Towards Formal Theory of Measure on Clans of Fuzzy Sets

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Abstract

Measures on clans of fuzzy sets are traditionally studied as certain functionals on collections of $[0, 1]$-valued functions. The goal of this paper is to make first steps towards establishing this part of fuzzy mathematics as a formal theory over a suitable predicate fuzzy logic. Two concepts of formal measure theory are sketched within Henkin-style ω-order fuzzy logic.

Keywords: Measure theory, Measures on clans, Fuzzy mathematics, Higher-order fuzzy logic.

1 Introduction

Classical measure theory [8] is dealing with real-valued set functions which are defined on Boolean algebras of (crisp) sets and satisfy some variant of additivity condition. Probability theory as well as a large part of modern analysis are only two examples of mathematical disciplines which are based on measure-theoretical apparatus. Research on measures of fuzzy sets was initiated by Zadeh [13]. The theory was further elaborated by Butnariu, Klement, Mesiar, Navara and others and applied to theory of fuzzy coalition games [5]. The most up-to-date treatment of measures on collections of fuzzy sets (so-called clans and their σ-completions, tribes) is [12] by Navara. We adopt exactly the concept of fuzzy measure theory developed in the above mentioned publications, that is, a treatment of fuzzy measures as real-valued functionals defined on certain collections of $[0, 1]$-valued functions.

This paper is based on the methodology proposed in [2]. Following this program means studying individual disciplines of fuzzy mathematics as formal theories within a general fuzzy logic formalism. This formal approach already proved to be fruitful in theory of fuzzy relations [4] and was applied to fuzzy real numbers [1].

The paper is structured as follows. In Section 2 we summarize basic definitions regarding higher-order fuzzy logic which are needed in the rest of the paper. Section 3 introduces a possible axiomatization of the notion of a clan of fuzzy sets. In Section 4 we study two approaches to formalization of measures on clans.

2 Higher-order Fuzzy Logic

We assume the familiarity with Hájek's Basic fuzzy logic BL and the logic ŁΠ which combines all three basic logics (Łukasiewicz, Gödel and product) and their multi-sorted and predicate variants. See [9] and [6] for details. We recall only basic and some of derived connectives of ŁΠ; they are listed with their standard semantics in the standard ŁΠ algebra $[0, 1]$:

0	0 (truth constant falsum)
$\varphi \to_{\text{Ł}} \psi$	$x \to_{\text{Ł}} y = \min(1, 1 - x + y)$
$\varphi \to_{\Pi} \psi$	$x \to_{\Pi} y = \min(1, \frac{y}{x})$
$\varphi \mathbin{\&_{\Pi}} \psi$	$x \mathbin{\&_{\Pi}} y = x \cdot y$
$\neg_{\text{Ł}} \varphi$	$\neg_{\text{Ł}} x = 1 - x$
$\Delta\varphi$	$\Delta x = 1$ if $x = 1$, otherwise 0
$\varphi \mathbin{\&_{\text{Ł}}} \psi$	$x \mathbin{\&_{\text{Ł}}} y = \max(0, x + y - 1)$
$\varphi \oplus_{\text{Ł}} \psi$	$x \oplus_{\text{Ł}} y = \min(1, x + y)$
$\varphi \leftrightarrow_{\text{Ł}} \psi$	$x \leftrightarrow_{\text{Ł}} y = 1 - \lvert x - y \rvert$
$\varphi \vee \psi$	$x \vee y = \max(x, y)$

Many other connectives based on continuous t-norms is definable within ŁΠ (or $Ł\Pi\frac{1}{2}$). In this case we use the notations $\neg_*, \&_*, \oplus_*, \rightarrow_*$ for a negation, conjunction, disjunction and implication, respectively, where $*$ denotes the respective t-norm. For instance, the symbol $\neg_Ł, \neg_\Pi$ and $\neg_G$ stands for Łukasiewicz, product and Gödel negation, respectively. Whenever the index of a connective does not matter (for example, if the connective is applied to crisp subformulae), we omit it for the sake of simplicity.

