On the use of the Choquet integral with fuzzy numbers in Multiple Criteria Decision Aiding

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Abstract

This paper presents a multiple criteria decision aiding approach in order to build a ranking on a set of alternatives. The partial evaluations of the alternatives on the points of view can be fuzzy numbers. The aggregation is performed through the use of a fuzzy extension of the Choquet integral. We detail how to assess the parameters of the aggregator by using alternatives which are well-known to the decision maker, and which originate from his domain of expertise.

Keywords: Multiple criteria decision aiding, fuzzy partial evaluations, Choquet integral, ranking

1 Introduction

This paper presents a multiple criteria decision aiding approach which aims to build a ranking on a set of alternatives A. The work is based on a multiple criteria sorting procedure proposed in [8]. Here we present an extension to alternatives which are described by fuzzy partial evaluations. In real life problems, this extension can be very useful, as very often it is difficult to determine an exact value for alternatives on certain points of view.

As an example one can consider the hypothetical problem of the establishment of a large factory in a country. The alternatives represent different areas and among the points of view one can find for example the number of direct and indirect working places which will be created. It seems obvious that for most of the areas it will be hard to determine their exact evaluation on this point of view. However some knowledge of a possible value distribution might be available. This small example shows that the use of imprecise information as evaluations can be useful in a lot of situations.

One possible way to represent such imprecise information is to use fuzzy numbers. Their aggregation by means of an extension of the Choquet integral is investigated in this paper in view of building a ranking on the set of alternatives A.

We briefly recall general results in Section 2. Then, in Section 3 we present the context of the multiple criteria decision aiding problem we are dealing with here. Section 4 recalls a few results on the aggregation by means of a Choquet integral [3]. Later, in Section 5 we show that it is possible to extend the Choquet integral by Zadeh's extension principle [15]. In Section 6 we present how to obtain a ranking on the alternatives of A. An important point is dealt with in Section 7 where we show how to assess the parameters of the Choquet integral from well-known alternatives.

2 Preliminary considerations

In this Section, we briefly recall general concepts on fuzzy sets and fuzzy numbers. Let X be a classical set. A fuzzy set $\widetilde{Y}$ in X can be defined by its membership function $\mu_{\widetilde{Y}} : X \to [0, 1]$. We will write $\widetilde{Y}(x) = \mu_{\widetilde{Y}}(x)$ for the degree of membership of the element x in the fuzzy set Y, for each x in X. For all $\lambda \in [0, 1]$, the $\lambda-$level set of a fuzzy set $\widetilde{Y}$ of X is written as $[\widetilde{Y}]^{\lambda}$.

A fuzzy number $\widetilde{x}$ of $\mathbb{R}$ is a fuzzy subset $\widetilde{Y}$ in $\mathbb{R}$ that is normal ($\max_x \widetilde{x}(x) = 1$), fuzzy convex (every $\lambda-$level set of $\widetilde{x}$ is an interval) and has an upper semi-continuous membership function of bounded support. Let $\mathcal{F}$ be the family of all fuzzy numbers. For a fuzzy number $\widetilde{x} \in \mathcal{F}$ we define $x_m(\lambda) = \min[\widetilde{x}]^\lambda$ and $x_M(\lambda) = \max[\widetilde{x}]^\lambda$.

In order to be able to use mathematical operations on the fuzzy numbers, Zadeh [15] introduces the sup-min extension principle which allows to work consistently with the crisp case. If $\widetilde{x_1}, \ldots, \widetilde{x_n} \in \mathcal{F}$ and $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then the sup-min extension principle is defined by

$$f(\widetilde{x_1}, \ldots, \widetilde{x_n})(y) = \sup_{f(x_1,\ldots,x_n)=y} \min\{\widetilde{x_1}(x_1), \ldots, \widetilde{x_n}(x_n)\}, \forall y \in \mathbb{R}.$$

It has been shown in [9] that $f(\widetilde{x_1}, \ldots, \widetilde{x_n})$ is a fuzzy number.

In the following section we present the multiple criteria decision aiding problem which we solve here.

