Algebraic Transfer Principle in Fuzzy Theory

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Abstract

We show that almost all results proved in many papers about fuzzy algebras can be proved uniformly and immediately by using so-called "Transfer Principle".
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1 Introduction

There are many papers about fuzzification of algebras so far ([1, 2, 7, 8, 9, 10, 11]). But almost all such results are extensions of those of the crisp theory which can be divided into the following four types:

type 0 : A subset A has a property P;

type 1 : If a subset A has a property P, then it has a property Q;

type 2 : Let $f : X \to Y$ be a homomorphism. If a subset B of Y has a property Q, then a subset $f^{-1}(B)$ of X has a property P;

type 3 : Let $f : X \to Y$ be a surjective homomorphism. If a subset A of X has a property P, then a subset $f(A)$ of Y has a property Q.

These results are obtained uniformly and immediately by using the *transfer principle* ([3, 6]). The area to which the principle can be applied is restricted to the property which are denoted by a certain formula described below, but almost all results obtained so far can be represented by such formulas. Thus, our principle is a powerful tool to investigate the fuzzy theory of algebras.

2 Algebras and Terms

To present our theorem accurately, we define terms on the algebras. A structure $(X; \omega_i)_{i \leq k}$ is called an algebra if X is a non-empty set and ω_i is an n_i-ary operation on X. A non-negative interger n_i called an *arity* corresponds to each operation ω_i. A sequence $(n_1, n_2, \cdots, n_k)$ of arities $(n_1 \geq n_2 \geq \cdots \geq n_k)$ is called a *type* of the algebra X. For two algebras $(X; \omega_i)_{i \leq k}$ and $(Y; \xi_j)_{j \leq l}$, they said to be similar if their types are identical.

Let $\mathfrak{A}$ be a class of similar algebras and $V = \{x, y, \cdots, \}$ be a countable set of variables. A *term* on an algebra $(X; \omega_i)_{i \leq k} \in \mathfrak{A}$ is defined as follows:

(t1) Each variable $x \in V$ is a term;

(t2) If $u_1, \cdots, u_i$ are terms and ω_i is an n_i-ary oeration, then so $\omega_i(u_1, \cdots, u_i)$ is.

A term $t(x, \cdots, y)$ is interpreted on X as follows:

(Int1) Each variable x is interpreted as an element of X, e.g., $a \in X$.

(Int2) If terms $u_1, \cdots, u_i$ are interpreted as $a_1, \cdots, a_i \in X$ respectively, then a term $\omega_i(u_1, \cdots, u_i)$ is done by $\omega_i(a_1, \cdots, a_i) \in X$.

For the sake of simplicity, we identify a term with the term which is interpreted on some algebra X.

Let $X \in \mathfrak{A}$. For every subset $A \subseteq X$, we define a first-order formula

$\mathcal{P}_A$: $\forall x \cdots \forall y (t_1(x, \cdots, y) \in A \wedge \cdots \wedge t_n(x, \cdots, y) \in A \to t(x, \cdots, y) \in A)$, where $t_i(x, \cdots, y)$ $(1 \le i \le n)$ and $t(x, \cdots, y)$ are terms which are constructed by variables $x, \cdots, y$. We say that a subset A satisfies the formula $\mathcal{P}_A$ if all $t_i(a, \cdots, b) \in A$ $(1 \le i \le n)$ imply $t(a, \cdots, b) \in A$ for every element $a, \cdots, b \in X$. The formula $\mathcal{P}_A$ represents a property of A. In the rest of paper, we use two statements "property $\mathcal{P}_A$" and "formula $\mathcal{P}_A$" with the same meaning.

Let $(X; \circ, ^{-1}, e)$ be a group. For every non-empty subset $A \subseteq X$, the formula

$$\mathcal{P}_A : \ \forall x \forall y (x \in A \wedge y \in A \rightarrow x \circ y^{-1} \in A)$$

means that A is a subgroup of X. We note that non-emptyness is equivalent to a formula

$$\forall x (x \in A \rightarrow 0 \in A).$$

Thus we can redefine the concept of subgroup by

$$\forall x (x \in A \rightarrow 0 \in A) \text{ and}$$
$$\forall x \forall y (x \in A \wedge y \in A \rightarrow x \circ y^{-1} \in A).$$

For every subset $A \subseteq X$, it is called a **$\mathcal{P}$-set** if it satisfies the formula $\mathcal{P}_A$.

