# Continuity of set defuzzification methods*

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## Abstract

The purpose of this work is to analyze the continuity of set defuzzification processes on $\mathbb{R}^n$ which are defined by Aumann integral methods.

**Keywords:** Set defuzzification, Fuzzy measures.

## 1 Introduction

The concept of fuzzy set was introduced by Zadeh in 1965 as a generalization of the concept of set (or crisp set), and it has very important applications to the treatment of non-deterministics real problems. In the most of these applications, it is necessary to realize a defuzzification process. Actually, defuzzification is the ultimate step in approximate reasoning and it consists in the replacement of a fuzzy set with a suitable non-negative real number. Ogura et al. [3] (2001) introduce a new defuzzification method $\Delta_\mu$ (associated to a suitable fuzzy measure $\mu$) of a fuzzy set $f \in \mathcal{F}(\mathbb{R}^n)$, the class of all compact fuzzy sets on $\mathbb{R}^n$, which is carried out in two steps:
a) From a fuzzy set to a crisp set via the Aumann integral, by means
$SD(f) = \int_0^1 L_\alpha f d\alpha$
b) From this crisp set $SD(f)$ to a point by means the formula
$\Delta_\mu(f) = \mu\left(\int_0^1 L_\alpha f d\alpha\right)$.

Recently Bouchon-Meunier et al. [1] (2004), in a very similar form define
b') $\Delta^*_\mu(f) = \int_0^1 \mu\left(L_\alpha f\right) d\alpha$.
Moreover, Theorem 1.1 in [1] establishes that $\Delta_\mu(f) = \Delta^*_\mu(f)$ for all $f \in \mathcal{F}_c(\mathbb{R}^n)$, the class of all compact and convex fuzzy sets on $\mathbb{R}^n$, under adequate conditions on the fuzzy measure $\mu$ (for instance, linearity). In relation to the step a), and in connection with the above set defuzzification process, Roventa&Spircu [6] analizes the compatibility of general "averaging procedures" on $\mathcal{F}(\mathbb{R}^n)$ with algebraic and order structures on $\mathcal{F}(\mathbb{R}^n)$, however we think that, additionally, a continuity type requirement must be also considered. That is to say, a property such as
"If $f_p \rightarrow f$ (in some suitable sense) then $Defuzz(f_p) \rightarrow Defuzz(f)$"
is also desirable.
Thus, because the Aumann integral is a compact (and convex) subset of $\mathbb{R}^n$, then the Hausdorff continuity of $\mu$ (shortly: H-continuity) plays an essential role for the continuity of set defuzzification processes defined in b) and b').

In this context, the aim of this work is to show some connections between the H-continuity of fuzzy measures and the continuity of set defuzzification processes on $\mathcal{F}(\mathbb{R}^n)$ which are defined by Aumann integral methods.

## 2 Main results

Let $\mathcal{K}(\mathbb{R}^n)$ be the class of all nonempty compact subsets of $\mathbb{R}^n$, and $\mathcal{K}_c(\mathbb{R}^n)$ the subclass of all nonempty compact and convex subsets of $\mathbb{R}^n$, respectively.

If $H$ is the Hausdorff metric on $\mathcal{K}(\mathbb{R}^n)$ induced by the euclidean norm $\|\cdot\|$ in $\mathbb{R}^n$, then it is well known that $(\mathcal{K}(\mathbb{R}^n), H)$ is a complete and separable metric space, and $\mathcal{K}_c(\mathbb{R}^n)$ is a closed subspace of $\mathcal{K}(\mathbb{R}^n)$. See [2].

**Theorem 1** *Let $A_p$, $A \in \mathcal{K}(\mathbb{R}^n)$ such that $A_p \xrightarrow{H} A$. If $\mu$ is an upper continuous Borel fuzzy measure on $\mathbb{R}^n$, then $\limsup_{p\to\infty} \mu(A_p) \le \mu(A)$.*

**Proof** If $\{A_p\}$ is an arbitrary sequence in $\mathcal{K}(\mathbb{R}^n)$ such that $A_p \xrightarrow{H} A$ then we have $\left\{\overline{\bigcup_{k\ge p} A_k}\right\}_p$ is a decreasing sequence and $A_p \subseteq \overline{\bigcup_{k\ge p} A_k}$, $\forall p$.