3 A particular multiple criteria decision aiding framework

Let A be a set of q alternatives and $N = \{1, \ldots, n\}$ be a label set of points of view. The partial evaluation of each alternative for each point of view is made on an interval scale. It accepts as admissible transformation function a positive linear transformation (see [11]). Moreover the evaluations are supposed to be commensurable. We can now accept without any loss of precision that the performance scale of each point of view is the closed continuous unit interval $[0, 1]$.

The particular case we are actually presenting here is where the evaluations of the alternatives on the points of view are fuzzy numbers. This means that each alternative $x \in A$ can be identified with its corresponding fuzzy profile

$$x \equiv (\widetilde{x_1}, \ldots, \widetilde{x_n}) \in [0, 1]^n$$

where, for any $i \in N$, $\widetilde{x_i}$ represents the partial evaluation of x on point of view i.

In order to suggest a ranking of the alternatives of A to the decision maker, one possibility is to first aggregate the partial evaluations of each alternative on the points of view in a global evaluation and then to order these global evaluations to obtain a ranking.

In case of crisp partial evaluations, the weighted sum is very often used as an aggregator. This additive representation however implies preferential independence of the points of view. One way to avoid this independence condition is to use the Choquet integral [3] [7] as an aggregator.

4 The Choquet integral as an aggregator

Let us consider an alternative $x \in A$ which is described by crisp partial evaluations $x_i, (i \in N)$. The Choquet integral of x is then defined by:

$$\mathcal{C}_v(x) := \sum_{i=1}^{n} x_i[v(A_{(i)}) - v(A_{(i+1)})]$$

where v represents a fuzzy measure on N, that is a monotone set function $v : 2^N \to [0, 1]$ fulfilling $v(\emptyset) = 0$ and $v(N) = 1$. This fuzzy measure merely expresses the importance of each subset of points of view. The parentheses used for indexes represent a permutation on N such that $x_1 \leq \ldots \leq x_n$ and $A_{(i)}$ represents the subset $\{(i), \ldots, (n)\}$.

If points of view cannot be considered as being independent, the importance of the coalitions $S \subseteq N$, namely $v(S)$ has to be taken into account.

The Choquet integral presents standard properties for aggregation [13]: it is continuous, non decreasing, located between min and max.

The major advantage linked to the use of the Choquet integral derives from the large number of parameters $(2^n - 2)$ associated with a fuzzy measure. On the other hand, this flexibility can also be considered as a serious drawback, especially when assigning real values to the importance of all possible combinations.

Let us present an equivalent definition of the Choquet integral which will be used in the context of fuzzy partial evaluations. Let v be a fuzzy measure on N. The Möbius transform of v is a set function $m : 2^N \to \mathbb{R}$ defined by

$$m(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T) \ \ (S \subseteq N).$$

This transformation is invertible and thus constitutes an equivalent form of a fuzzy measure and v can be recovered from m by using

$$v(S) = \sum_{T \subseteq S} m(T) \ \ (S \subseteq N).$$

This transformation can be used to redefine the Choquet integral without reordering the partial scores:

$$\mathcal{C}_m(x) = \sum_{T \subseteq N} m(T) \bigwedge_{i \in T} x_i. \qquad (1)$$

In order to be able to use fuzzy partial evaluations, we must define the fuzzy extension of the Choquet integral in the following Section.

5 An extension of the Choquet integral to the fuzzy case

We can see that definition (1) of the Choquet integral in terms of a set function m is a combination of functions which are continuous on $\mathbb{R} \times \mathbb{R}$, namely the addition $(+)$, the multiplication $(\cdot)$ and the minimum $(\wedge)$ functions. By using the extension principle of Zadeh described in Section 2 one can extend these three functions to their fuzzy versions $\widetilde{+}$, $\widetilde{\cdot}$ and $\widetilde{\wedge}$.

The fuzzy extension of the Choquet integral can be written as

$$\widetilde{\mathcal{C}_m}(x) = \widetilde{\sum_{T \subseteq N}} m(T) \,\widetilde{\cdot}\, \widetilde{\bigwedge_{i \in T}} x_i.$$

where m is the set function which is obtained by a Möbius transform of the fuzzy measure v. According to [9] the Choquet integral of a vector of fuzzy partial evaluations is a fuzzy number. In case of partial evaluations which are trapezoidal or triangular fuzzy numbers, the resulting fuzzy number has piecewise linear side functions.

