ON SOME CHARACTERIZATIONS OF COMPLETE FUZZY PREORDERS

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Summary

We study different characterizations for the transitivity of reflexive fuzzy relations under completeness. We depart from some properties equivalent to the definition of a crisp preorder. We study if the same conditions characterize the transitivity of reflexive relations in the fuzzy set context. Instead of restricting ourselves to t-norms, we work in the more general setting of conjunctors. We consider two different completeness conditions: strong completeness and absence of incomparability.

Keywords: Preorder, transitivity, negative transitivity, fuzzy preference relation, completeness condition.

1 Introduction

Transitivity is one of the most important properties in ordering structures and a basic point in the setting of preference modelling. Every crisp reflexive relation can be decomposed into three (two, if the relation is complete) parts: the symmetric, asymmetric and if it is complete, dual symmetric components. The transitivity of a complete reflexive relation is characterized by the transitivity of its symmetric and asymmetric components. Although this is the best known, there exist other characterizations for crisp complete preorders.

In this paper we focus on the different properties presented in [1] characterizing complete preorders. We study their definition and relationship with complete preorders in the setting of fuzzy relations. In the crisp case, the concept of completeness is equivalent to the absence of its third component, called *incomparability relation* in the preference modelling terminology. This does not hold onto the fuzzy framework: there exist several different concepts of completeness, which are not equivalent in general to the absence of incomparability. In this paper, we shall study the most restrictive case: strongly complete reflexive fuzzy relations. We shall also prove that some of the results obtained for that particular case do not hold under a more general situation: in absence of incomparability.
Although transitivity is traditionally defined by t-norms, we consider a more general type of binary operators we have called conjunctors (see [6, 7]). Therefore, we study the characterization of complete fuzzy f-preorders, for f a conjunctor.

The paper is structured in six sections. In Section 2 we recall the characterizations known for complete preorders. Section 3 contains notions concerning fuzzy relations necessary to understand the rest of the paper. We discuss in Sections 4 and 5 the relationship between the different properties presented in Section 2 and complete preorders in the fuzzy set context. Section 4 includes the results concerning strong completeness and Section 5, the study of the same properties in absence of incomparability. Finally, Section 6 contains some conclusions.

2 Characterizations for a crisp preorder

Every reflexive relation R can be regarded as a preference relation under the interpretation *aRb if and only if a is preferred or indifferent to b.* The reflexive relation R is then called *large preference relation.* Let us denote the transpose, the complement and the dual of a relation R by R^t, R^c and R^d, respectively. Reflexive relations can be split into three disjoint parts: the asymmetric component $P = R \cap R^d$, the symmetric component $I = R \cap R^t$ and the dual symmetric component $J = R^c \cap R^d$, called respectively strict preference, indifference and incomparability relations in preference modelling. The triplet (P, I, J), called *preference structure*, allows to reconstruct the original reflexive relation: $R = P \cup I$. The completeness of R is equivalent to the absence of incomparability ($J = \emptyset$) in the associated preference structure.

The composition of two relations Q_1 and Q_2 is the binary relation $Q_1 \circ Q_2$ defined by $Q_1 \circ Q_2(a, c) = \sup_b(\min(Q_1(a, b), Q_2(b, c)))$. The transitivity of Q is equivalent to $Q \circ Q \subseteq Q$.

The relation Q is called negatively transitive if it holds that $aQc \Rightarrow (aQb \vee bQc)$ for all a, b, c.

A binary relation Q defined on a set A, can be represented by the graph (A, Q) where A is the set of nodes and Q the set of arcs, i.e. there is an arc from the node a to the node b if and only if aQb and it is represented as (a, b). A path of length n in such a graph is a set of n arcs $(a_0, a_1), \ldots, (a_{n-1}, a_n)$ in (A, Q). A *circuit* in (A, Q) is a path for which $a_0 = a_n$.

For crisp reflexive relations the following statements are equivalent under completeness [1]:

(I) R is transitive;

(II) $\begin{cases} P \text{ is transitive,} \\ I \text{ is transitive;} \end{cases}$

(III) $\begin{cases} P \text{ is transitive,} \\ P \circ I \subseteq P; \end{cases}$

(IV) P is negatively transitive;

(V) every circuit of length ≤ 3 in (A, R) contains no P.