So, by upper continuity, we have

$$\begin{aligned} \limsup_{p\to\infty} \mu(A_p) &\le \limsup_{p\to\infty} \mu\left(\overline{\bigcup_{k\ge p} A_k}\right) \\ &= \mu\left(\bigcap_{p=1}^{\infty}\left(\overline{\bigcup_{k\ge p} A_k}\right)\right) = \mu(A), \end{aligned}$$

and the proof is complete. □

We recall that a (partially ordered by inclusion) subclass $\mathcal{L} \subseteq \mathcal{K}(\mathbb{R}^n)$ is an *upper semi-lattice* if all finite subset of $\mathcal{L}$ have a least upper bound in $\mathcal{L}$.

**Example 1** *a) If $\mathcal{L} = \mathcal{K}(\mathbb{R}^n)$ then the least upper bound of $\{K_1, ..., K_p\}$ is $K = \cup K_p$. Therefore $\mathcal{K}(\mathbb{R}^n)$ is an upper semi-lattice.*

*b) If $\mathcal{L} = \mathcal{K}_c(\mathbb{R}^n)$ then the least upper bound of $\{K_1, ..., K_p\}$ is $K = \overline{co}(\cup K_p)$, where $\overline{co}(A)$ denotes the closed convex hull of $A$. Therefore $\mathcal{K}_c(\mathbb{R}^n)$ is also an upper semi-lattice.*

**Theorem 2** *Let $\mu$ be a Borel fuzzy measure on $\mathbb{R}^n$ and consider an upper semi-lattice $\mathcal{L} \subseteq \mathcal{K}(\mathbb{R}^n)$. If we suppose that*

*i) $\mu$ is upper continuous on $\mathcal{L}$;*

*ii) $\mu(\{\mathbf{0}\}) = 0$;*

*iii) for each $K \in \mathcal{L}$ and $\epsilon \ge 0$*

$$\mu(K + \epsilon\mathbf{B}) \le \mu(K) + \mu(\epsilon\mathbf{B}) + p_K(\epsilon), \quad (1)$$

*where $\mathbf{B}$ denotes the closed unitary ball in $\mathbb{R}^n$ and $p_K : [0,\infty) \to [0,\infty)$ is a nonnegative and monotone (in both $\epsilon$ and $K$) function such that $p_K(\epsilon) \to 0$ as $\epsilon \searrow 0$.*

*Then $\mu$ is $H$-continuous on $\mathcal{L}$.*

**Proof** Let $K_p$ , $K \in \mathcal{L}$ such that $K_p \xrightarrow{H} K$. Then, due to Theorem 1 and hypothesis i), we have $\limsup_{p\to\infty} \mu(K_p) \le \mu(K)$.

Thus, only remains to show that $\mu(K) \le \liminf_{p\to\infty}\mu(K_p)$.

In fact, if $\epsilon > 0$ is given then there exists $p_0 \in \mathbb{N}$ such that $H(K, K_p) < \epsilon$ for all $p \ge p_0$ and, by definition, we have

$$K \subset N(K_p, \epsilon) \subset K_p + \epsilon\mathbf{B},\ \forall p \ge p_0. \quad (2)$$

where $N(K_p, \epsilon)$ denotes the $\epsilon$-neighborhood of $K_p$. On the other hand, if $K^*$ is the least upper bound of $\{K_1, ..., K_{p_0-1}, cl[N(K, \epsilon)]\}$ then $K^* \in \mathcal{L}$ and, also by (2), $K^*$ contain $K$ and $K_p$ for all $p$.

So, by (2) and hypothesis iii), we obtain

$$\begin{aligned} \mu(K) &\le \mu(K_p + \epsilon\mathbf{B}) \\ &\le \mu(K_p) + \mu(\epsilon\mathbf{B}) + p_{K_p}(\epsilon). \\ &\le \mu(K_p) + \mu(\epsilon\mathbf{B}) + p_{K^*}(\epsilon) \end{aligned}$$

for all $p \ge p_0$.
Thus,

$$\mu(K) \le \liminf_{p\to\infty} \mu(K_p) + \mu(\epsilon\mathbf{B}) + p_{K^*}(\epsilon),$$

and, finally, taking limit $\epsilon \searrow 0$ we conclude that $\mu(K) \le \liminf_{p\to\infty} \mu(K_p)$, and the proof is complete. □

**Example 2** *The Lebesgue measure is a $H$-continuous fuzzy measure on $\mathcal{L} = \mathcal{K}_c(\mathbb{R}^n)$ (details are discussed in [5]) .*

**Example 3** *Consider the monotone set function:*

$$\mu(A) = \begin{cases} 0 & if \quad A = \emptyset \\ \sup_{a\in A} \parallel a \parallel & if \quad A \neq \emptyset, \end{cases}$$

*for every $A \subseteq \mathbb{R}^n$. Then $\mu$ is a fuzzy measure which verifies conditions of Theorem 1 on $\mathcal{L} = \mathcal{K}(\mathbb{R}^n)$. In fact:*
*i) Let $A_p \supseteq A_{p+1} \supseteq ...$, $p \ge 1$, a decreasing sequence in $\mathcal{L} = \mathcal{K}(\mathbb{R}^n)$. Then it is clear that*

$$A_p \xrightarrow{H} A = \bigcap_{p\ge 1} A_p.$$

*Consequently, because* $A \subseteq A_p$ *for every* $p$*, then* $\mu(A) \leq \liminf_{p\to\infty} \mu(A_p)$.

