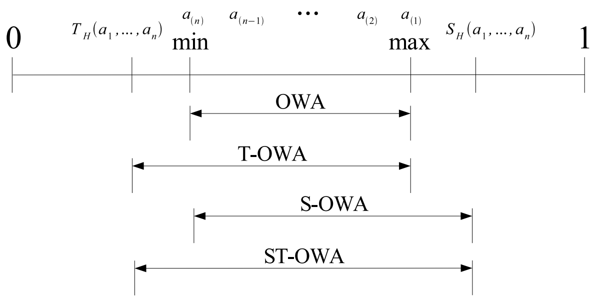

Figure 2: ST-OWA range of variation

Proposition 14. *For any choice of T and S, we get*

$$AC(w,S,T) = (1-\sigma)AC(w,T) + \sigma AC(w,S) \tag{68}$$

We can choose a dual pair $< T, S >$. In this case the attitudinal character is

$$\begin{aligned} AC(w,S,T) &= AC(w,T) + \sigma - 2\sigma AC(w,T) = \\ &= 1 - (AC(w,S) + \sigma - 2\sigma AC(w,S)) \end{aligned} \tag{69}$$

In particular,

$$AC(w,S,T) = \begin{cases} 0 & ,\sigma = 0 \\ 1/2 & ,\sigma = 1/2 \\ 1 & ,\sigma = 1 \end{cases} \tag{70}$$

If we also consider that in a dual pair $< T, S >$ $\pi(T) = \tau(S)$, we have that the variation range of ST-OWA operators is symmetric as in the case of ordinary OWA operators, as depicted in Fig.2. Moreover, similarly to ordinary OWA operators, we can easily prove the following proposition.

Proposition 15. *Given a pair of weight vectors $\hat{w}$ and w, if $\hat{w}$ is dual of w, then $M_{ST[\hat{w}]}$ and $M_{ST[w]}$ are dual.*

If the dual pair is $< \min, \max >$ we get the ST-OWA with the minimal variation range: it corresponds to the case of ordinary OWA operators. If the dual pair is $< T_d, S_d >$, we get the ST-OWA with maximum variation. In this case, it is easy to verify that

$$M_{S_d,T_d[w_i]}(a_1,\ldots,a_n) = \sum_{i=1}^{m} w_i \tag{71}$$

where m is the number of criteria that are fully satisfied ($a_i = 1$) by an alternative. For this reason we will term $M_{S_d,T_d[w_i]}$ as *drastic OWA operator*.

6 Conclusions

The deeper analysis of the structure of OWA operators suggests that it is possible to extend the definition of ordinary OWA operators, mixing the OWA operators with t-norms and t-conorms. The extension can be justified at the linguistic level, and it provides a means to introduce the reinforcement property provided by t-norms and t-conorms in the context of the OWA operators that are compensatory. In this paper, we presented three particular extensions: T-OWA, S-OWA and ST-OWA, discussing assumptions and properties. In particular, ST-OWA operators present a natural symmetry in analogy to ordinary OWA operators.

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