Let us note that, given the definition of P, every circuit of length 2 or 1 in (A, R) contains no P for any reflexive relation. Property (V) can be written as *every circuit of length* 3 *in* (A, R) *contains no* P.

Property (III) can be written in a different way given that $P \circ I \subseteq P$ is equivalent to property $I \circ P \subseteq P$. Therefore, we can also write

(IIIa) $\begin{cases} P \text{ is transitive,} \\ I \circ P \subseteq P; \end{cases}$

and it is still equivalent to all the previous five conditions.

A complete reflexive relation R is a *preorder* if it satisfies the transitive property. The six conditions above provide six different characterizations of a preorder.

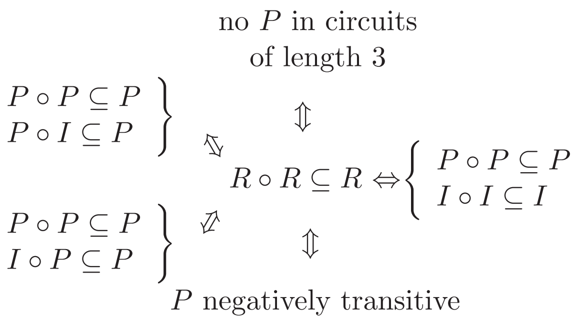

Figure 1: Different characterizations for the transitivity of a complete reflexive relation.

3 Fuzzy set context

3.1 Fuzzy preference structures

In fuzzy set theory, a reflexive fuzzy relation R on A can also be decomposed into the so-called (additive) fuzzy preference structure, by means of a generator i. This was defined by De Baets and Fodor in [3] as a symmetric (commutative) $[0, 1]^2 \to [0, 1]$ mapping bounded by the Łukasiewicz t-norm, $T_{\mathbf{L}}$, and the minimum operator, $T_{\mathbf{M}}$, i.e. $T_{\mathbf{L}} \leq i \leq T_{\mathbf{M}}$.

Given a reflexive fuzzy relation R and a generator i, the three components of an additive fuzzy preference structure are defined as follows:

$$\begin{aligned} P(a,b) &= p(R(a,b),R(b,a)) \\ &= R(a,b) - i(R(a,b),R(b,a))\,, \\ I(a,b) &= i(R(a,b),R(b,a))\,, \\ J(a,b) &= j(R(a,b),R(b,a)) \\ &= I(a,b) - (R(a,b)+R(b,a)-1)\,. \end{aligned}$$

They satisfy the additive property: $P(a,b) + I(a,b) + P^t(a,b) + J(a,b) = 1$ for all a,b. The corresponding large preference relation R from which they are defined is then given by $R(a,b) = P(a,b) + I(a,b)$.

The concept of completeness for fuzzy relations is usually defined by a t-conorm. The most usual completeness conditions considered are the *strong completeness*, defined by the maximum t-conorm: Q is strongly complete if $\max(Q(a,b),Q(b,a)) = 1$ for all a,b; and the *weak completeness* defined by the Łukasiewicz t-conorm: Q is weakly complete if $Q(a,b) + Q(b,a) \geq 1$ for all a,b. The absence of the associated incomparability relation is not equivalent to any completeness condition over the reflexive fuzzy relation R. Strong completeness is a more restrictive condition than the absence of incomparability relation and this is a stronger property than weak completeness.

3.2 Fuzzyfication of properties

There is no unique way of defining the composition of fuzzy relations. Any t-norm and more generally, any conjunctor, leads to a definition. A conjunctor f is a binary non-decreasing $[0,1]^2 \to [0,1]$ mapping such that $f(0,0) = f(1,0) = f(0,1) = 0$ and $f(1,1) = 1$. Given the conjunctor f, the f-composition of two fuzzy relations Q_1 and Q_2 on A is defined by $Q_1 \circ_f Q_2\,(a,c) = \sup_b f(Q_1(a,b),Q_2(b,c))$. Conjunctors and in particular t-norms, allow to generalize the definition of transitivity. Given the conjunctor f, Q is called *f-transitive* if

$$f(Q(a,b),Q(b,c)) \leq Q(a,c)\,, \quad \forall\, a,b,c.$$

As for crisp relations, the f-transitivity of a fuzzy relation is equivalent to $Q \circ_f Q \subseteq Q$. The definition of f-transitivity is a too general notion in a particular case: only conjunctors upper bounded by the minimum can define the transitivity for reflexive fuzzy relations (see [8]).