*Now, if* $\epsilon > 0$ *is given then there exists* $p_0 \in \mathbb{N}$ *such that* $H(A, A_p) < \epsilon$ *for all* $p \geq p_0$ *and, by (1), we have* $A_p \subset N(A, \epsilon) \subset A + \epsilon\mathbf{B}$, $\forall p \geq p_0$.

*Thus,* $x \in A_p$ *implies* $x = a + \epsilon b$*, for some* $a \in A$ *and* $b \in \mathbf{B}$*. Therefore*

$$\| x \| \leq \| a \| + \epsilon \leq \| A \| + \epsilon$$

*for all* $x \in A_p$*. Consequently,*

$$\sup_{x\in A_p} \| x \| = \| A_p \| \leq \| A \| + \epsilon,$$

*for every* $p$*, which implies that*

$$\limsup_{p\to\infty} \mu(A_p) \leq \mu(A).$$

*Therefore,* $\mu$ *is upper continuous on* $\mathcal{L} = \mathcal{K}(\mathbb{R}^n)$.

*ii) It is clear that, by definition,* $\mu(\{\mathbf{0}\}) = 0$.

*iii) If* $K \in \mathcal{K}(\mathbb{R}^n)$ *we have*

$$\begin{aligned} \mu(K + \epsilon\mathbf{B}) &= \sup_{x\in K,\, b\in\mathbf{B}} \| x + \epsilon b \| \\ &\leq \sup_{x\in K,\, b\in\mathbf{B}} \| x \| + \| \epsilon b \| \\ &= \mu(K) + \mu(\epsilon\mathbf{B}). \end{aligned}$$

*So, in this case we can take* $p_K \equiv 0$ *verifying condition iii) in Theorem 1 and, consequently,* $\mu$ *is* $H$*-continuous on* $\mathcal{L} = \mathcal{K}(\mathbb{R}^n)$.

**Definition 1** *Let* $f \in \mathcal{F}(\mathbb{R}^n)$*. The set defuzzification of* $f$ *is defined ([3]) by*

$$SD(f) = \int_0^1 L_\alpha f d\alpha \tag{3}$$

*where in the above formula, the integral is the Aumann integral with respect to the Lebesgue measure on* $[0,1]$ *(see [2]).*

**Remark 1** *a) Because* $L_\alpha f \in \mathcal{K}(\mathbb{R}^n)$ *for every* $\alpha \in [0,1]$ *and the Lebesgue measure is a nonatomic measure, then by using the properties of the Aumann integral, we obtain that* $SD(f)$ *is a compact-convex and nonempty subset of* $\mathbb{R}^n$ *(see [2]).*

*b) Also, if we consider the application* $SD : \mathcal{F}(\mathbb{R}^n) \to \mathcal{K}_c(\mathbb{R}^n)$ *then, due to linearity of the Aumann integral, we have* $SD(f+g) = SD(f) + SD(g)$ *and* $SD(\lambda f) = \lambda SD(f)$.

**Definition 2** *If* $\mu$ *is a borelian fuzzy measure on* $\mathbb{R}^n$*, then the value of the defuzzification of* $f$ *by* $\mu$ *is defined ([3]) as*

$$\Delta_\mu(f) = \mu(SD(f)) = \mu\left(\int_0^1 L_\alpha f d\alpha\right). \tag{4}$$

**Definition 3** *(D-convergence) Let* $f_p,\ f_0 \in \mathcal{F}(\mathbb{R}^n)$*. We say that that* $f_p$ *D-converges to* $f_0$ *iff*

$$D(f_p, f_0) \to 0 \text{ as } p \to \infty.$$

For details on $D$-convergence and its properties see [4].