Next we define a *fuzzy theory* $\bar{\mathfrak{A}}$ of $\mathfrak{A}$. For every algebra $X \in \mathfrak{A}$, a map $\mu : X \rightarrow [0,1]$ is called a *fuzzy subset* of X. The class of all fuzzy subsets of $X \in \mathfrak{A}$ is said to be a fuzzy theory of $\mathfrak{A}$ and denoted by $\bar{\mathfrak{A}}$. For every fuzzy subset μ of $X \in \mathfrak{A}$, we define a formula $\mathcal{P}_\mu$ by $\forall x \cdots \forall y (\mu(t_1(x, \cdots, y)) \wedge \cdots \wedge \mu(t_n(x, \cdots, y)) \le \mu(t(x, \cdots, y))$.

Similarly to the case of crisp theory, the formula $\mathcal{P}_\mu$ represents a property of the fuzzy subset μ. We say that μ satisfies $\mathcal{P}_\mu$ whenever

$$\mu(t_1(a, \cdots, b)) \wedge \cdots \wedge \mu(t_n(a, \cdots, b)) \le \mu(t(a, \cdots, b))$$

for all elements $a, \cdots, b \in X$.

In the theory of groups, for example, this means that, for every fuzzy subset μ of X, a *fuzzy subgroup* μ of X is defined by

$$\mu(x) \le \mu(0) \text{ and } \mu(x) \wedge \mu(y) \le \mu(x \circ y^{-1}).$$

For every fuzzy subset μ of X, we call μ a ***fuzzy $\mathcal{P}$-set*** if it satisfies the formula $\mathcal{P}_\mu$. For $0 \le \alpha \le 1$, we put

$$U(\mu; \alpha) = \{a \in X | \mu(a) \ge \alpha\}.$$

3 Transfer Principle

In this section we prove the general and fundamental theorems by use of the transfer principle defined below. At first we consider the basic form of a transfer principle. Since this type is basic to develop our theory, we call it **type 0**. In the rest of paper, let $\mathfrak{A}$ be an arbitrary algebra.

Theorem 1 (Transfer Principle (type 0)). *Let $(X; \omega_i) \in \mathfrak{A}$. For every fuzzy subset μ of X, μ is a fuzzy $\mathcal{P}$-set if and only if $U(\mu; \alpha)$ is a $\mathcal{P}$-set provided that $U(\mu; \alpha) \neq \emptyset$ for all $\alpha \in [0,1]$.*

Proof. $(\Longrightarrow)$ Suppose that $U(\mu; \alpha)$ is not a $\mathcal{P}$-set for some $\alpha \in [0,1]$. There are elements $a, \cdots, b \in X$ such that $t_i(a, \cdots, b) \in U(\mu; \alpha)$ but $t(a, \cdots, b) \notin U(\mu; \alpha)$. Since, $\mu(t_i(a, \cdots, b)) \ge \alpha$ but $\mu(t(a, \cdots, b)) \not\ge \alpha$, we have $\mu(t(a, \cdots, b)) \not\ge \bigwedge_i \mu(t_i(a, \cdots, b))$. This means that μ is not the fuzzy $\mathcal{P}$-set.

$(\Longleftarrow)$ Conversely, assume that μ is not a fuzzy $\mathcal{P}$-set. There exist $a, \cdots, b \in X$ such that $\mu(t(a, \cdots, b)) \not\ge \bigwedge_i \mu(t_i(a, \cdots, b))$. If we take $\alpha = \bigwedge_i \mu(t_i(a, \cdots, b))$, then we have $\alpha \in [0,1]$, $U(\mu; \alpha) \neq \phi$ and $t_i(a, \cdots, b) \in U(\mu; \alpha)$, $t(a, \cdots, b) \notin U(\mu; \alpha)$. This indicates that $U(\mu; \alpha)$ is not the $\mathcal{P}$-set. □

