Stronger version of standard completeness theorem for MTL

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Abstract

As it was shown in [7], Monoidal t-norm based logic (MTL) is a logic of left-continuous t-norms. In other words, this means that MTL enjoys the standard completeness theorem. In this paper we present a different proof of this theorem. In fact, we prove even more since we show that MTL is complete w.r.t. the class of standard MTL-algebras with finite congruence lattices or equivalently with finitely many Archimedean classes.

Keywords: monoidal t-norm based logic, left-continuous t-norm, fuzzy logic.

1 Introduction

Monoidal t-norm based logic (MTL) was introduced by Esteva and Godo in [2] as a logic that should be a logic of left-continuous t-norms and their residua. The fact that this is really the case was proved later by Jenei and Montagna in [7], i.e., they proved that MTL satisfies the standard completeness theorem. In other words, they showed that MTL is complete w.r.t. the class of all standard MTL-algebras (MTL-algebras in the real interval $[0,1]$).

In this paper we are going to present an alternative proof of the standard completeness theorem for MTL. Moreover, our approach gives a stronger version of this theorem since we are able to show that MTL is complete w.r.t. the class of all standard MTL-algebras with finite congruence lattice or equivalently with finitely many Archimedean classes. In Section 2 we recall the definition of MTL and some basic results. Section 3 describes the connection between congruence lattice of an MTL-algebra and its Archimedean classes. Finally, Section 4 presents the proof of the standard completeness theorem.

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2 Monoidal t-norm based logic

The language of MTL consists of a countable set of propositional variables, a conjunction &, an implication $\Rightarrow$, the truth constant $\bar{0}$, and the minimum conjunction $\wedge$. Derived connectives are defined as follows:

$$\begin{array}{rll} \varphi \vee \psi & \text{is} & ((\varphi \Rightarrow \psi) \Rightarrow \psi) \wedge ((\psi \Rightarrow \varphi) \Rightarrow \varphi)\,, \\ \neg\varphi & \text{is} & \varphi \Rightarrow \bar{0}\,, \\ \varphi \equiv \psi & \text{is} & (\varphi \Rightarrow \psi) \& (\psi \Rightarrow \varphi)\,, \\ \bar{1} & \text{is} & \neg\bar{0}\,. \end{array}$$

In [2] the authors introduced a Hilbert style calculus for MTL with the following axiomatization:

$$\begin{array}{ll} \text{(A1)} & (\varphi \Rightarrow \psi) \Rightarrow ((\psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow \chi))\,, \\ \text{(A2)} & \varphi \& \psi \Rightarrow \varphi\,, \\ \text{(A3)} & \varphi \& \psi \Rightarrow \psi \& \varphi\,, \\ \text{(A4)} & (\varphi \wedge \psi) \Rightarrow \varphi\,, \\ \text{(A5)} & (\varphi \wedge \psi) \Rightarrow (\psi \wedge \varphi)\,, \\ \text{(A6)} & (\varphi \& (\varphi \Rightarrow \psi)) \Rightarrow (\varphi \wedge \psi)\,, \\ \text{(A7a)} & (\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow (\varphi \& \psi \Rightarrow \chi)\,, \\ \text{(A7b)} & (\varphi \& \psi \Rightarrow \chi) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \chi))\,, \\ \text{(A8)} & ((\varphi \Rightarrow \psi) \Rightarrow \chi) \Rightarrow \\ & \quad \Rightarrow (((\psi \Rightarrow \varphi) \Rightarrow \chi) \Rightarrow \chi)\,, \\ \text{(A9)} & \bar{0} \Rightarrow \varphi\,. \end{array}$$

The deduction rule of MTL is *modus ponens*.

Algebras of truth values for MTL are so-called MTL-algebras and they form a subvariety of bounded integral residuated lattices.