**Theorem 3** *Let* $f_p,\ f_0 \in \mathcal{F}(\mathbb{R}^n)$ *such that* $f_p \xrightarrow{D} f_0$*. Then*

$$SD(f_p) \xrightarrow{H} SD(f_0) \text{ in } \mathcal{K}_c(\mathbb{R}^n).$$

**Proof** Due to Remark 1 we know that $SD(f_p), SD(f_0) \in \mathcal{K}_c(\mathbb{R}^n)$. Now, by Definition 3, $f_p \xrightarrow{D} f_0$ implies $H(L_\alpha f_p, L_\alpha f_0) \to 0$ for every $\alpha \in [0,1]$. So, defining $\Gamma_p(\alpha) = L_\alpha f_p$ and $\Gamma_0(\alpha) = L_\alpha f_0$, then $\Gamma_p$ converges pointwise to $\Gamma_0$ on $[0,1]$.

On the other hand, there exists a compact $K \in \mathcal{K}(\mathbb{R}^n)$ such that $L_\alpha f_p \subseteq K$, for all $\alpha \in [0,1]$ and $p \geq 0$. So, taking $h : [0,1] \to R$ defined by $h(x) = \| K \|$, we have that $h \in L^1([0,1], \mathbb{R})$ (actually, $\int_0^1 h(x)dx = \| K \|$) and $\sup_{x\in\Gamma_p(x)} \| x \| \leq h(x)$ , $\forall p \geq 0$.

Consequently, by Dominated Convergence Theorem for Aumann integral (see [2]), we obtain

$$SD(f_p) = \int_0^1 \Gamma_p(\alpha)d\alpha \xrightarrow{H} \int_0^1 \Gamma(\alpha)d\alpha = SD(f_0)$$

and the proof is complete. □

Now we want to close the circle around of the set defuzzification process and its continuity.

**Theorem 4** *If* $\mu$ *is a Borel fuzzy measure on* $\mathbb{R}^n$ *verifying the same conditions of Theorem 2, with* $\mathcal{L} = \mathcal{K}_c(\mathbb{R}^n)$*, then the defuzzification process* $\Delta_\mu$ *is continuous on* $(\mathcal{F}_c(\mathbb{R}^n), D)$.

**Proof** If $f_p \xrightarrow{D} f$ then $SD(f_p) \xrightarrow{H} SD(f)$ in $\mathcal{K}_c(\mathbb{R}^n)$ (Theor.3) and, by H-continuity of $\mu$, we have $\Delta_\mu(f_p) = \mu(SD(f_p)) \xrightarrow{H} \mu(SD(f)) = \Delta_\mu(f)$. $\square$

Finally, we present the following equivalence:

**Theorem 5** *Let $\mu$ be a Borel fuzzy measure on $\mathbb{R}^n$. Then the following conditions are equivalents*
*a) $\mu$ is H-continuous on $\mathcal{K}_c(\mathbb{R}^n)$*
*b) $\Delta_\mu$ is continuous on $(\mathcal{F}_c(\mathbb{R}^n), D)$*

**Proof** *a)* → *b)*. Theorem 4.
*b)* → *a)*. If we suppose that $A_p \xrightarrow{H} A_0$ in $\mathcal{K}_c(\mathbb{R}^n)$ then, taking $f_p = \chi_{A_p}$ the characteristic function of $A_p$ for all $p \geq 0$, we have $f_p \xrightarrow{D} f_0$ in $\mathcal{F}_c(\mathbb{R}^n)$. In fact, because $L_\alpha f_p = A_p$, for every $\alpha \in [0,1]$ and for all $p \geq 0$, we have

$$\begin{aligned} D(f_p, f_0) &= \sup_{\alpha\in[0,1]} H(L_\alpha f_p, L_\alpha f_0) \\ &= \sup_{\alpha\in[0,1]} H(A_p A_0) \to 0 \end{aligned}$$

as $p \to \infty$.

Thus, by hypothesis,

$$\Delta_\mu(f_p) = \mu\left(\int_0^1 A_p d\alpha\right)$$
$$\to \mu\left(\int_0^1 A_0 d\alpha\right) = \Delta_\mu(f_0)$$

as $p \to \infty$, which implies that $\mu(A_p) \to \mu(A_0)$, and $\mu$ is H-continuous. $\square$

**Example 4** *a) If $\mu$ is the Lebesgue measure on $\mathbb{R}^n$ then $\Delta_\mu$ is a continuous defuzzification process on $(\mathcal{F}_c(\mathbb{R}^n), D)$.*
*b)Let $\mu$ be as in Example 3. Then $\Delta_\mu$ is also continuous on $(\mathcal{F}_c(\mathbb{R}^n), D)$.*
*c)Let $\mu$ be the Dirac measure concentred in $x_0 \in \mathbb{R}^n$, i.e.,*

$$\mu(E) = \begin{cases} 1 & if \quad x_0 \in E \\ 0 & if \quad x_0 \notin E, \end{cases}$$