Let us recall the definition of negative S-transitivity (see [2]). Given a t-conorm S, the fuzzy relation Q is negatively S-transitive if $Q(a,c) \leq S(Q(a,b),Q(b,c))$ for all a,b,c. This definition can be generalized to conjunctors since every t-conorm is the dual of a t-norm, $S(x,y) = T^d(x,y) = 1 - T(1-x,1-y)$. Given a conjunctor f, we say that a fuzzy relation Q is *negatively f^d-transitive* if

$$Q(a,c) \leq f^d(Q(a,b),Q(b,c)), \quad \forall\, a,b,c.$$

The concept of negative f^d-transitivity must be restricted to conjunctors upper bounded by the minimum when the property is applied to irreflexive fuzzy relations, and in particular to strict preference relations, P. Indeed, for such a P it holds that

$$\begin{aligned} &P(a,b) \leq f^d(P(a,b),P(b,b)) \\ &\quad \Leftrightarrow 1 - P(a,b) \geq f(1-P(a,b),1) \\ &P(a,b) \leq f^d(P(a,a),P(a,b)) \\ &\quad \Leftrightarrow 1 - P(a,b) \geq f(1,1-P(a,b)) \end{aligned}$$

Hence, f must be bounded by the minimum when its arguments are values taken by P.

Since only conjunctors upper bounded by the minimum are appropriate to define the transitivity of reflexive fuzzy relations and negative transitivity of strict preference relations, we will only consider these conjunctors from here on.

Concerning the absence of strict preference in cycles of length 3, this property states that for any three alternatives a,b,c such that aRb, bRc and cRa, it holds that $a \not\!P b$, $b \not\!P c$ and $c \not\!P a$, i.e.

$$aRb \wedge bRc \wedge cRa \quad \Rightarrow \quad a \not\!P b \wedge b \not\!P c \wedge c \not\!P a.$$

The intersection of two fuzzy relations is not unique. Every conjunctor leads to a definition. When the intersection concerns more than two elements, the only property required is associativity. We can write ($a\not\!P b$ *and* $b\not\!P c$ *and* $c\not\!P a$) by means of $(f(P^c(a,b),P^c(b,c),P^c(c,a)))$, for any associative conjunctor, or equivalently, $1 - f^d(P(a,b),P(b,c),P(c,a))$. Property (V) reads for fuzzy relations as follows

$$\begin{aligned} 1 - f^d&(P(a,b),P(b,c),P(c,a)) \\ &\geq f(R(a,b),R(b,c),R(c,a))\,, \end{aligned}$$

with f an associative conjunctor. For the particular case of t-norms,

$$1 - S(P(a,b), P(b,c), P(c,a)) \geq T(R(a,b), R(b,c), R(c,a)),$$

where S is the dual t-conorm of the t-norm T.

We have identified Property (V) for fuzzy relations with the absence of strict preference in cycles of length 3. The absence of strict preference in cycles of length 1 and 2 is always guaranteed. Since P is asymmetric, it cannot be involved in cycles of length 1. Concerning cycles of length 2, by the additive property, $R(a,b) \leq 1 - P(b,a)$ for all a, b, and this implies $1 - f^d(P(a,b), P(b,a)) \geq f(R(a,b), R(b,a))$.

4 Strongly complete fuzzy preorders

The only characterization known for the transitivity of a reflexive fuzzy relation R is proven by De Baets, Van De Walle and Kerre [4] under strong completeness. It involves the transitivity of P and I and the two compositions between them,

$$R \circ_T R \subseteq R \quad \Leftrightarrow \quad \begin{cases} P \circ_{T_\mathrm{M}} P \subseteq P, \\ I \circ_T I \subseteq I, \\ P \circ_{T_\mathrm{L}} I \subseteq P, \\ I \circ_{T_\mathrm{L}} P \subseteq P. \end{cases}$$

There is a particular case for which this characterization can be improved. We have proven that the implication from the transitivity of R to the transitivity of P and I can be generalized.