An algebraic structure $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is called *commutative residuated lattice* if $(L, *, \mathbf{1})$ is a commutative monoid, $(L, \wedge, \vee)$ is a lattice, and $(*, \rightarrow)$ form a residuated pair, i.e.,

$$x * y \leq z \text{ iff } x \leq y \rightarrow z\,.$$

The operation $\rightarrow$ is called a *residuum.* A residuated lattice is called *integral* if the neutral element $\mathbf{1}$ is also a top element.

Definition 2.1 An MTL-*algebra* is a structure $(L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ where the following conditions are satisfied:

1. $(L, *, \rightarrow, \wedge, \vee, \mathbf{1})$ is an integral residuated lattice,
2. $(L, \wedge, \vee, \mathbf{0}, \mathbf{1})$ is a bounded lattice,
3. $(x \rightarrow y) \vee (y \rightarrow x) = \mathbf{1}$ for all $x, y \in L$.

A totally ordered MTL-algebra is called an MTL-*chain.* An MTL-chain whose underlying set is the real interval $[0,1]$ with the usual order is referred to as a *standard* MTL-chain. Throughout the text we will use without mentioning also the alternative signature of an MTL-algebra where the lattice operations $\wedge$ and $\vee$ are substituted by the corresponding order $\leq$.

In [2, Theorem 2] Esteva and Godo proved that MTL is complete w.r.t. the class of all MTL-chains.

Theorem 2.2 (Completeness) *Let φ be a formula over* MTL. *Then* MTL $\vdash \varphi$ *iff φ is a tautology over each* MTL-*chain.*

As we mentioned in the introduction, the fact that MTL is complete w.r.t. the class of standard MTL-chains was proved by Jenei and Montagna in [7].

Theorem 2.3 (Standard Completeness) *Let φ be a formula over* MTL. *Then* MTL $\vdash \varphi$ *iff φ is a tautology over each standard* MTL-*chain.*

3 Archimedean classes and congruences

In this section we introduce the notion of an Archimedean class in an MTL-chain and show how it is related to the congruence lattice of this MTL-chain. Firstly, let us recall how the congruences are characterized by means of filters.

Definition 3.1 Let $\mathbf{L} = (L, *, \rightarrow, \wedge, \vee, \mathbf{0}, \mathbf{1})$ be an MTL-algebra. A *filter* F in $\mathbf{L}$ is a subset of L satisfying:

1. if $x, y \in F$, then $x * y \in F$,
2. if $x \in F$, $x \leq y$, then $y \in F$.

Throughout the paper the collection of all filters of an MTL-algebra $\mathbf{L}$ will be denoted by $\mathcal{F}_{\mathbf{L}}$. It was shown in [2] that the congruence lattice of $\mathbf{L}$ (Con $\mathbf{L}$) is isomorphic to $\mathcal{F}_{\mathbf{L}}$. The isomorphism is given by the mutually inverse maps:

$$F \mapsto \theta_F = \{\langle a, b\rangle \mid a \rightarrow b \in F \text{ and } b \rightarrow a \in F\}\,,$$

$$\theta \mapsto [\mathbf{1}]_\theta\,.$$

Thus we can arbitrarily work either with Con $\mathbf{L}$ or with $\mathcal{F}_{\mathbf{L}}$.

Lemma 3.2 *Let* $\mathbf{L}$ *be an* MTL-*chain. Then any union of filters of* $\mathbf{L}$ *is again a filter.*

The next trivial result characterizes the *principal* filters, i.e., the filters generated by a single element. A principal filter F generated by b is denoted by F^b. The set of all principal filters of an MTL-chain $\mathbf{L}$ will be denoted by $\mathcal{P}_{\mathbf{L}}$.

Lemma 3.3 *Let* $\mathbf{L}$ *be an* MTL-*chain and $b \in L$. Then the principal filter generated by b is of the form:*

$$F^b = \{z \in L \mid (\exists n \in \mathbb{N})(b^n \leq z)\}\,,$$

where $b^n = b * \cdots * b$ (n-times).