The next definition introduces a general logic formalism of [3].

Definition 1 *The* Henkin-style second-order fuzzy logic $Ł\Pi_2$ *is a theory over multi-sorted predicate logic ŁΠ with sorts for objects (lowercase letters) and classes (uppercase letters). Both of the sorts subsume subsorts for n-tuples ($n \geq 1$). Apart from the usual function symbols and axioms for tuples (tuples equal iff their respective constituents equal), the only primitive symbol is the membership predicate ∈ between objects and classes. The axioms for ∈ are the following:*

1. *Comprehension axiom*

$$(\exists X)\Delta(\forall x)(x \in X \leftrightarrow_* \varphi),$$

where φ does not contain X, enables the (eliminable) introduction of comprehension terms $\{x \mid \varphi\}$ with the axiom

$$y \in \{x \mid \varphi(x)\} \leftrightarrow_* \varphi(y)$$

(allowing φ to contain other comprehension terms).

2. *Extensionality axiom*

$$(\forall x)\Delta(x \in X \leftrightarrow_* x \in Y) \leftrightarrow X = Y.$$

Standard models of classes encompass Zadeh-like fuzzy sets. In this paper we are going to deal with classes of classes and thus we actually need a fuzzy logic whose order is higher than two.

Definition 2 Henkin-style fuzzy logic of higher orders *is obtained by repeating the previous definition on each level of the type hierarchy. Obviously, defined symbols of any type can then be shifted to all higher types as well. (Consequently, all theorems are preserved by uniform upward type-shifts.) Types may be allowed to subsume all lower types. Henkin-style fuzzy logic ŁΠ of order n will be denoted by $Ł\Pi_n$, the whole hierarchy by $Ł\Pi_\omega$.*

Convention 1 *Classes on the third level of the hierarchy are denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$*

Convention 2 *A formula $(\forall x)(x \in X \rightarrow_* \varphi)$ may be abbreviated as $(\forall x \in X)_*\varphi$ and an analogous notation may be used for binary predicates. An alternative notation for $x \in A$ and $\langle x_1, \ldots, x_n\rangle \in R$ is Ax and $Rx_1 \ldots x_n$, respectively.*

Convention 3 *A formula $x \notin X$ stands for $\neg_Ł(x \in X)$.*

We will need some of the elementary fuzzy set operations definable in $Ł\Pi_2$:

$$\begin{aligned}
\emptyset &=_{\mathrm{df}} \{x \mid 0\}\\
\mathrm{V} &=_{\mathrm{df}} \{x \mid 1\}\\
-X &=_{\mathrm{df}} \{x \mid x \notin X\}\\
X \cap_* Y &=_{\mathrm{df}} \{x \mid x \in X \,\&_*\, y \in Y\}\\
X \cup_* Y &=_{\mathrm{df}} \{x \mid x \in X \oplus_* y \in Y\}\\
X - Y &=_{\mathrm{df}} \{x \mid x \in X \,\&_Ł\, y \notin Y\}
\end{aligned}$$

In particular, V is a so-called *universe of discourse.* The unary operation − is *involutive complement* of a fuzzy set. A meaning of the below introduced (fuzzy) relations is clear:

$$\begin{aligned}
\mathrm{Crisp}(X) &\equiv_{\mathrm{df}} (\forall x)\Delta(x \in X \vee x \notin X)\\
X \subseteq_* Y &\equiv_{\mathrm{df}} (\forall x \in X)_*(x \in Y)
\end{aligned}$$

In the sequel, the symbol = denotes crisp equality. For any fuzzy relation R, we put:

$$\begin{aligned}
\mathrm{Dom}(R) &\equiv_{\mathrm{df}} \{x \mid \langle x, y\rangle \in R\}\\
\mathrm{Rng}(R) &\equiv_{\mathrm{df}} \{y \mid \langle x, y\rangle \in R\}\\
\mathrm{Fnc}_*(R) &\equiv_{\mathrm{df}} (\forall x)(\forall y)(\forall z)\\
&\qquad (Rxy \,\&_*\, Rxz \rightarrow_* y = z)
\end{aligned}$$

In particular, Fnc_* is a property of being a function with respect to $*$. Observe that if $\mathrm{Fnc}_*(f)$ for some fuzzy relation f, then we may write $y = f(x)$ instead of fxy as usual.