In [14] a fuzzy extension of the Choquet integral is presented, based on the classical representation of the Choquet integral by means of a fuzzy measure. The authors of [2] present an interval-based Choquet integral to derive preferences on multicriteria alternatives.

6 Ranking the alternatives

Let us suppose that the partial evaluations of the elements of A have been aggregated into a fuzzy number for each alternative. We will see later in Section 7 how to determine the values of the parameters of the fuzzy measure v (or equivalently of its Möbius transform m). The global evaluations obtained by aggregation will allow us to determine a ranking on the set A which can be proposed to the decision maker. The goal is to build an order on the set of alternatives to obtain a ranking. To build such a ranking, we suggest two possibilities.

The first approach extracts a degree of credibility of the preference of x over y (for each $x, y \in A$). This leads to a fuzzy preference relation on A which can be exploited to extract a ranking [12]. The second approach uses results from [6] to compare the alternatives by the concept of area compensation. These two approaches should be compared in order to control the robustness of the ranking. Other ways of ranking fuzzy numbers have been explored.

6.1 Ranking by a fuzzy preference

This first approach, in order to build a ranking on the alternatives, extracts a degree of credibility of the preference of one alternative on another one. In [5] this credibility is defined as the possibility Π that an alternative x is not worse than y (let's write $x \succeq y$) in the following way:

$$\Pi(x \succeq y) = \sup_{u \geq v}[\min(\widetilde{x}(u), \widetilde{y}(v))]. \qquad (2)$$

Figure 1 illustrates the meaning of this degree of credibility of the preference of x over y $(x, y \in A)$. We have $\Pi(x \succeq y) = h$ and $\Pi(y \succeq x) = 1$.

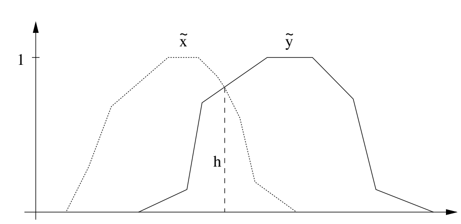

Figure 1: Degree of credibility of the preference of x over y

Roubens and Vincke [12] have shown that the credibility Π as defined in (2) is a fuzzy interval order (a reflexive, complete[1] and Ferrers[2] valued relation). If one calculates these credibilities for each pair of alternatives of A, it is possible to consider a valued digraph where the nodes represent the alternatives and the arcs the valued relation Π. This graph is called a fuzzy possibility digraph. Starting from this credibility Π, Roubens and Vincke [12] have defined a preference relation by

$$P(x,y) = \min(\Pi(x,y), 1-\Pi(y,x)) \quad \forall (x,y) \in A^2.$$

The associated digraph is called a fuzzy preference digraph. The goal is now to extract some useful information from this digraph, in order to build a ranking on the alternatives of A. It is possible to do this by an iterative heuristic which uses the concept of the *core* of a digraph. In order to define the core of a digraph, we first need to define the concept of the score of non-domination of an alternative x of A [10] by $ND(x) = 1 - \max_{y \neq x} P(y,x)$.

The core Y_0 of A is a subset $Y_0 \subset A$ such that its elements all have a score of non-domination of 1. In other words, $Y = \{x \in A | ND(x) = 1\}$. It has been shown in [10] that the core is non-empty.

Y_0 gives a solution to the choice problem (the subset of best actions among A). In order to obtain a ranking we restrict ourselves to a subgraph by considering the vertices of $A \setminus Y_0$. The core of this subgraph is non empty and is called Y_1. An iteration of this procedure leads to the cores $Y_0, Y_1, Y_2, \ldots$. A possible ranking is given by the crisp total preorder $Y_0 > Y_1 > Y_2 > \ldots$, where the elements in each core are considered as indifferent.