Let $\mathbf{L}$ be an MTL-chain. Observe, that each filter F in $\mathbf{L}$ is a union of principal filters since $F = \bigcup_{b \in F} F^b$. Moreover, if Con $\mathbf{L}$ is finite (i.e., $\mathbf{L}$ has only finite number of filters) then all filters are principal. Indeed, as the collection of all filters forms a chain, we get $F = \bigcup_{b \in F} F^b = F^c$ for some $c \in F$.

Let F^b be a principal filter. Then by Lemma 3.2 the union of all filters not containing b is a filter. Clearly, it is the largest filter not containing b. Thus we obtain the following lemma.

Lemma 3.4 *Let* $\mathbf{L}$ *be an MTL-chain. Then each principal filter in* $\mathbf{L}$ *has a predecessor in* $\mathcal{F}_{\mathbf{L}}$.

We will denote the predecessor of F^b by F_b.

The notion of an Archimedean class comes from the theory of ℓ-groups and ℓ-monoids (see e.g.[4]).

Definition 3.5 Let $\mathbf{L}$ be an MTL-chain, a, b be elements of L, and $\sim$ be an equivalence on L defined as follows:

$$a \sim b \text{ iff there exists an } n \in \mathbb{N} \text{ such that } a^n \leq b \leq a \text{ or } b^n \leq a \leq b.$$

Then for any $a \in L$ the equivalence class $[a]_\sim$ is called an *Archimedean class.*

Archimedean classes correspond to the subsets of L where the elements behave like in an Archimedean ℓ-monoid, i.e., for any pair of elements $x, y \in [a]_\sim$, such that $x \leq y$, there is an $n \in \mathbb{N}$ such that $y^n \leq x$.

Further, we list several easy results about Archimedean classes.

Lemma 3.6 *Let* $\mathbf{L}$ *be an* MTL*-chain and* $a, b \in L$*. Then the following holds:*

1. $[a]_\sim$ *is closed under* $*$.
2. $[a]_\sim$ *is an interval.*
3. $[a * b]_\sim = [a \wedge b]_\sim$.

Note that $L/\sim$ is totally ordered thanks to Lemma 3.6(2). We have $[a]_\sim < [b]_\sim$ if $a \notin [b]_\sim$ and $a < b$. The chain of all Archimedean classes of $\mathbf{L}$ is denoted by $\mathcal{C}_{\mathbf{L}}$.

As we mentioned at the beginning, the Archimedean classes are related to the congruence lattice. This connection is described by the next theorem.

Theorem 3.7 *Let* $(\mathcal{C}_{\mathbf{L}}, \leq)$ *be the chain of all Archimedean classes of an* MTL*-chain* $\mathbf{L}$*. Then the dual chain of* $\mathcal{C}_{\mathbf{L}}$ *is isomorphic to the chain of all principal filters* $\mathcal{P}_{\mathbf{L}}$*. Let* $C \in \mathcal{C}_{\mathbf{L}}$*. The order-isomorphism* $\phi : \mathcal{C}_{\mathbf{L}} \to \mathcal{P}_{\mathbf{L}}$ *is defined as follows:*

$$\phi(C) = F^b\,, \text{ for some } b \in C\,.$$

It can be shown that the inverse of isomorphism between $\mathcal{C}_{\mathbf{L}}$ and $\mathcal{P}_{\mathbf{L}}$ from Theorem 3.7 is $\phi^{-1}(F^b) = F^b - F_b$ where F_b is the predecessor of F^b.

As a corollary of the previous theorem, we obtain the following statement.