Remark : The transfer principle means that it is sufficient to ckeck whether a set $U(\mu; \alpha) \neq \emptyset$ satisfies the property $\mathcal{P}$ for all $\alpha \in [0,1]$ if we want to know whether a fuzzy subset μ satisfies a property fuzzy $\mathcal{P}$. Hence we can show the property fuzzy $\mathcal{P}$ of a fuzzy subset μ in the crisp theory of algebras. Moreover we note that the interpretation of the symbol $\wedge$ in the formula $\mathcal{P}$ have to be obeyed to the usual crisp (or classical) sense, that is, it is interpreted by *min*, because, the word "and" in the property described by $\mathcal{P}$ is not in the object logic, i.e., fuzzy logic, but in the meta-logic, i.e., classical logic. Thus it is interpreted in the classical sense. We do not confuse the meta-logic with the object logic.

4 Application to type 1

We apply the transfer principle to statements of **type 1** and prove a general theorem of that type. The statement of **type 1** has a form:

For every subset A of X, if A has a property $\mathcal{P}$ (or A is a $\mathcal{P}$-set), then it has a property $\mathcal{Q}$ (or it is a $\mathcal{Q}$ set).

We denote the statement simply by

$$\mathfrak{A} \models (X : A) : \mathcal{P} \Longrightarrow (X : A) : \mathcal{Q}.$$

If the statement $\mathfrak{A} \models (X : A) : \mathcal{P} \Longrightarrow (X : A) : \mathcal{Q}$ holds for every $X \in \mathfrak{A}$ and $A \subseteq X$, we denote it by

$$\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}.$$

Hence the two formal statements

For every $X \in \mathfrak{A}$ and $A \subseteq X$, $\mathfrak{A} \models (X : A) : \mathcal{P} \Longrightarrow (X : A) : \mathcal{Q}$ holds.

and $\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}$ have the same meaning.

The statement can be extended to the following in the fuzzy theory of algebras:

For every fuzzy subset μ of X, if μ has a property $\mathcal{P}_\mu$ (or μ is a fuzzy $\mathcal{P}$-set) , then it has a property $\mathcal{Q}_\mu$ (or it is a fuzzy $\mathcal{Q}$-set).

We also denote the statement simply by

$$\bar{\mathfrak{A}} \models (X : \mu) : \text{fuzzy}\mathcal{P} \Longrightarrow (X : \mu) : \text{fuzzy}\mathcal{Q}.$$

If the statement $\bar{\mathfrak{A}} \models (X : \mu) : \text{fuzzy}\mathcal{P} \Longrightarrow (X : \mu) : \text{fuzzy}\mathcal{Q}$ holds for every $X \in \mathfrak{A}$ and fuzzy subset μ of X, then we denote it by

$$\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}.$$

Thus as in the case of above there is no difference between the statement

For every $X \in \mathfrak{A}$ and μ, we have $\bar{\mathfrak{A}} \models (X : \mu) : \text{fuzzy}\mathcal{P} \Longrightarrow (X : \mu) : \text{fuzzy}\mathcal{Q}$

and the statement $\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}$.

We have an example of the statements of **type 1**. In the theory of groups, if A is a normal subgroup then it is a subgroup.

A general theorem concerning to the statements of **type 1** is represented as follows:

Theorem 2. *If* $\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}$ *then* $\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}$

Proof. Suppose $\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}$. Let $(X; \omega_i)$ be an arbitrary algebra in $\mathfrak{A}$ and μ a fuzzy subset of X. If μ has a property $\mathcal{P}_\mu$ (i.e., μ is a fuzzy $\mathcal{P}$-set), then it follows from transfer principle that $U(\mu; \alpha) \subseteq X$ is a $\mathcal{P}$-set for all $\alpha \in [0,1]$ such that $U(\mu; \alpha) \neq \emptyset$. Hence $U(\mu; \alpha)$ is a $\mathcal{Q}$-set from assumption. This yields that if $U(\mu; \alpha) \neq \emptyset$ then $U(\mu; \alpha)$ is the $\mathcal{Q}$-set for all $\alpha \in [0,1]$. It follows from transfer principle that μ is the fuzzy $\mathcal{Q}$-set. That is,

$$\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}$$

□

Example Let $\mathfrak{A}$ be the class of groups and $(X; \circ, ^{-1}, e) \in \mathfrak{A}$. For every $A \subseteq X$, we define $\mathcal{P}_A, \mathcal{Q}_A$ by

$$\mathcal{P}_A : \forall x \forall y (x \in A \to y \circ x \circ y^{-1} \in A);$$

$$\mathcal{Q}_A : \forall x \forall y (x \in A \wedge y \in A \to x \circ y^{-1} \in A).$$

These formulas indicate that

A is a normal subgroup of X;

A is a subgroup of X.