3 Theory of Clans

Clans of fuzzy sets are generalizations of Boolean algebras of sets. We are going to introduce a formal theory of clans as a theory over $Ł\Pi_\omega$. In addition, we relax the assumption that clans are crisp at the very beginning.

Definition 3 *Let $\mathcal{C}$ be a constant standing for a fuzzy set of fuzzy sets. The* theory of fuzzy clans *is a theory with the following axioms:*

(C1) $\emptyset \in \mathcal{C}$

(C2) $(\forall A \in \mathcal{C})_*(-A \in \mathcal{C})$

(C3) $(\forall A, B \in \mathcal{C})_*(A \cup_L B \in \mathcal{C})$

The last formula should be read as

$$(\forall A)(\forall B)((A \in \mathcal{C} \,\&_*\, B \in \mathcal{C}) \rightarrow_* (A \cup_Ł B \in \mathcal{C})).$$

For the sake of simplicity, we confined to Łukasiewicz union although its replacement by any other definable operation is possible. The constant $\mathcal{C}$ is represented in models of theory of fuzzy clans by a fuzzy set of fuzzy sets which contains the empty set in the degree 1 and satisfies conditions given by (C2) and (C3).

Proposition 1 *These are provable formulae in the theory of fuzzy clans:*

1. $\mathrm{V} \in \mathcal{C}$
2. $(\forall A, B \in \mathcal{C})_*(A \cap_L B \in \mathcal{C})$
3. $(\forall A, B \in \mathcal{C})_*(A \cup_G B \in \mathcal{C})$
4. $(\forall A, B \in \mathcal{C})_*(A \cap_G B \in \mathcal{C})$

Proof. 1. Putting together axioms (C1) and (C2) from Definition 3, we get immediately $\mathrm{V} \in \mathcal{C}$.
2. Since the formula $A \cap_Ł B \in \mathcal{C}$ is provably equivalent to $-(-A \cup_Ł -B) \in \mathcal{C}$, it follows from (C2) and (C3) that the second formula is provable.
3. The expression $A \cap_G B$ may be rewritten as $A \cap_Ł (-A \cup_Ł B)$.
4. $A \cup_G B$ is the same as $-(-A \cap_G -B)$. □

We can also prove the formulae:

$$(\forall A_1, \ldots, A_n \in \mathcal{C})_*(A_1 \cup_Ł \ldots \cup_Ł A_n \in \mathcal{C}),$$

$$(\forall A_1, \ldots, A_n \in \mathcal{C})_*(A_1 \cap_Ł \ldots \cap_Ł A_n \in \mathcal{C}),$$

$$(\forall A_1, \ldots, A_n \in \mathcal{C})_*(A_1 \cup_G \ldots \cup_G A_n \in \mathcal{C}),$$

$$(\forall A_1, \ldots, A_n \in \mathcal{C})_*(A_1 \cap_G \ldots \cap_G A_n \in \mathcal{C}).$$

Since the previous formulae can be rewritten as implications each having the same conjunction

$$A_1 \in \mathcal{C} \,\&_* \ldots \&_*\, A_n \in \mathcal{C}$$

in the antecedent, the respective implications can be quite weak as, for instance, in case of Łukasiewicz conjunction $\&_Ł$. This difficulty can be simply overcome by dealing rather with crisp sets of fuzzy sets than fuzzy sets of fuzzy sets. This assumption is in fact the traditional approach adopted in the field of measures on clans of fuzzy sets. Moreover, we avoid in this way any discussion inevitably related to a concept of 'fuzzy measurability' of elements belonging to fuzzy clans.