6.2 Ranking by area compensation

This method, presented in [6], is based on the area compensation determined by the membership functions of two fuzzy numbers. Let $\widetilde{x}$ and $\widetilde{y}$ be two fuzzy numbers representing the global evaluation of two alternatives of A. Fortemps and Roubens [6] define

$$S_l(\widetilde{x} \geq \widetilde{y}) = \int_{U(\widetilde{x},\widetilde{y})} [x_m(\lambda) - y_m(\lambda)] d\lambda \quad \text{and}$$

$$S_r(\widetilde{x} \geq \widetilde{y}) = \int_{V(\widetilde{x},\widetilde{y})} [x_M(\lambda) - y_M(\lambda)] d\lambda,$$

where $U(\widetilde{x},\widetilde{y}) = \{\lambda \in [0,1] : x_m(\lambda) \geq y_m(\lambda)\}$ and $V(\widetilde{x},\widetilde{y}) = \{\lambda \in [0,1] : x_M(\lambda) \geq y_M(\lambda)\}$. Figure 2 represents these four areas when comparing $\widetilde{x}$ to $\widetilde{y}$. Area (1) represents $S_l(\widetilde{x} \geq \widetilde{y})$, (2) represents $S_l(\widetilde{y} \geq \widetilde{x})$, (3) represents $S_r(\widetilde{y} \geq \widetilde{x})$ and finally (4) represents $S_r(\widetilde{x} \geq \widetilde{y})$. For example, $S_l(\widetilde{x} \geq \widetilde{y})$

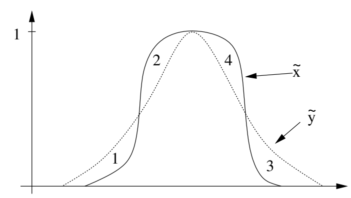

Figure 2: Comparing $\widetilde{x}$ to $\widetilde{y}$

is the area which claims that the left slope of $\widetilde{x}$ is greater to the corresponding part of $\widetilde{x}$.

The degree to which $\widetilde{x}$ is greater than $\widetilde{y}$ is then defined in [6] as

$$C : \mathcal{F} \times \mathcal{F} \to \mathbb{R} : C(\widetilde{x} \geq \widetilde{y}) = \tfrac{1}{2}[S_l(\widetilde{x} \geq \widetilde{y}) + S_r(\widetilde{x} \geq \widetilde{y}) - S_l(\widetilde{y} \geq \widetilde{x}) - S_r(\widetilde{y} \geq \widetilde{x})].$$

One considers that $\widetilde{x} > \widetilde{y} \iff C(\widetilde{x} \geq \widetilde{y}) > 0$, $\widetilde{x} \geq \widetilde{y} \iff C(\widetilde{x} \geq \widetilde{y}) \geq 0$ and one can see that in that case $C(\widetilde{y} \geq \widetilde{x}) < 0$.

From this relation it is possible to induce a complete ranking of the global evaluations of the alternatives of A.

One important question has been put aside until now. It concerns the parameters of the set function m used in the Choquet integral. Section 7 deals with this question and shows how information can be extracted from the decision maker's expertise.

7 Assessment of the parameters of m

The method described in Section 6 presents how to build two possible rankings on alternatives which have fuzzy partial evaluations on a set of

[1] $x \Pi y$ or $y \Pi x$ for each $x \neq y$
[2] $w \Pi x, y \pi z \Rightarrow w \Pi z$ or $z \Pi x$

criteria. These evaluations are aggregated by means of a fuzzy extension of the Choquet integral presented in Section 5. One important point concerns the assessment of the parameters of the set function m which is the Möbius transform of the fuzzy measure v (see Section 4).

The number of parameters which have to be determined for the aggregation is very important. It seems obvious that neither the parameters of the fuzzy measure v, nor those of its Möbius transform m have a clear meaning to the decision maker. Therefore the large number of parameters linked to an unclear semantics of v or m lead us to think that an alternative possibility has to be explored.

In [8] we have considered the sorting of alternatives by means of a Choquet integral in a universe of crisp partial evaluations. The idea is to ask the decision maker to provide information on a set of prototypes. A prototype is an alternative which is well known to the decision maker and which belongs to his domain of expertise.