Corollary 3.8 *An* MTL*-chain* $\mathbf{L}$ *has a finite number of Archimedean classes iff* Con $\mathbf{L}$ *is finite.*

4 Standard completeness theorem

Now we are going to prove Standard Completeness Theorem for MTL, i.e., we want to show that a formula φ is provable in MTL iff φ is a tautology over each standard MTL-chain. The proof given here is a modified version of the proof of the standard completeness theorem for ΠMTL which we presented in [5, 6]. It is clear that one direction of this statement follows already from Theorem 2.2. Thus, we will start with a formula φ which is not valid in an MTL-chain $\mathbf{L}$. Then we construct a new MTL-chain $\mathbf{S}$ such that φ is not valid in $\mathbf{S}$, too, and $\mathbf{S}$ has a more transparent structure. Further, we extend $\mathbf{S}$ to a continuum. Finally, we will show that this extension of $\mathbf{S}$ is order-isomorphic to $[0, 1]$.

We know by Theorem 2.2 that whenever φ is not provable in MTL then there exists an MTL-chain $\mathbf{L} = (L, *_L, \to_L, \leq, \mathbf{0}, \mathbf{1})$ and an $\mathbf{L}$-evaluation $e_{\mathbf{L}}$ such that $e_{\mathbf{L}}(\varphi) < \mathbf{1}$. Let us denote the set of all subformulas of φ by B. Since B is finite, we can assume that $B = \{\psi_1, \ldots, \psi_n\}$. Then let us define the following set:

$$G = \{a_i \in L \mid a_i = e_{\mathbf{L}}(\psi_i),\ \psi_i \in B,\ 1 \leq i \leq n\}\,.$$

Let $\mathbf{S}$ be the submonoid of $\mathbf{L}$ generated by G, i.e. $\mathbf{S} = (S, *, \leq, \mathbf{0}, \mathbf{1})$, where

$$S = \{a_1^{k_1} *_L \cdots *_L a_n^{k_n} \mid a_i \in G,\ k_i \in \mathbb{N}\} \cup \{\mathbf{0}, \mathbf{1}\}\,,$$

and $*$ denotes the restriction of $*_L$ to S.

The proof of the following lemma is based on Dickson's lemma stating that there are only finitely many minimal elements in any subset of $(\mathbb{N}, \leq)^n$.

Lemma 4.1 *The monoid* $\mathbf{S}$ *is countable and inversely well ordered (i.w.o.), i.e., each subset of* S *has a maximum.*

Due to Lemma 4.1, $\mathbf{S}$ is i.w.o. and we can introduce a residuum on $\mathbf{S}$ as follows:

$$a \to b = \max\{z \in S \mid a * z \leq b\}\,.$$

Theorem 4.2 *The enriched monoid* $\mathbf{S} = (S, *, \to, \leq, \mathbf{0}, \mathbf{1})$ *is a* MTL*-chain and there exists an* $\mathbf{S}$*-evaluation* $e_{\mathbf{S}}$ *such that* $e_{\mathbf{S}}(\varphi) = e_{\mathbf{L}}(\varphi)$.

Note that $\mathbf{S}$ need not be a subalgebra of $\mathbf{L}$ since $\mathbf{S}$ arises only from a submonoid of $\mathbf{L}$. However, the existence of the evaluation $e_{\mathbf{S}}$ such that $e_{\mathbf{S}}(\varphi) < \mathbf{1}$ is sufficient for us.

Since $\mathbf{S}$ is finitely generated using only $*$, there must be only finitely many Archimedean classes in $\mathbf{S}$ by Lemma 3.6(3).

Lemma 4.3 *There are only finitely many Archimedean classes in* $\mathbf{S}$.

Now we have the MTL-chain $\mathbf{S}$ which is countable and the evaluation $e_{\mathbf{S}}$ such that $e_{\mathbf{S}}(\varphi) < \mathbf{1}$. The next step is to build a new MTL-chain $\mathbf{S}'$ order-isomorphic to $[0, 1]$ in which $\mathbf{S}$ can be embedded.

At this point we could use the same construction as in the proof of the standard completeness theorem for MTL from [7, Theorem 3.1]. In that paper the authors take for the non-provable formula φ the MTL-chain $\mathbf{L}$ such that φ is not valid in $\mathbf{L}$. Since φ is finite, they suppose that $\mathbf{L}$ is finitely generated and hence countable. Then they extend the ℓ-monoid reduct of $\mathbf{L}$ so that it is dense. Moreover, the operation in the extended monoid is left-continuous. Nevertheless, they cannot prove that this extension is a residuated lattice since it need not be complete and thus residua need not exist.