*for any $E \subseteq \mathbb{R}^n$. Is not difficult to see that $\mu$ is not H-continuous on $\mathcal{K}_c(\mathbb{R}^n)$. In fact, taking a sequence $(x_p) \in \mathbb{R}^n$ such that $x_p \neq x_0$ for every $p$ and $x_p \to x_0$ as $p \to \infty$, then $\mu(\{x_0\}) = 1$ whereas $\mu(\{x_p\}) = 0$ for all $p > 0$ and, consequently, $\mu$ is not H-continuous on $\mathcal{K}_c(\mathbb{R}^n)$.*
*On the other hand, taking $f_p = \chi_{x_p}$ the characteristic function of $\{x_p\}$ for all $p \geq 0$, then it is clear that $f_p \xrightarrow{D} f_0$ in $\mathcal{F}_c(\mathbb{R}^n)$, but $\Delta_\mu(f_0) = 1$ whereas $\Delta_\mu(f_p) = 0$ for all $p > 0$ and, consequently, $\Delta_\mu$ is not continuous on $(\mathcal{F}_c(\mathbb{R}^n), D)$.*

**Remark 2** *It is interesting to note that hypothesis in Theorem 2 are not verified by the Dirac measure $\mu$. In fact, if $x_0 = \mathbf{0}$ then $\mu(\{\mathbf{0}\}) = 1 \neq 0$ which implies that condition ii) fails. Now, if $x_0 \neq \mathbf{0}$ with $\| x_0 \| = \delta > 0$ we can choose two positive real numbers $\epsilon_1$ and $\epsilon$ such that $0 < \epsilon_1 < \delta$, and $0 < \epsilon < \delta$, and $\epsilon_1 + \epsilon > \delta$. Then, choosing a compact $K \in \mathcal{K}_c(\mathbb{R}^n)$ such that $\epsilon_1\mathbf{B} \subset K$ and $\| K \| < \delta$, we obtain that $x_0 \notin K$ and $x_0 \notin \epsilon\mathbf{B}$, but $x_0 \in K + \epsilon\mathbf{B}$. Consequently,*

$$\mu(K + \epsilon\mathbf{B}) = 1 > \mu(K) + \mu(\epsilon\mathbf{B}) = 0, \quad (5)$$

*for all arbitrarily small $\epsilon$ (for this it is sufficient to note that, because $\epsilon_1$ can be choosed arbitrarily near of $\delta$, then $\epsilon$ can be choosed arbitrarily near of 0). Finally, it is clear that inequality (5) is not compatible with condition iii) in Theorem 2.*

## 3 Concluding remarks

a)The study of H-continuity of fuzzy measures (or monotone set functions) is an interesting problem, and it is connected whith several theoretical and applied fields. For instance, the theory of H-continuous valuations on $\mathcal{K}_c(\mathbb{R}^n)$ is one the most studied topics in convex geometry in the last years (see [7]).

In this work, we give sufficient conditions for the H-continuity of borel fuzzy measures on $\mathcal{K}_c(\mathbb{R}^n)$ (see Theorem 2) and, in addition, we present some ilustrative examples.

On the other hand, a new class of defuzzification process (set defuzification, averaging procedures) have been recently introduced. These new defuzzification methods can be used in a wide class of real problems (see [1, 3]).

In this context, we analyze the continuity of the set defuzzification process (see Theorems 3, 4 and 5) defined ([3]) via Aumann integration by means
$\Delta_\mu(f) = \mu\left(\int_0^1 L_\alpha f d\alpha\right)$ , $f \in \mathcal{F}(\mathbb{R}^n)$,
and we prove that continuity of $\Delta_\mu$ is equivalent to the H-continuity of fuzzy measure $\mu$.

b) In [1], the authors have defined another set defuzzification method, via Choquet integral, by means
$\Delta_\mu^*(f) = \int_0^1 \mu\left(L_\alpha f\right) d\alpha$.
Moreover, Theorem 1.1 in [2] establishes that $\Delta_\mu(f) = \Delta_\mu^*(f)$ for all $f \in \mathcal{F}_c(\mathbb{R}^n)$, under adequate conditions on the fuzzy measure $\mu$ (for instance, linearity).
In this context, is interesting to note that, independently of above equality, the H-continuity of $\mu$ implies (Lebesgue Convergence Theorem) the continuity of $\Delta_\mu^*$.

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