Extending the Monoidal T-norm Based Logic with an Independent Involutive Negation

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Abstract

In this paper we investigate the logic $\mathbf{MTL}_{\sim}$ obtained by extending Esteva and Godo's logic **MTL** with an involutive negation $\sim$ not dependent on the t-norm and with the Baaz's operator Δ. Moreover, we also introduce and study the related predicate calculus $\mathbf{MTL}\forall_{\sim}$. Algebraic and standard completeness results are shown to hold for both $\mathbf{MTL}_{\sim}$ and $\mathbf{MTL}\forall_{\sim}$.

Keywords: Many-valued logics, Left-continuous t-norms, Involutive negations.

1 Introduction

The Monoidal T-norm based Logic **MTL** was introduced by Esteva and Godo in [4], and was shown in [7], by Jenei and Montagna, to be standard complete w.r.t. **MTL**-algebras over the real unit interval, i.e. algebras defined by left-continuous t-norms and their residua. In this logic a negation is definable from the implication and the truth constant $\overline{0}$, so that $\neg\varphi$ stands for $\varphi \to \overline{0}$. This negation behaves quite differently depending on the chosen left-continuous t-norm and in general is not an involution. Such an operator can be forced to be involutive by adding the axiom $\neg\neg\varphi \to \varphi$ to **MTL**. The system so obtained was called in [4] **IMTL** (Involutive Monoidal T-norm based Logic). However, in such a logic the involution does depend on the t-norm, so that **IMTL** singles out only those left-continuous t-norms which yield an involutive negation. Clearly, operators like Gödel and Product t-norms are ruled out. This motivated then the interest in studying a logic of left-continuous t-norms with an independent involutive negation. Our approach is somehow related to the one carried out in [5]. Indeed we investigate the logic obtained by adding to **MTL** a unary connective $\sim$ and axioms which capture the behavior of involutive negations. Moreover, we also introduce in **MTL** the operator Δ [1], which resulted to be fundamental to prove the subdirect representation theorem for $\mathbf{MTL}_{\sim}$-algebras.

In the first part of this paper we introduce the logic $\mathbf{MTL}_{\sim}$, the variety of $\mathbf{MTL}_{\sim}$-algebras (its algebraic structures) and we provide algebraic and standard completeness results. Finally, we introduce the predicate calculus $\mathbf{MTL}\forall_{\sim}$ and prove both standard and algebraic completeness.

2 $\mathbf{MTL}_{\sim}$ and $\mathbf{MTL}_{\sim}$-algebras

Definition 1 *The logic* $\boldsymbol{MTL}_{\sim}$ *is obtained by adding to* ***MTL*** *the operator* Δ *and the related axioms* $(\Delta 1) - (\Delta 5)$ *(see* [1, 6]*), and the unary connective* $\sim$ *and the axioms:*

(~ 1) $\sim \overline{0}$,

(~ 2) $\sim\sim \varphi \equiv \varphi$,

(~ 3) $\Delta(\varphi \to \psi) \to (\sim \psi \to \sim \varphi)$,

where, for any formula ψ *and* γ, $\psi \equiv \gamma$ *stands for* $(\psi \to \gamma)\&(\gamma \to \psi)$.

Deduction rules for $\boldsymbol{MTL}_{\sim}$ *are* modus ponens $(\frac{\varphi \quad \varphi\to\psi}{\psi})$ *and* generalization $(\frac{\varphi}{\Delta\varphi})$.

Definition 2 *An* $\boldsymbol{MTL}_{\sim}$*-algebra is a structure* $\mathcal{L} = \langle L, \sqcap, \sqcup, \star, \Rightarrow, n, \delta, 0, 1\rangle$ *such that* $\langle L, \sqcap, \sqcup, \star, \Rightarrow, 0, 1\rangle$ *is an* ***MTL****-algebra; for every*

$x, y \in L$, δ satisfies the properties of Baaz's projection [1]*; moreover the following properties hold:*

$(n1)$ $n(0) = 1$,

$(n2)$ $n(n(x)) = x$,

$(n3)$ $\delta(x \Rightarrow y) \leq (n(y) \Rightarrow n(x))$.