In our approach we can do that since we are extending the MTL-chain $\mathbf{S}$ which is i.w.o. and hence complete. However, the extension of $\mathbf{S}$ by this method is not suitable for us because the extended chain possesses infinitely many Achimedean classes. Consequently, we would not obtain the stronger version of the standard completeness theorem. Thus, we have to proceed in a different way.

The new universe is defined as follows:

$$S' = \{\langle a, x\rangle \mid a \in S - \{\mathbf{0}\},\ x \in (0, 1]\} \cup \{\langle \mathbf{0}, 1\rangle\}\,.$$

The order $\leq'$ on S' is lexicographic, i.e.,

$$\langle a, x\rangle \leq' \langle b, y\rangle \text{ iff } a \leq b \text{ or } [a = b \text{ and } x \leq y]\,.$$

The monoid operation is defined by the following formula:

$$\langle a, x\rangle *' \langle b, y\rangle = \begin{cases} \langle a * b, 1\rangle \text{ if } a * b < \min\{a, b\}\,, \\ \langle a, xy\rangle \text{ if } a = b \text{ and } a^2 = a\,, \\ \min\{\langle a, x\rangle, \langle b, y\rangle\} \text{ otherwise}\,. \end{cases}$$

Lemma 4.4 $\mathbf{S}' = (S', *', \leq', \langle \mathbf{0}, 1\rangle, \langle \mathbf{1}, 1\rangle)$ *forms an integral totally ordered monoid, where* $\langle \mathbf{1}, 1\rangle$ *is the neutral element and the top element as well,* $\langle \mathbf{0}, 1\rangle$ *is the bottom element, and* $*'$ *is monotone w.r.t.* $\leq'$*, i.e.,* $\langle a, x\rangle \leq' \langle b, y\rangle$ *implies* $\langle a, x\rangle *' \langle c, z\rangle \leq' \langle b, y\rangle *' \langle c, z\rangle$*. Moreover,* $*'$ *is left-continuous w.r.t. the order topology on* $(S', \leq')$.

Since $*'$ is left-continuous and in addition S' is a complete lattice, the residuum in $\mathbf{S}'$ always exists, i.e.,

$$\langle a, x\rangle \to' \langle b, y\rangle = \\ = \max\{\langle c, z\rangle \mid \langle a, x\rangle *' \langle c, z\rangle \leq' \langle b, y\rangle\}\,.$$

Thus $\mathbf{S}' = (S', *', \to', \leq', \langle \mathbf{0}, 1\rangle, \langle \mathbf{1}, 1\rangle)$ is an MTL-chain.

Finally, the mapping $\Psi : S \to S'$ defined by $\Psi(x) = \langle x, 1\rangle$ is an MTL-homomorphism since it satisfies the following equalities:

$$\Psi(x*y) = \langle x*y, 1\rangle = \langle x, 1\rangle *' \langle y, 1\rangle = \Psi(x) *' \Psi(y)\,,$$

and

$$\begin{aligned} \Psi(x \to y) &= \langle x \to y, 1\rangle = \langle x, 1\rangle \to' \langle y, 1\rangle \\ &= \Psi(x) \to' \Psi(y)\,. \end{aligned}$$

Moreover, Ψ obviously preserves the order, i.e., $x \leq y$ implies $\Psi(x) \leq' \Psi(y)$.

The remaining step is to find an order-isomorphism $\Phi : S' \to [0,1]$. Since S is countable and has a minimum and a maximum, there exists an order-preserving mapping $\nu' : S \to \mathbb{Q} \cap [0,1]$ such that $\nu'(\mathbf{0}) = 0$ and $\nu'(\mathbf{1}) = 1$. Moreover, as S is i.w.o., for each $a \in S - \{\mathbf{0}\}$ there is a predecessor a^- of a. We want to map each set of the form $\{\langle a, x\rangle \mid x \in (0,1]\}$ on the interval $(\nu'(a^-), \nu'(a)]$. However, if we do that immediately, the resulting mapping from S' to $[0,1]$ need not be onto. Thus we have to firstly change the values of ν' a little bit.