A decisive hypothesis has to be done here. As a prototype is a well-known alternative for the decision maker, we suppose that it is also well-defined, and has therefore crisp partial evaluations.

Let us suppose that the decision maker has provided a set P of such prototypes. We then ask him to determine a total order on the elements of P. Once again, as he is an expert of the knowledge field we are working on, he should be able to provide such an order.

The next step is to determine the parameters of the set function m used in the definition of the Choquet integral. As we have no guarantee that the total order provided by the decision maker is compatible with a discrimant function of the Choquet integral type[3], the objective is to determine a set function m which will provide a satisfactory ranking on the prototypes. Therefore, the following procedure tends to minimise the gap between the ranking given by the decision maker and the one resulting from the aggregation. Intuitively, for a given alternative $x \in P$, its Choquet integral $\mathcal{C}_m(x)$ should be as close as possible to an unknown global evaluation $y(x)$, which respects the total order imposed by the decision maker on the prototypes.

The parameters of the set function m are determined by solving the following quadratic program

$$\min \sum_{x \in P} [\mathcal{C}_m(x) - y(x)]^2,$$

where the unkowns are the parameters of the set function m and some global evaluations $y(x)$ for each $x \in P$.

In order to ensure boundary and monotonicity conditions imposed on the fuzzy measure v, its Möbius transform m must satisfy:

$$\begin{cases} m(\emptyset) = 0, \quad \sum_{T \subseteq N} m(T) = 1 \\ \sum_{T : i \in T \subseteq S} m(T) \geq 0, \quad \forall S \subseteq N, \forall i \in S. \end{cases}$$

The global evaluations $y(x)$ must verify the ranking imposed by the decision maker. In other words, for every ordered pair $(x, x') \in P \times P$, the condition $y(x) - y(x') \geq \varepsilon$, $\varepsilon > 0$ must be satisfied[4].

The solution of the quadratic program gives the parameters of the set function m and the global evaluations $y(x), \forall x \in P$. If the objective function equals 0, then for each alternative, its global evaluation corresponds to its aggregated evaluation by a Choquet integral. In that case, the ranking obtained by ordering the alternatives according to their aggregated evaluations is consistant with the ranking on the prototypes. If the objective function is strictly positive, then it was not possible to determine the parameters of the set function m in order to respect the ranking on the prototypes. In the ranking obtained by ordering the alternatives according to their aggregated evaluations, certain pairs of alternatives may be reversed compared to the ranking imposed on the prototypes.

It is possible to measure the adequation of both rankings. The symmetric difference distance δ de-

[3]The partial order may violate the triple cancellation axiom ([13]) for example.

[4]In practice, one of the main problems is to fix the value of ε. We think that it has to be chosen in $]0, \frac{1}{p}[$ where p is the number of prototypes considered by the decision maker.

fined as

$$\delta(R,S) = |\{(x,y) \in A^2 : [xSy \text{ and } \neg(xTy)] \text{ or } [\neg(xSy) \text{ and } (xTy)]\}|,$$

where R and S are two binary relations on A, measures the number of disagreements between R and S. Barthlemy [1] shows that δ satisfies a number of naturally desirable properties. In our case, we can measure the symmetric difference distance between the ranking imposed on the prototypes, and the one obtained by the use of a Choquet Integral. By using δ it is now quite easy to measure the adequation of both rankings.

If the decision maker is satisfied with the model built on the prototypes , the next step is to determine the global evaluations of the alternatives of A which have fuzzy partial evaluations. The parameters of the set function m, determined by the procedure we just described, are used to aggregate the fuzzy partial evaluations by means of the extension of the Choquet integral presented in Section 5.

8 Concluding remarks

In this paper we have presented an extension of the Choquet integral to fuzzy partial evaluations in the case of multiple criteria decision aiding. We have presented two possibilities to build a ranking on the alternatives and we have shown how it is possible to determine the parameters of the Choquet integral. Further investigations need to be done concerning the comparison of both rankings that we presented in Section 6. Besides, it would certainly be useful to implement this proposal in order to test it on real cases.

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