The notions of *theory*, *proof* and *model* are as usual. The notion of *evaluation* e from formulas into an $\mathbf{MTL}_\sim$-algebra $\mathcal{L}$ is extended by requiring: $e(\sim \varphi) = n(e(\varphi))$.
Similarly to $\mathbf{BL}_\Delta$ ([6]), for $\mathbf{MTL}_\sim$ a deduction theorem in its usual form fails ($\varphi \vdash \Delta\varphi$, but for each $n \in \mathbb{N}$, $\not\vdash \varphi^n \to \Delta\varphi$). Anyway we get the following (weaker) formulation:

Theorem 1 *Let T be a theory over $\boldsymbol{MTL}_\sim$ and let φ, ψ be two formulas. Then $T \cup \{\varphi\} \vdash \psi$ iff $T \vdash \Delta\varphi \to \psi$.*

Since all the axioms of $\mathbf{MTL}_\sim$-algebras are expressed in an equational way, it is clear that the class of all $\mathbf{MTL}_\sim$-algebras constitutes a variety. Moreover, since the axioms for $\mathbf{MTL}_\sim$-algebras are just the algebraic translation of the axioms of $\mathbf{MTL}_\sim$, it is clear that the Lindenbaum sentence algebra of $\mathbf{MTL}_\sim$ is (termwise equivalent to) the free $\mathbf{MTL}_\sim$-algebra on countably many generators. Therefore $\mathbf{MTL}_\sim$ is sound and strongly complete w.r.t. the class of all $\mathbf{MTL}_\sim$-algebras.
Now we are going to prove that $\mathbf{MTL}_\sim$ is complete w.r.t. the class of all linearly ordered $\mathbf{MTL}_\sim$-algebras. In the sequel we will use the following notation: $\mathbf{MTL}_\Delta$ stands for the logic obtained by adding to $\mathbf{MTL}$ the connective Δ and the axioms $(\Delta 1 - \Delta 5)$. Similarly $\mathbf{MTL}_\Delta$-algebras are the algebraic counterpart of $\mathbf{MTL}_\Delta$.
We follow the line of [6] for the completeness of $\mathbf{BL}_\Delta$ w.r.t. the class of all linearly ordered $\mathbf{BL}_\Delta$-algebras, and introduce the notion of *filter over an* $\mathbf{MTL}_\Delta$-algebra $\mathcal{L}$ by requiring F to be a filter over the $\mathbf{MTL}$-reduct of $\mathcal{L}$, plus the further condition

$$x \in F \text{ implies } \delta(x) \in F. \tag{1}$$

It is now easy to show that the following holds:

Theorem 2 *The following are equivalent:*

(i) $\boldsymbol{MTL}_\Delta \vdash \varphi$,

(ii) *φ is an $\mathcal{L}$-tautology for each $\boldsymbol{MTL}_\Delta$-algebra,*

(iii) *φ is an $\mathcal{L}$-tautology for each linearly ordered $\boldsymbol{MTL}_\Delta$-agebra.*

The next step is to use Theorem 2 in order to show that each $\mathbf{MTL}_\sim$-algebra can be decomposed as subdirect product of a class of linearly ordered $\mathbf{MTL}_\sim$-algebras. We use the following:

Lemma 1 *Let $\mathcal{L}$ be any $\boldsymbol{MTL}_\sim$-algebra and let $\mathcal{L}^-$ be its underlying $\boldsymbol{MTL}_\Delta$-algebra. Then $\mathcal{L}$ and $\mathcal{L}^-$ have the same congruences.*

Proof: As usual it is possible to define an isomorphism between the lattice of filters of an $\mathbf{MTL}_\Delta$-algebra (ordered by inclusion) and the congruence lattice of any $\mathbf{MTL}_\Delta$-algebra. In particular, for every congruence θ, we define the filter $F_\theta = \{x : (x, 1) \in \theta\}$ and the inverse, if F is a filter over an $\mathbf{MTL}_\Delta$-algebra, then we consider $\theta_F = \{(x, y) : x \Rightarrow y \in F \text{ and } y \Rightarrow x \in F\}$. Hence in order to prove the claim is sufficient to prove that for every filter over $\mathcal{L}^-$, θ_F is a congruence of $\mathcal{L}$. We already know that θ_F is a congruence of $\mathcal{L}^-$, thus we only need to prove that if $(x, y) \in \theta_F$, then $(n(x), n(y)) \in \theta_F$. Now if $(x, y) \in \theta_F$, then $x \Rightarrow y \in F$ and by (1) $\delta(x \Rightarrow y) \in F$. Since $\delta(x \Rightarrow y) \leq (n(y) \Rightarrow n(x))$, we obtain $(n(y) \Rightarrow n(x)) \in F$. Similarily we can prove that $y \Rightarrow x \in F$ implies $(n(x) \Rightarrow n(y)) \in F$, therefore $(n(x), n(y)) \in \theta_F$ and thus the Lemma is proved. □