As S is i.w.o., the elements of S can be indexed by ordinals. We have to fix the values of ν' on the limit ordinals. Let us define $\nu : S \to [0,1]$ as follows:

$$\nu(s_\alpha) = \begin{cases} \nu'(s_\alpha) & \text{if } \alpha = \beta + 1\,, \\ \bigwedge_{\beta<\alpha} \nu'(s_\beta) & \text{if } \alpha \text{ is a limit ordinal}\,. \end{cases}$$

Now we can define a mapping Φ by the following formulas:

$$\begin{aligned} \Phi(\mathbf{0}, 1) &= 0\,, \\ \Phi(a, x) &= \nu(a^-) + (\nu(a) - \nu(a^-))x\,. \end{aligned}$$

Since ν is order-preserving and the elements of S' are lexicographically ordered, we obtain the following result.

Theorem 4.5 *The mapping Φ is an order-isomorphism between S' and the real interval $[0,1]$.*

Finally, we define the operations in $[0,1]$ as usual:

$$a \odot b = \Phi(\Phi^{-1}(a) *' \Phi^{-1}(b))\,,$$

$$a \to_\odot b = \Phi(\Phi^{-1}(a) \to' \Phi^{-1}(b))\,.$$

Then $[\mathbf{0},\mathbf{1}]_\odot = ([0,1], \odot, \to_\odot, \leq, 0, 1)$ is an MTL-chain and $[\mathbf{0},\mathbf{1}]_\odot \not\models \varphi$, i.e.,

$$\Phi(\Psi(e_{\mathbf{S}}(\varphi))) < 1\,.$$

Thus the proof of Standard Completeness Theorem is done.

Theorem 4.6 (Standard Completeness) *A formula φ is provable in* MTL *if and only if φ is a tautology in all standard* MTL-*chains.*

Since **S** has finitely many Archimedean classes by Lemma 4.3, we can a little bit strengthen the latter theorem. It can be shown that by the construction of $\mathbf{S}'$ the number of Archimedean classes cannot become infinite.

Lemma 4.7 *Let k be the number of Archimedean classes of* **S**. *Then the number of Archimedean classes in $\mathbf{S}'$ is less than $2k$.*

Lemma 4.7 together with Corollary 3.8 gives us the following version of Standard Completeness Theorem.

Theorem 4.8 *Let φ be an* MTL *formula. Then the following are equivalent:*

1. MTL $\vdash \varphi$.
2. *φ is a tautology in all standard* MTL-*chains with finitely many Archimedean classes.*
3. *φ is a tautology in all standard* MTL-*chains with finite congruence lattice.*

5 Conclusions and remarks

We present an alternative proof of the standard completeness theorem for MTL. The proof uses a different construction which is interesting on its own. Moreover, it helps us to strengthen this theorem and show that MTL is complete w.r.t. the class of all standard MTL-chains with finite congruence lattice.

We should also mention that the construction of the MTL-chain **S** from Theorem 4.2 can be replaced by the construction published in [1, Section 5]. In this case we would obtain not only a countable and i.w.o. MTL-chain but even a finite one. However, the construction is more complicated.

As a possible future task it would be interesting whether our proof can be generalized also for other schematic extension of MTL similarly as the proof of Jenei and Montagna which was generalized for several schematic extensions of MTL in [3]. It can be easily checked that our proof can be used also for SMTL without any change. A modification for ΠMTL was presented in [5, 6].

However, it seems that there is no straightforward generalization for IMTL.

Due to lack of the space we have presented only main ideas and skipped the technical details. The full proofs will be published in a journal article.

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