Proposition 4.1 *Let R be a strongly complete reflexive fuzzy relation R with strict preference relation P and indifference relation I, and f a conjunctor such that* $\max(f(1,x), f(x,1)) = x$, $\forall x$. *It holds that*

$$R \circ_f R \subseteq R \quad \Rightarrow \quad \begin{cases} P \circ_{T_\mathrm{M}} P \subseteq P, \\ I \circ_f I \subseteq I. \end{cases}$$

The converse implication does not hold for all of these conjunctors.

Proposition 4.2 *Let R be a strongly complete reflexive fuzzy relation and P and I its asymmetric and symmetric components. It holds that*

$$\left. \begin{array}{l} P \circ_{T_\mathrm{M}} P \subseteq P \\ I \circ_f I \subseteq I \end{array} \right\} \quad \Rightarrow \quad R \circ_f R \subseteq R,$$

for all f such that $\max(f(x,y), f(y,x)) = \min(x,y), \forall (x,y)$.

This implication does not hold for any other conjunctor f involved in Proposition 4.1. Thus, the equivalence only holds for the conjunctors considered in Proposition 4.2. The only commutative conjunctor (and therefore the only t-norm) that satisfies the equivalence is the minimum t-norm.

Property (III) does not characterize the transitivity of R. On the one hand, we have proven that

$$R \circ_f R \subseteq R \not\Rightarrow \begin{cases} P \circ_{T_\mathrm{M}} P \subseteq P \\ P \circ_{T_\mathrm{L}} I \subseteq P \end{cases}, \quad \forall f < T_\mathrm{L}.$$

On the other hand,

$$\left. \begin{array}{l} P \circ_{T_\mathrm{M}} P \subseteq P \\ P \circ_{T_\mathrm{L}} I \subseteq P \end{array} \right\} \not\Rightarrow R \circ_f R \subseteq R, \quad \forall f \geq T_\mathrm{L}.$$

If we replace the composition $P \circ_{T_\mathrm{L}} I$ by $I \circ_{T_\mathrm{L}} P$, the equivalence neither holds for any conjunctor f. Property (IIIa) neither characterizes the transitivity of a reflexive fuzzy relation R.

Let us study now the relationship between f-transitivity and negative f^d-transitivity. The following equivalence can be found in [2, 9].

Proposition 4.3 *Let R be a fuzzy relation. It holds that*

R is T-transitive $\Leftrightarrow$ R^d is negatively T-transitive.

The asymmetric component of a crisp complete reflexive relation is its dual, $P = R^d$. The same equality holds for any strongly complete reflexive fuzzy relation. Then, for any strongly complete reflexive fuzzy relation, it holds that it is T-transitive if and only if its asymmetric component is negatively S-transitive. It is easy to check that the property holds for any commutative conjunctor.

Corollary 4.4 *Let R be a strongly complete reflexive fuzzy relation, P the corresponding strict preference relation and f a commutative conjunctor. It holds that*

R is f-transitive $\Leftrightarrow$ P is negatively f^d-transitive.

Concerning Property (V), the f-absence of strict preference in cycles of length 3 ensures the f-transitivity of R.

Proposition 4.5 *Let R be a strongly complete reflexive fuzzy relation and P its asymmetric component. It holds that*

$$1 - f^d(P(a,b), P(b,c), P(c,a)) \geq f(R(a,b), R(b,c), R(c,a)), \quad \forall\, a,b,c$$

$$\Downarrow$$

$$R \text{ is } f\text{-transitive},$$

for every associative conjunctor f such that $f(1,\cdot) = f(\cdot,1)$ is strictly increasing.

Observe that the implication holds for any conjunctor with neutral element 1 and in particular, for any t-norm.

However, the converse implication does not hold for any of these operators except for the minimum.

Proposition 4.6 *Let R be a strongly complete reflexive fuzzy relation and P its asymmetric component. It holds that*

(i)

$$R \text{ is } T_{\rm M}\text{-transitive}$$

$$\Downarrow$$

$$1 - S_{\rm M}(P(a,b), P(b,c), P(c,a)) \geq T_{\rm M}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a,b,c.$$

(ii)

$$R \text{ is } f\text{-transitive}$$

$$\not\Downarrow$$

$$1 - f^d(P(a,b), P(b,c), P(c,a)) \geq f(R(a,b), R(b,c), R(c,a)), \quad \forall\, a,b,c$$

for any associative conjunctor f, such that $f(1,\cdot) = f(\cdot,1)$ is strictly increasing.