Theorem 3 *Every $\boldsymbol{MTL}_\sim$-algebra is isomorphic to a subdirect product of a family of linearly ordered $\boldsymbol{MTL}_\sim$-algebras.*

Proof: By the Birkhoff's subdirect representation theorem (see [8]), every $\mathbf{MTL}_\sim$-algebra is isomorphic to a subdirect product of a family of subdirectly irreducible $\mathbf{MTL}_\sim$-algebras. Being subdirectly irreducible only depends on the congruence lattice, therefore by the previous Lemma, an $\mathbf{MTL}_\sim$-algebra is subdirectly irreducible iff its underlying $\mathbf{MTL}_\Delta$-algebra is subdirectly irreducible. Since any subdirectly irreducible $\mathbf{MTL}_\Delta$-algebra is linearly ordered, the claim follows. □

Finally, Theorem 3 implies the following:

Theorem 4 *Let Γ be any set of $\boldsymbol{MTL}_\sim$-sentences and let φ be any $\boldsymbol{MTL}_\sim$-sentence. If $\boldsymbol{MTL}_\sim \cup \Gamma \not\vdash \varphi$, then there are a linearly ordered $\boldsymbol{MTL}_\sim$-algebra and an evaluation v of $\boldsymbol{MTL}_\sim$ into $\mathcal{L}$ such that $v(\gamma) = 1$ for each $\gamma \in \Gamma$ and $v(\varphi) \neq 1$*

3 Rational and standard completeness

We begin by recalling some definitions and results.

Definition 3 ([3]) *Let $\mathcal{C}$ be any extension of $\boldsymbol{MTL}$, and let $\mathbb{V}(\mathcal{C})$ be the variety generated by the Lindenbaum sentence algebra over $\mathcal{C}$.*

(1) *$\mathcal{C}$ has the* rational (real) embedding *property if any linearly ordered finite or countable structure of $\mathbb{V}(\mathcal{C})$ can be embedded into a structure in $\mathbb{V}(\mathcal{C})$ whose lattice reduct is* $\mathbb{Q} \cap [0,1]$ (*the real interval* $[0,1]$).

(2) *$\mathcal{C}$ is* rational (standard) complete *iff it is sound and complete w.r.t. interpretations in a class $\boldsymbol{K}$ of $\boldsymbol{MTL}$-algebras whose lattice reduct is the rational interval* $[0,1]$ (*the real interval* $[0,1]$).

(3) *$\mathcal{C}$ is* (finite) strong complete *w.r.t. a class $\boldsymbol{K}$ iff for every (finite) set of formulas T and every formula A, one has: $T \vdash_{\mathcal{C}} A$ iff $e(A) = 1$ for every evaluation e into any $\mathcal{A} \in \boldsymbol{K}$.*

(4) *$\mathcal{C}$ is* (finite) strong rational (standard) complete *iff $\mathcal{C}$ is (finite) strong complete w.r.t. a class $\boldsymbol{K}$ of $\boldsymbol{MTL}$-algebras whose lattice reduct is the rational interval* $[0,1]$ (*the real interval* $[0,1]$).

Lemma 2 ([3]) *Let $\mathcal{C}$ be any axiomatic extension of $\boldsymbol{MTL}$. Then:*

(i) If $\mathcal{C}$ has the real embedding property, then it has the rational embedding property.

(ii) If $\mathcal{C}$ is finite strong standard complete, then $\mathcal{C}$ is finite strong rational complete.

(iii) $\mathcal{C}$ has the rational (real respectively) embedding property, then $\mathcal{C}$ is finite strong rational (standard respectively) complete.

Our first step, now, will consist in showing that any countable $\mathbf{MTL}_\sim$-chain can be embedded into a countable linearly ordered dense monoid X which will be order-isomorphic to $\mathbb{Q} \cap [0,1]$.