The *theory of clans* is an extension of theory of fuzzy clans by the axiom of crispness for $\mathcal{C}$:

(C4) $\mathrm{Crisp}(\mathcal{C})$

Intended models of the theory of clans are exactly the clans of fuzzy sets in the sense of Butnariu and Klement [5], that is, collections of fuzzy sets containing the empty set and closed with respect to involutive complement and Łukasiewicz union. Since clans were introduced to generalize Boolean algebras of sets, we can expect that any clan contains one. We put

$$\mathbf{B}(\mathcal{C}) =_{\mathrm{df}} \{A \mid A \in \mathcal{C} \,\&\, A \cup_Ł A = A\}$$

and call $\mathbf{B}(\mathcal{C})$ a *Boolean skeleton* of $\mathcal{C}$. Observe that $\mathbf{B}(\mathcal{C}) \subseteq \mathcal{C}$ and $\mathbf{B}(\mathcal{C})$ is crisp from the definition. In any model of the theory of clans, the constant $\mathbf{B}(\mathcal{C})$ is represented by a Boolean algebra of sets. The next theorem enables to interpret algebras of sets as particular cases of clans.

Proposition 2 *This formula is provable in the theory of clans:*

$$(\forall A \in \mathcal{C})(\mathrm{Crisp}(A)) \rightarrow \mathcal{C} = \mathbf{B}(\mathcal{C})$$

Proof. We want to show that $\mathcal{C} \subseteq \mathbf{B}(\mathcal{C})$ which is provably equivalent to $A \in \mathcal{C} \to A \in \mathbf{B}(\mathcal{C})$. It follows by modus ponens from the premises $A \in \mathcal{C}$ and $A \in \mathcal{C} \to \mathrm{Crisp}(A)$ that $\mathrm{Crisp}(A)$. Since $\mathrm{Crisp}(A) \to A \cup_{\text{Ł}} A = A$, we obtain immediately $A \in \mathbf{B}(\mathcal{C})$. □

4 Theory of Measure

We are going to present two different formalizations (theories TM_1 and TM_2) capturing the notion of measure of fuzzy sets. We restrict our considerations only to finitely additive measures; it was demonstrated in [11] that a reasonably deep theory can be developed even in this general case.

Since the ω-order fuzzy logic ŁΠ_ω contains the classical type theory, we may introduce arbitrary theories definable in classical type theory within ŁΠ_ω. In the sequel we use the theory of the linearly ordered group $\langle \mathbb{R}, \leq, + \rangle$ of real numbers.

Definition 4 *The* theory of measure TM_1 *is an extension of the theory of clans obtained by adding a symbol* m *for a fuzzy relation together with the following axioms:*

(M1) $\mathrm{Dom}(m) = \mathcal{C}$

(M2) $\mathrm{Crisp}(\mathrm{Rng}(m))$

(M3) $\mathrm{Rng}(m) \subseteq \mathbb{R}$

(M4) $\mathrm{Fnc}(m)$

(M5) $m(\emptyset) = 0$

(M6) $(\forall A, B \in \mathcal{C})_*(A \cap_L B = \emptyset \to m(A \cup_L B) = m(A) + m(B))$

Axioms (M1)-(M4) ensure that the symbol m can be viewed as a mapping $\mathcal{C} \to \mathbb{R}$ in models of TM_1. Axiom (M5) together with (M5) means that models of the theory of measure are just (Łukasiewicz) measures on clans of fuzzy sets in the sense of Butnariu and Klement [5].