Therefore, the minimum is the only t-norm satisfying the equivalence.

5 In absence of incomparability

In the context of fuzzy sets, the absence of incomparability is a weaker condition than the strong completeness condition considered in the previous section, and stronger than weak completeness. We proved (see [8]) that the incomparability relation associated to a weakly complete reflexive fuzzy relation is empty if and only if the preference structure is generated from R by means of the Łukasiewicz t-norm, $i = T_{\rm L}$. In this section we study the characterizations obtained for the transitivity of strongly complete reflexive fuzzy relations under this condition.

When we consider reflexive fuzzy relations without incomparability, the $T_{\rm M}$-transitivity of P does not follow from the $T_{\rm M}$-transitivity of R ([6]). The converse implication holds for any conjunctor considered in Proposition 4.2. We have proven that absence of incomparability is equivalent to strong completeness when P is $T_{\rm M}$-transitive.

Lemma 5.1 *Let R be a reflexive fuzzy relation R with corresponding preference structure $(P, I, \emptyset)$. It holds that*

$$P \text{ is } T_{\rm M}\text{-transitive} \Rightarrow R \text{ is strongly complete}.$$

As a consequence, Proposition 4.2 holds in absence of incomparability.

Figure 2: Example E10 presented by Dasgupta and Deb in [5].

R	a	b	c
a	1	0.4	0.2
b	0.6	1	0.4
c	0.8	0.8	1

P	a	b	c
a	0	0.4	0.8
b	0.6	0	0.2
c	0.2	0.6	0

This result seems to contradict one given by Dasgupta and Deb in [5]. They present a counterexample showing that the $T_{\rm M}$-transitivity of P and I, together with the conditions $P \circ_{T_{\rm M}} I \subseteq P$ and $I \circ_{T_{\rm M}} P \subseteq P$, are not sufficient to ensure the $T_{\rm M}$-transitivity of R. However, the strict preference relation P recalled in Figure 2 fails to be $T_{\rm M}$-transitive,

$$P(a,a) = 0 < T_{\rm M}(0.4, 0.6) = T_{\rm M}(P(a,b), P(b,a)).$$

We have checked that if we add the two conditions on the $T_{\rm M}$-composition of P and I to the $T_{\rm M}$-transitivity of P and I, the $T_{\rm M}$-transitivity of R is guaranteed even in absence of completeness.

Proposition 5.2 *Let R be a reflexive fuzzy relation and P and I the corresponding strict preference and indifference relations generated from R by means of a generator i. It holds that*

$$\left.\begin{array}{l} P \circ_{T_{\rm M}} P \subseteq P \\ I \circ_{T_{\rm M}} I \subseteq I \\ P \circ_{T_{\rm M}} I \subseteq P \\ I \circ_{T_{\rm M}} P \subseteq P \end{array}\right\} \Rightarrow R \circ_{T_{\rm M}} R \subseteq R\,.$$

The absence of the strict preference relation in cycles, i.e. Property (V), does not guarantee the transitivity of R for any associative conjunctor f, in particular, for any t-norm.

$$1 - f^d(P(a,b), P(b,c), P(c,a)) \geq$$
$$f(R(a,b), R(b,c), R(c,a)), \quad \forall\, a, b, c$$
$$\not\Downarrow$$
$$R \text{ is } f\text{-transitive.}$$

The implication from the $T_{\rm M}$-transitivity of R to the absence of cycles proven in the strongly complete case, holds in this more general situation.

Proposition 5.3 *Let R be a weakly complete reflexive fuzzy relation and P the corresponding strict preference relation generated from R by means of $i = T_{\rm L}$. It holds that*

$$R \textit{ is } T_{\rm M}\textit{-transitive}$$
$$\Downarrow$$
$$1 - S_{\rm M}(P(a,b), P(b,c), P(c,a)) \geq$$
$$T_{\rm M}(R(a,b), R(b,c), R(c,a)), \quad \forall\, a, b, c.$$

6 Conclusion

We have studied different characterizations known for crisp complete preorders in the fuzzy set context. We have considered two completeness conditions: strong completeness and absence of incomparability. We have checked that the equivalences known for crisp relations only hold for particular conjunctors in the strongly complete case. We have also shown that some of those equivalences get lost in absence of incomparability.

Acknowledgments

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