Theorem 5 *For every countable linearly ordered $\boldsymbol{MTL}_\sim$-algebra $\mathcal{S} = \langle S, \star, \Rightarrow, n, \delta, \leq_S, 0_S, 1_S \rangle$, there are a countably ordered set $\langle X, \preceq \rangle$, a binary operation $\circ$, two monadic operations $\bullet$ and $\diamond$, and a mapping $\Phi : S \to X$ such that the following conditions hold:*

(1) X is densely ordered, and has a maximum M and a minimum m.

(2) $\langle X, \circ, \preceq, M \rangle$ is a commutative linearly ordered integral monoid.

(3) $\circ$ is left-continuous w.r.t. the order topology on $\langle X, \preceq \rangle$.

(4) $\bullet$ is an order reversing involutive mapping.

(5) For any $x \in X$, $x^\diamond = \begin{cases} M & \text{if } x = M \\ m & \text{otherwise} \end{cases}$.

(6) Φ is an embedding of the structure $\langle S, \star, n, \delta, \leq_S, 0_S, 1_S \rangle$ into $\langle X, \circ, \bullet, \diamond, \preceq, m, M \rangle$, and for all $s, t \in S$, $\Phi(s \Rightarrow t)$ is the residuum of $\Phi(s)$ and $\Phi(t)$ in $\langle X, \circ, \bullet, \diamond, \preceq, m, M \rangle$.

Proof: Notice that (1), (2) and (3) have been proved in [7]. Then we just have to show (4) and (5), and extend the proof of (6) given in [7] so as to cope with the operations $\bullet$ and $\diamond$.

For any $s \in S$, let $\varsigma(s)$ be the successor of s if it exists, and take $\varsigma(s) = s$ otherwise. Let

$$X = \{(s,1) \mid s \in S\} \cup \{(s,r) \mid \exists s', s = \varsigma(s') >_S s', r \in \mathbb{Q} \cap (0,1)\}.$$

For any $(s,q), (t,r) \in X$, let

$(s,q) \preceq (t,r)$ iff either $s <_S t$, or $s = t$ and $q \leq r$.

Clearly, $\preceq$ is a linear lexicographic order with a maximum $(1_S, 1)$ and a minimum $(0_S, 1)$.

To prove (4), define for any $(s,q) \in X$:

$$(s,q)^\bullet = \begin{cases} (n(s), 1) & \text{if } q = 1; \\ (\varsigma(n(s)), 1-q) & \text{otherwise.} \end{cases}$$

It is easy to see that $\bullet$ is indeed an order-reversing and involutive mapping.

To prove (5), let, for any $(s, q) \in X$:

$$(s,q)^{\diamond} = \begin{cases} (\delta(s), 1) & \text{if } q = 1; \\ (0_S, 1) & \text{otherwise.} \end{cases}$$

Obviously $(s,q)^{\diamond} = (1_S, 1)$ iff $s = 1$ and $q = 1$, otherwise we have the minimum $(0_S, 1)$.

To prove (6), let for every $s \in S$, $\Phi(s) = (s, 1)$. Clearly $\Phi(0_S) = (0_S, 1)$ and $\Phi(1_S) = (1_S, 1)$. Moreover, $\Phi(s) \circ \Phi(t) = (s,1) \circ (t,1) = (s \star t, 1) = \Phi(s \star t)$. Finally, let $\Phi(n(s)) = (n(s), 1)$ and $\Phi(\delta(s)) = (\delta(s), 1)$. Clearly, $\Phi(n(s)) = (s,1)^{\bullet}$, and $\Phi(\delta(s)) = (s,1)^{\diamond}$. Thus, Φ is an embedding of partially ordered monoids equipped with an order-reversing involutive mapping. To conclude, notice that the fact that for all $s, t \in S$, $\Phi(s \Rightarrow t)$ is the residuum of $\Phi(s)$ and $\Phi(t)$, is shown in [7]. □

We now prove that X is order-isomorphic to $\mathbb{Q} \cap [0,1]$, and define a structure $\langle \mathbb{Q} \cap [0,1], \leq, \circ', \bullet', \diamond', 0, 1\rangle$ which satisfies (1-6). Such a structure can be embedded over the real unit interval. The resulting structure $\langle [0,1], \hat{\circ}, \hat{\bullet}, \hat{\diamond}, \leq, 0, 1\rangle$ will be a linearly ordered $\mathbf{MTL}_{\sim}$ algebra where the initial $\mathbf{MTL}_{\sim}$-chain can be embedded. This clearly means that $\mathbf{MTL}_{\sim}$ has the real embedding property.