The fact that every normal subgroup is a subgroup can be represented by

$$\mathfrak{A} \models \mathcal{P} \Longrightarrow \mathcal{Q}.$$

Hence it follows from the above that

$$\bar{\mathfrak{A}} \models \text{fuzzy}\mathcal{P} \Longrightarrow \text{fuzzy}\mathcal{Q}.$$

The statement represents that

For every fuzzy subset μ of X, if μ is a fuzzy normal subgroup then it is a fuzzy subgroup. That is, every normal fuzzy group is a fuzzy group.

5 Application to type 2

In this section we apply the transfer principle to the statement of **type 2** which has a follwing form:

> Let $X, Y \in \mathfrak{A}$ and f be a homomorphism from X to Y. For every subset B of Y, if B is a $\mathcal{Q}$-set then $f^{-1}(B)$ is a $\mathcal{P}$-set.

We denote it formally by

$$\mathfrak{A} \models (Y : B) : \mathcal{Q} \Longrightarrow (X : f^{-1}(B)) : \mathcal{P},$$

If the statement holds for any $X, Y \in \mathfrak{A}$, homomorphism $f : X \to Y$ and subset $B \subseteq Y$, then we describe it by

$$\mathfrak{A} \models B : \mathcal{Q} \Longrightarrow f^{-1}(B) : \mathcal{P}.$$

We extend the statement to the case of the fuzzy theory of algebras:

> Let $X, Y \in \mathfrak{A}$ and f be a homomorphism from X to Y. For every fuzzy subset ν of Y, if ν is a *fuzzy $\mathcal{Q}$-set* then $f^{-1}(\nu)$ is a *fuzzy $\mathcal{P}$-set.*

We denote formally the statement by

$$\bar{\mathfrak{A}} \models (Y : \nu) : \text{fuzzy } \mathcal{Q} \Longrightarrow (X : f^{-1}(\nu)) : \text{fuzzy } \mathcal{P},$$

and if it holds for every algebra $X, Y \in \mathfrak{A}$, every homomorphism $f : X \to Y$, and every fuzzy subset ν of Y, then we do by

$$\bar{\mathfrak{A}} \models \nu : \text{fuzzy } \mathcal{Q} \Longrightarrow f^{-1}(\nu) : \text{fuzzy } \mathcal{P}.$$

The statements of **type 2** have a new concept. It is an inverse image of a fuzzy subset by a homomorphism. To extend the concept to fuzzy theory, we have to define an inverse image of a fuzzy subset by a homomorphism. This is defined as follows: Let X, Y be algebras and f be a homomorphism from X to Y. For every fuzzy subset ν of Y, we define an inverse image $f^{-1}(\nu)$ of ν by

$$f^{-1}(\nu)(x) = \nu(f(x)) \quad (x \in X).$$

It follows from transfer principle that

Theorem 3. *Let $X, Y \in \mathfrak{A}$ be algebras and $f : X \to Y$ be a homomorphism. Then,*

$$\text{if } \mathfrak{A} \models B : \mathcal{Q} \Longrightarrow f^{-1}(B) : \mathcal{P} \text{ then}$$
$$\bar{\mathfrak{A}} \models \nu : \text{fuzzy } \mathcal{Q} \Longrightarrow f^{-1}(\nu) : \text{fuzzy } \mathcal{P}.$$

6 Application to type 3

A statement of **type 3** has a form:

> Let X, Y be algebras, $f : X \to Y$ be a homomorphism, and A be a subset of X. If A is a $\mathcal{P}$-set, then $f(A)$ is a $\mathcal{Q}$-set.