Proposition 3 *The following are provable formulae in the theory of measure* TM_1*:*

1. $(\forall A \in \mathcal{C})_*(m(-A) = m(\mathrm{V}) - m(A))$
2. $(\forall A \in \mathcal{C})_*(0 \leq m(A)) \to (\forall A, B \in \mathcal{C})(A \subseteq B \to m(A) \leq m(B))$
3. $(\forall A, B \in \mathcal{C})_* (m(A \cup_L B) + m(A \cap_L B) = m(A) + m(B))$
4. $(\forall A, B \in \mathcal{C})_* (m(A \cup_G B) + m(A \cap_G B) = m(A) + m(B))$
5. $(\forall A, B \in \mathcal{C})_*(A \cap_G B = \emptyset \to m(A \cup_G B) = m(A) + m(B))$

Proof.
1. We have $A \cup_{\text{Ł}} -A = \mathrm{V}$ and $A \cap_{\text{Ł}} -A = \emptyset$. By applying (M6) to A and $-A$, we get $m(\mathrm{V}) = m(A) + m(-A)$.
2. We obtain from elementary relations between Łukasiewicz operations and involutive complement

$$A \cap_{\text{Ł}} (B \cap_{\text{Ł}} (-A)) = \emptyset \qquad (1)$$

and

$$A \cup_{\text{Ł}} (B \cap_{\text{Ł}} (-A)) = A \cup_G B. \qquad (2)$$

Hence

$$m(A) + m(B \cap_{\text{Ł}} (-A)) = m(A \cup_G B)$$

from (M6). Since $A \subseteq B$ (assumption), we get $A \cup_G B = B$. Consequently, $m(A \cup_G B) = m(B)$ and

$$m(A) + m(B \cap_{\text{Ł}} (-A)) = m(B).$$

The premise $0 \leq m(B \cap_{\text{Ł}} (-A))$ finally implies $m(A) \leq_{\text{Ł}} m(B)$.
3. Elementary relations between Łukasiewicz operations combined with (M6) imply the provability of the identities below:

$$\begin{aligned} m(A \cup_{\text{Ł}} B) + m(A \cap_{\text{Ł}} B) &= \\ m(A - B) + 2m(A \cap_{\text{Ł}} B) + m(B - A) &= \\ m(A) + m(A \cap_{\text{Ł}} B) + m(B - A) &= \\ m(A) + m(B). \end{aligned}$$

4. From the premise $A \in \mathcal{C}$ and $B \in \mathcal{C}$ we obtain the following list of provable identities:

$$\begin{aligned} m(A \cup_G B) + m(A \cap_G B) &= \\ m((A \cup_G B) \cup_{\text{Ł}} ((A \cap_G B)) + & \\ m((A \cup_G B) \cap_{\text{Ł}} (A \cap_G B)) &= \\ m(A \cup_{\text{Ł}} B) + m(A \cap_{\text{Ł}} B) &= \\ m(A) + m(B). \end{aligned}$$

5. A direct consequence of 4. and (M5). □

Thus, in particular, measures on clans of fuzzy sets are monotone whenever they are non-negative (property 2.) and properties 3. and 4. mean that measures are so-called *valuations* both with respect to Łukasiewicz and Gödel operations.

Results obtained in Proposition 3 were already known and proved as theorems of classical mathematics as assertions about certain properties of functionals defined on collections of $[0,1]$-valued functions. Yet we believe that by stating those results in a purely formal framework may result in some further insight and lead towards their systematization by virtue of a sound axiomatic background based on huge expressive power of the presented higher-order fuzzy logic.

We will present an alternative concept to TM_1. As a starting point, consider the following motivation. Any measure μ on a Boolean algebra of sets $\mathcal{A}$ attaining only values $\{0,1\}$ is in one-to-one correspondence with the family of 'large' sets (a so-called *filter*) in $\mathcal{A}$; namely, those where μ takes a value 1. An analogous version of this claim holds true also for measures on clans; certain measures may thus be identified with crisp subsets of clans. In the light of this fact, introducing measure as a fuzzy set of fuzzy sets does not necessarily seem unnatural. A similar approach was already presented by Höhle in [10].