Theorem 6 *Every countable $\boldsymbol{MTL}_{\sim}$-chain can be embedded into a standard algebra.*

Proof: As shown in the above theorem $\langle X, \preceq\rangle$ is a countable, dense, linearly-ordered set with maximum and minimum. Then $\langle X, \preceq\rangle$ is order-isomorphic to the rationals in $[0,1]$ with the natural order $\langle \mathbb{Q} \cap [0,1], \leq\rangle$. Let Ψ be such an isomorphism. Suppose that (1-6) hold, and let for $\alpha, \beta \in [0,1]$,

- $\alpha \circ' \beta = \Psi(\Psi^{-1}(\alpha) \circ \Psi^{-1}(\beta))$,
- $\alpha^{\bullet'} = \Psi((\Psi^{-1}(\alpha))^{\bullet})$,
- $\alpha^{\diamond'} = \Psi((\Psi^{-1}(\alpha))^{\diamond})$,

and let for all $s \in S$, $\Phi'(s) = \Psi(\Phi(s))$. Hence, we have a structure $\langle \mathbb{Q} \cap [0,1], \leq, \circ', \bullet', \diamond', 0, 1\rangle$, that, along with Φ', satisfies (1-6).

Now, we can assume, without loss of generality, that $X = \mathbb{Q} \cap [0,1]$ and that $\preceq$ is $\leq$. It is shown in [7, 9] that such a structure is embeddable into an analogous structure $\langle [0,1], \hat{\circ}, \leq\rangle$ over the real unit interval, where, for any $\alpha, \beta \in [0,1]$:

$$\alpha \hat{\circ} \beta = \sup_{x \in X : x \leq \alpha} \; \sup_{y \in X : y \leq \beta} x \circ y.$$

The operation $\hat{\circ}$ is shown to be a left-continuous t-norm which extends $\circ$.

Now, define for any $\alpha \in [0,1]$

$$\alpha^{\hat{\bullet}} = \inf_{x \in X : x \leq \alpha} x^{\bullet}.$$

We show that such $\hat{\bullet}$ is an order-reversing involutive mapping which extends $\bullet$. First let

$$\alpha^{\hat{\bullet}'} = \sup_{y \in X : \alpha \leq y} y^{\bullet}.$$

We prove that $\alpha^{\hat{\bullet}} = \alpha^{\hat{\bullet}'}$, which means the negation defined is continuous. In general we have that $\alpha^{\hat{\bullet}'} \leq \alpha^{\hat{\bullet}}$. Suppose the inequality is strict: i.e. $\alpha^{\hat{\bullet}'} < \alpha^{\hat{\bullet}}$. This means that there is some $z \in \mathbb{Q}$ such that $\alpha^{\hat{\bullet}'} < z^{\bullet} < \alpha^{\hat{\bullet}}$. Therefore, for any $x \leq \alpha$, $z^{\bullet} < x^{\bullet}$ and, for any $y \geq \alpha$, $y^{\bullet} < z^{\bullet}$. Hence we have that, for any $x \leq \alpha$, $x < z$ and, for any $y \geq \alpha$, $z < y$. Then z must equal α, but $\alpha \in [0,1] \backslash \mathbb{Q} \cap [0,1]$, so we obtain a contradiction. Notice that if $\alpha \in \mathbb{Q}$, then the above equivalence clearly holds. Thus we have proved $\alpha^{\hat{\bullet}} = \alpha^{\hat{\bullet}'}$.

It is easy to see that $0^{\hat{\bullet}} = 1$, $1^{\hat{\bullet}} = 0$ and that $\hat{\bullet}$ is order-reversing.