We formally denote the above by

$$\mathfrak{A} \models (X : A) : \mathcal{P} \Longrightarrow (Y : f(A)) : \mathcal{Q}.$$

If the representation holds for all algebras $X, Y \in \mathfrak{A}$, and homomorphism $f : X \to Y$, and $A \subseteq X$, we denote simply

$$\mathfrak{A} \models A : \mathcal{P} \Longrightarrow f(A) : \mathcal{Q}.$$

The statement of **type 3** can be generalized to the fuzzy theory. It has the following representation:

> Let X, Y be algebras, $f : X \to Y$ be a homomorphism, and μ be a fuzzy subset of X. If μ is a fuzzy $\mathcal{P}$-set, then $f[\mu]$ is a fuzzy $\mathcal{Q}$-set.

The statement is represented by

$$\bar{\mathfrak{A}} \models (X : \mu) : \text{fuzzy } \mathcal{P} \Longrightarrow (Y : f[\mu]) : \text{fuzzy } \mathcal{P}$$

and if the representation holds for all algebras $X, Y \in \mathfrak{A}$, homomorphism $f : X \to Y$, and fuzzy subset of X, then we denote it by

$$\bar{\mathfrak{A}} \models \mu : \text{fuzzy } \mathcal{P} \Longrightarrow f[\mu] : \text{fuzzy } \mathcal{P}.$$

There is also a new concept called an *image* $f[\mu]$ of a fuzzy subset μ by a (surjective) homomorphism f. We have to define $f[\mu]$ for a homomorphism f and a fuzzy subset μ. Let $X, Y \in \mathfrak{A}$ and $f : X \to Y$ be a map from X to Y. For every fuzzy subset μ of X, we define an image $f[\mu]$ of μ as

$$f[\mu](y) = \bigvee_{u \in f^{-1}(y)} \mu(x), \quad y \in Y.$$

If $f^{-1}(y) = \emptyset$, then we put $f[\mu](y) = 0$. We note that the image $f[\mu]$ is also a fuzzy subset of Y. For images of fuzzy subsets we have the fundamental result which plays an important role in the theory of **type 3**.

Lemma 1. *Let $f : X \to Y$ be a surjective homomorphism. For every $\alpha \in [0,1]$, we have*

$$U(f[\mu];\alpha) = \bigcap_{\epsilon>0} f(U(\mu;\alpha-\epsilon))$$

In order to develop our theory to wider classes of algebras, we need to consider a property whcih is carried over from sets to the intersection of those sets. A property $\mathcal{P}$ is called to have an ***intersection property*** if the intersection $\bigcap_\lambda A_\lambda$ has a property $\mathcal{P}$ for every set A_λ with the property $\mathcal{P}$. This means that if each set A_λ has a property $\mathcal{P}$ then the intersection $A = \bigcap_\lambda A_\lambda$ has the property $\mathcal{P}$, that is,

$\forall x \cdots \forall y (t_1(x,\cdots,y) \in A_\lambda \wedge \cdots \wedge t_n(x,\cdots,y) \in A_\lambda \to t(x,\cdots,y) \in A_\lambda)$ for every $\lambda \in \Lambda$ imply $\forall x \cdots \forall y (t_1(x,\cdots,y) \in A \wedge \cdots \wedge t_n(x,\cdots,y) \in A \to t(x,\cdots,y) \in A)$.

Using the intersection property we can show the general theorem about the statements of **type 3**.

Theorem 4. *Let $X, Y \in \mathfrak{A}$ and f be a surjective homomorphism from X to Y. If $\mathfrak{A} \models A : \mathcal{P} \Longrightarrow f(A) : \mathcal{Q}$ and $\mathcal{Q}$ has an intersection property, then $\bar{\mathfrak{A}} \models \mu : \text{fuzzy } \mathcal{P} \Longrightarrow f[\mu] : \text{fuzzy } \mathcal{Q}$.*

7 Other Properties

Let X be a group. A subset A of X is not always a subgroup of X. In this case we often consider the subgroup $\langle A\rangle$ generated by A, that is, the least subgroup containing A. In this case there is a question whether we can extend such concept to the fuzzy theory of groups. Or more generally, how do we extend the concept to the fuzzy theory of algebras? In this section we think about the question and give a certain solution by use of the transfer principle.