Definition 5 *The* theory of measure TM_2 *is a theory over $Ł\Pi_\omega$ with an additional symbol $\mathcal{M}$ standing for a fuzzy set of fuzzy sets and satisfying the following axioms:*

(M1') $\mathrm{V} \in \mathcal{M}$

(M2') $(\forall A, B)(A \subseteq B \rightarrow$
$((A \in \mathcal{M} \oplus_L B - A \in \mathcal{M}) \leftrightarrow_L B \in \mathcal{M}))$

(M3') $(\forall A, B)(A \cap_L B = \emptyset \rightarrow$
$(A \in \mathcal{M} \rightarrow_L B \notin \mathcal{M}))$

A model of TM_2 is a fuzzy set of 'large' fuzzy sets: we may introduce a unary predicate Large by

$$\mathrm{Large}(A) \equiv_{\mathrm{df}} A \in \mathcal{M}.$$

Axiom (M1') means that the universal set is large (in degree 1) and axiom (M2') postulates that a superset B is large iff its subset A is large or the involutive difference $B - A$ is large. Finally, axiom (M3') says that disjoint sets cannot be both large.

Proposition 4 *The following are provable formulae in the theory of measure TM_2:*

1. $\emptyset \notin \mathcal{M}$
2. $(\forall A, B)(A \subseteq B \rightarrow (A \in \mathcal{M} \rightarrow_L B \in \mathcal{M}))$
3. $(\forall A, B)(A \cap_L B = \emptyset \rightarrow$
   $(A \cup_L B \in \mathcal{M} \leftrightarrow_L (A \in \mathcal{M} \oplus_L B \in \mathcal{M})))$
4. $(\forall A)(A \in \mathcal{M} \leftrightarrow_L -A \notin \mathcal{M})$

Proof.
1. Apply (M3') with $A = \mathrm{V}$ and $B = \emptyset$.
2. The premises are $A \subseteq B$ and $A \in \mathcal{M}$. We get $A \in \mathcal{M} \oplus_{Ł} B - A \in \mathcal{M}$ from $(M2')$ together with the second premise and hence $B \in \mathcal{M}$.
3. From the assumption $A \cap_{Ł} B = \emptyset$ we obtain $(A \cup_{Ł} B) - A = B$. The conclusion then follows from (M2') as $A \subseteq A \cup_{Ł} B$ and thus

$$A \cup_{Ł} B \in \mathcal{M} \leftrightarrow_{Ł} (A \in \mathcal{M} \oplus_{Ł} B \in \mathcal{M}).$$

4. The implication $A \in \mathcal{M} \rightarrow_{Ł} -A \notin \mathcal{M}$ is a consequence of (M3') since $A \cap_{Ł} -A = \emptyset$. Further, we have

$$A \cup_{Ł} -A \in \mathcal{M} \leftrightarrow_{Ł} (A \in \mathcal{M} \oplus_{Ł} -A \in \mathcal{M})$$

from the previous proof and hence $A \in \mathcal{M} \oplus_{Ł} -A \in \mathcal{M}$, which can be rewritten as $-A \notin \mathcal{M} \rightarrow_{Ł} A \in \mathcal{M}$. □

The interpretation of properties in the proposition above is evident:

1. empty set is not large;
2. superset of a large subset is large;
3. 'additivity';
4. set is large iff its involutive complement is not large.

Observe that some of the properties from Proposition 4 are in fact analogues of properties of TM_1 from Proposition 3.

5 Conclusions

It is evident that the two introduced formal axiomatizations TM_1 and TM_2 give rise to same models of measures as real-valued mappings defined on fuzzy sets. While TM_1 completely mimicks the classical definition of measure as a mapping, the theory TM_2 accentuates correspondence of measures and filters and the notion of a 'large' set and can be thus viewed a a certain fuzzification of a notion of filter. The theory TM_1 is based on (classical) theory of real ordered group; on the other hand, measures according to TM_2 are purely internal objects (fuzzy sets) of the Henkin-style higher-order fuzzy logic. This approach is in a way similar to [7].

We presented first steps directed to a formalization of measure theory; we are convinced that a formal study of measures and clans may lead to new concepts (such as the theory TM_2) and shed a new light even on the well-known results. In the future we intend to generalize TM_1 to fuzzy number-valued measure on clans. Last but not least, observe that the proofs of theorems (in fact formal proofs within fuzzy logic) appearing in this paper strongly resemble classical proofs.

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