It remains to prove that $(\alpha^{\hat{\bullet}})^{\hat{\bullet}} = \alpha$. Notice that

$$(\alpha^{\hat{\bullet}})^{\hat{\bullet}} = (\inf_{x \leq \alpha} x^{\bullet})^{\hat{\bullet}} = \sup_{x \leq \alpha}((x^{\bullet})^{\bullet}) = \sup_{x \leq \alpha} x = \alpha.$$

Now, define for any $\alpha \in [0,1]$

$$\alpha^{\hat{\diamond}} = \begin{cases} 1 & \text{if } \alpha = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\langle \mathbb{Q} \cap [0,1], \leq, \circ', \bullet', \diamond' 0, 1\rangle$ embeds into $\langle [0,1], \hat{\circ}, \hat{\bullet}, \hat{\diamond}, \leq, 0, 1\rangle$. Given left-continuity of $\hat{\circ}$ over $[0,1]$, $\langle [0,1], \hat{\circ}, \Rightarrow_{\hat{\circ}}, \hat{\bullet}, \hat{\diamond}, \leq, 0, 1\rangle$ is a linearly ordered $\mathbf{MTL}_{\sim}$-algebra, where the residuum $\Rightarrow_{\hat{\circ}}$ always exists. Hence the initial $\mathbf{MTL}_{\sim}$-chain $\mathcal{S}$ can be embedded into the standard algebra $\langle [0,1], \hat{\circ}, \Rightarrow_{\hat{\circ}}, \hat{\bullet}, \hat{\diamond}, \leq, 0, 1\rangle$. □

Hence, given Definition 3 and Lemma 2, we easily obtain the following result.

Theorem 7 ***MTL***$_\sim$ *has the real (rational) embedding property, is strong standard (rational) complete, and consequently standard (rational) complete.*

4 Predicate calculus: MTL$\forall_\sim$

We begin by enlarging the propositional language with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$ together with the two classical quantifiers $\forall$ and $\exists$. The notion of formula is trivially generalized by saying that if φ is a formula and $x \in Var$, then both $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas.

Definition 4 *Let $\mathcal{L}$ be an* ***MTL***$_\sim$*-algebra. An $\mathcal{L}$-interpretation for a predicate language $\mathscr{L}$ is a structure $\mathcal{M} = \langle M, (r_P)_{P \in Pred}, (m_c)_{c \in Const}\rangle$, where:*

- *M is a non-empty set,*
- *$r_P : M^{ar(P)} \to A$ for any $P \in Pred$, where $ar(P)$ stands for the ariety of the predicate P,*
- *$m_c \in M$ for each $c \in Const$.*

For every evaluation of variables $v : Var \to M$, the truth value of a formula φ ($\|\varphi\|^{\mathcal{L}}_{M,v}$) is inductively defined as follows:

- *$\|P(x, \ldots, c, \ldots)\|^{\mathcal{L}}_{M,v} = r_P(v(x), \ldots, m_c, \ldots)$, where $v(x) \in M$ for each variable x,*
- *The truth value commutes with connectives of* ***MTL***$_\sim$*,*
- *$\|(\forall x)\varphi\|^{\mathcal{L}}_{M,v} = \inf\{\|\varphi\|^{\mathcal{L}}_{M,v'} : v(y) = v'(y)$ for all variables, except $x\}$ and $\|(\exists x)\varphi\|^{\mathcal{L}}_{M,v} = \sup\{\|\varphi\|^{\mathcal{L}}_{M,v'} : v(y) = v'(y)$ for all variables, except $x\}$.*

The infimum and the supremum might not exist in $\mathcal{L}$. In that case the truth value remains undefined. A structure $\mathcal{M}$ is called $\mathcal{L}$-safe if all infima and suprema needed for definition of the truth value of any formula exist in $\mathcal{L}$. In such a case the truth value of a formula φ in an $\mathcal{L}$-safe structure $\mathcal{M}$ corresponds to

$$\|\varphi\|^{\mathcal{L}}_{M} = \inf\{\|\varphi\|^{\mathcal{L}}_{M,v} : v : Var \to M\}.$$

Definition 5 *The axioms for* ***MTL***$\forall_\sim$ *are those of* ***MTL***$_\sim$ *plus the following axioms with quantifiers (with x not free in v for ($\forall$2), ($\forall$3) and ($\exists$2); and with t substitutable for x in $\varphi(x)$ for ($\forall$1) and ($\exists$1)):*

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$,

($\forall$2) $(\forall x)(v \to \varphi) \to (v \to (\forall x)\varphi)$,

($\forall$3) $(\forall x)(\varphi \vee v) \to ((\forall x)\varphi \vee v)$,

($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$,

($\exists$2) $(\forall x)(\varphi \to v) \to ((\exists x)\varphi \to v)$.