Let $X \in \mathfrak{A}$ be an arbitrary algebra and A be a subset of X. For a formula $\mathcal{P}$ with an intersection property, a $\mathcal{P}$-set $\langle A\rangle$ generated by A is defined as the least $\mathcal{P}$-set which contains A. It can also be represented by

$$\langle A\rangle = \bigcap_\lambda \{B_\lambda \mid A \subseteq B_\lambda,\ B_\lambda : \mathcal{P}\text{-set}\}$$

Hence we can extend the $\mathcal{P}$-set $\langle A\rangle$ generated by A to the fuzzy subset μ of X as follows:

$$\langle\mu\rangle = \bigwedge_\lambda \{\nu_\lambda \mid \mu \le \nu_\lambda,\ \nu_\lambda : \text{fuzzy } \mathcal{P}\text{-set}\},$$

where $\langle\mu\rangle$ is a fuzzy subset of X defined by

$$\langle\mu\rangle(x) = \inf_\lambda \{\nu_\lambda(x) \mid \mu \le \nu_\lambda,\ \nu_\lambda : \text{fuzzy } \mathcal{P}\text{-set }\} \quad (x \in X).$$

It has another representation: For $\alpha \in [0,1]$,

$$U(\langle\mu\rangle;\alpha) = \bigcap_\lambda \{U(\nu_\lambda;\alpha) \mid \mu \le \nu_\lambda,\ \nu_\lambda : \text{fuzzy } \mathcal{P}\text{-set }\}$$

Using the representation we can get all facts obtained so far by transfer principle and of course we can get new results.

At last we consider a direct product of fuzzy subsets of X. Let μ_i be a fuzzy subset of X_i $(i \in I)$. A map

$$\mu : \Pi_{i\in I} X_i \to [0,1]^I$$

of $\Pi_{i\in I} X_i$ satisfying the condition

$$\mu(a)(j) = \mu_j(a(j)), \quad (a \in \Pi_{i\in I} X_i, \quad j \in I)$$

is called a direct product of μ_i and denoted by $\mu = \Pi_{i\in I}\mu_i$.

As the direct product $\Pi_{i\in I}\mu_i$ of fuzzy subsets μ_i, we have the following.

Lemma 2. *For all $\alpha \in [0,1]^I$,*

$$\Pi_j\, U(\mu_j;\alpha(j)) = U(\Pi_i \mu_i;\alpha)$$

A property $\mathcal{P}$ is said to have a *direct product property* if for every subset X_i with a property $\mathcal{P}$ the direct product $\Pi_i X_i$ also has the property $\mathcal{P}$. Thus, we can get a general theorem about direct product of fuzzy subsets by transfer principle.

Theorem 5. *If $\mathcal{P}$ has a direct product property and $\mathfrak{A} \models A_\lambda : \mathcal{P} \Longrightarrow \Pi_\lambda A_\lambda : \mathcal{P}$, then $\bar{\mathfrak{A}} \models \mu_\lambda : \text{fuzzy } \mathcal{P} \Longrightarrow \bar{\mathfrak{A}} \models \Pi_\lambda \mu_\lambda : \text{fuzzy } \mathcal{P}$.*

It follows from the theorem that if a class of crisp algebras is closed under the direct product then a class of extended fuzzy subsets of those algebras is also closed under the direct product.

8 Conclusion

We prove the fundamental and general results that any property about crisp subsets expressed by special formulas can be extended to that of the fuzzy subsets. Thus the direction of the research of fuzzy theory of algebras aims to consider other properties which are not expressed by our formulas. It is also important to investigate the properties **H,S,P** of a class of fuzzy algebras. Of course, these mean that a class of algebras is closed under the operation of *homomorphic images, subalgebras, direct product*, respectively. In this paper we consider the properties **S** and **P**. The rest of ones **H** is very important to consider the quotient algebras. Because to consider the quotient algebras are correspond to do congruences. As to *fuzzy congruences*, their fundamental results are obtained in [4, 5], which are restricted to the case of BCI-algebras and groups but they can be developped to the general case of algebras.

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