Rules for ***MTL***$\forall_\sim$ *are those of* ***MTL***$_\sim$ *plus generalization for the quantifier: $\frac{\varphi}{(\forall x)\varphi}$*

By using the connective $\sim$ it is easy to see that a quantifier is definable by the other one, for instance $(\exists x)\varphi(x)$ is $\sim (\forall x)(\sim \varphi(x))$. Therefore the above set of axioms can surely be simplified.

Theorem 8 *Let T be a theory over* ***MTL***$\forall_\sim$ *and φ a formula. T proves φ over* ***MTL***$\forall_\sim$ *iff $\|\varphi\|^{\mathcal{L}}_{M,v} = 1_{\mathcal{L}}$ for each* ***MTL***$_\sim$*-algebra $\mathcal{L}$, each $\mathcal{L}$-safe $\mathcal{L}$-model $\mathcal{M}$ of T and each v.*

Proof: (Sketch). Just inspect the corresponding proof given in [6] and see that the proof for **MTL**$_\sim$ is similar (using the Deduction Theorem 1 given here). □

5 Standard completeness for MTL$\forall_\sim$

First, we need the following Lemma.

Lemma 3 ([9]) *Let $\mathcal{S}$ be a countably linearly ordered* ***MTL***$_\sim$ *algebra, and let X be the dense linearly ordered commutative monoid above defined. Let Φ be the embedding of $\mathcal{S}$ into X, as above. Then, Φ is a complete lattice embedding, i.e., if $a = \sup\{a_i : i \in I\} \in \mathcal{S}$, then $\sup\{\Phi(a_i) : i \in I\} = \Phi(a)$; if $c = \inf\{a_i : i \in I\} \in \mathcal{S}$, then $\inf\{\Phi(a_i) : i \in I\} = \Phi(c)$.*

Theorem 9 *For every* $\boldsymbol{MTL}\forall_\sim$ *formula* φ*, the following are equivalent:*

(i) $\boldsymbol{MTL}\forall_\sim \vdash \varphi$.

(ii) For every left-continuous t-norm $\odot$ *on* $[0,1]$ *and for every evaluation* v *in the linearly ordered* $\boldsymbol{MTL}_\sim$*-algebra* $\langle [0,1], \odot, \Rightarrow_\odot, n, \delta, \leq, 0, 1\rangle$*, where* $\Rightarrow_\odot$ *is the residuum of* $\odot$*,* $v(\varphi) = 1$.

Proof: (i)⇒(ii) is immediate. The converse is an easy adaptation of the proof given in [9], Theorem 5.4 (iii). Indeed, suppose that $\mathbf{MTL}\forall_\sim \not\vdash \varphi$, then there is a countable $\mathbf{MTL}_\sim$-chain $\mathcal{S}$ where $v(\varphi) \neq 1$. As done in the proof of standard completeness for the $\mathbf{MTL}_\sim$-logic we can define an embedding Ψ of $\mathcal{S}$ into a standard $\mathbf{MTL}_\sim$-algebra over the real unit interval. Such an embedding is complete by the above Lemma. Then we define an evaluation v' over the standard algebra by letting for every atomic formula ψ, $v'(\psi) = \Psi(v(\psi))$. Hence for any closed formula χ we have that $v'(\chi) = \Psi(v(\chi))$. This clearly implies that $v'(\varphi) \neq 1$. □

6 Final remarks

In this paper we investigated an extension of the Monoidal T-norm based Logic **MTL** with an involutive negation that does not depend on the t-norm operation. We introduced the propositional logic $\mathbf{MTL}_\sim$ and the predicate calculus $\mathbf{MTL}\forall_\sim$, and proved standard and algebraic completeness for both systems.

The logic obtained is interesting, since by defining a new connective

$$\varphi \underline{\vee} \psi \equiv \sim (\sim \varphi \& \sim \psi),$$

it allows to represent by means of left-continuos t-norms and involutions all dual t-conorms. This is not possible in any other residuated fuzzy logic. Moreover, notice that we can also define S-implications as follows:

$$\varphi \rightsquigarrow \psi \equiv \sim \varphi \underline{\vee} \psi.$$

This suggests that the work carried out in [2] might be recovered under our framework.

Future work will focus on proving Kripke-style completeness for the systems obtained and on introducing an involutive negation in other extensions of **MTL** and especially in **BL**.

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