Aggregation and Non-Contradiction

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Abstract

This paper investigates the satisfaction of the Non-Contradiction Law (NC) within the domain of aggregation operators. It provides characterizations both for those aggregation operators that satisfy NC with respect to (w.r.t.) some given strong negation, as well as for those satisfying the law w.r.t. any strong negation. The results obtained are applied to some of the most important known classes of aggregation operators.

Keywords: Non-Contradiction law, Aggregation Operators, Strong Negations.

1 Introduction

Information aggregation is a crucial issue in the construction of many intelligent systems, and it is used in different application domains, such as medicine, economics, engineering, statistics or decision-making processes. It is particularly useful in situations presenting some degree of uncertainty or imprecision, a feature that explains the great development that this discipline has experimented in recent years within the field of Fuzzy Logic. It is a well assorted research field, whose topics of interest range from theoretical aspects to the use of the different aggregation methods and techniques in practical situations. A large collection of distinguished classes of aggregation operators and construction methods is nowadays available, and different potential application fields have been explored (see for example [4], [1] or the recent overview on aggregation theory given in [2]).

When using aggregation techniques in practical situations, one of the first problems that one has to face up is the choice, among all the aggregation operators that are available, of the most suited one. Clearly, there is not a universal answer to this problem, since the decision is largely context-dependent. Notwithstanding, there are several criteria that may help in making this decision, such as the achievement of empirical experiments, the analysis of the expected operator's behavior (tolerant, intolerant, compensatory) or the need of some mathematical/logical properties (e.g. idempotency, symmetry, associativity, the existence of neutral or annihilator elements, etc).

This paper deals with the satisfaction of one of these mathematical properties, namely the well-known *Non-Contradiction law* (NC), $p \wedge \neg p = 0$, once the later has been appropriately translated to the aggregation operators field. This is performed as follows: if A is a binary aggregation operator acting on $[0, 1]$ (i.e., an operator of the form $A : [0, 1]^2 \rightarrow [0, 1]$ fulfilling some basic properties that will be recalled later), the NC law can be interpreted as $A(x, N(x)) = 0$ for any $x \in [0, 1]$, where N belongs to the family of *strong negations*, commonly used for modelling fuzzy complements.

The paper is organized as follows. It begins by briefly recalling the main issues on aggregation operators and negations that are needed later on. Section 3 provides some conditions and characterizations regarding the satisfaction of the NC law, and section 4 applies these results to some concrete families of aggregation operators. Finally, the paper ends with some conclusions and pointers to future work.

2 Preliminaries

Although aggregation operators are defined for the general multidimensional case ([2]), in this paper we will only deal with *binary aggregation operators*, i.e., non-decreasing operators $A : [0,1]^2 \to [0,1]$ verifying the boundary conditions $A(0,0) = 0$ and $A(1,1) = 1$. Aggregation operators may be compared pointwise, that is, given two operators A_1 and A_2, it is said that A_1 is *weaker* than A_2 (or A_2 is *stronger* than A_1), and it is denoted $A_1 \leq A_2$, when it is $A_1(x,y) \leq A_2(x,y)$ for any $x, y \in [0,1]$. The relation $\leq$ is clearly a partial order, i.e., there are couples of aggregation operators which are non-comparable. Aggregation operators may be classified, by means of the relation $\leq$ and the distinguished operators Min and Max, in the following four categories:

- *Conjunctive* operators, which are those verifying $A \leq Min$. This class includes the well-known *triangular norms* (*t-norms*) as well as *copulas* (see [8] for an exhaustive overview on t-norms).
- *Disjunctive* operators, verifying $Max \leq A$, such as *triangular conorms (t-conorms)* and *dual copulas*.
- *Averaging* operators (or *mean* operators) which verify $Min \leq A \leq Max$. These operators are always idempotent (i.e., $A(x,x) = x$ holds for any $x \in [0,1]$), and some distinguished ones in this class are those based on the arithmetic mean, such as *quasi-linear means* or *OWA operators*, as well as those based on integrals, such as *Lebesgue, Choquet* or *Sugeno* integral-based aggregations.
- Finally, the class of *hybrid* aggregation operators contains all the operators that do not belong to any of the three previous categories, i.e., operators that are not comparable with Min and/or are not comparable with Max. This class includes different aggregation operators related to t-norms and t-conorms (such as *uninorms*, *nullnorms* or *compensatory operators*) as well as *symmetric sums*.

In order to translate the Non-Contradiction law to the aggregation operators' field, a way for representing the logic negation is needed. The later is usually done by means of the so-called *strong negations* ([10]), i.e., non-increasing functions $N : [0,1] \to [0,1]$ which are involutive, that is, verify $N(N(x)) = x$ for any $x \in [0,1]$. Due to their definition, strong negations are continuous and strictly decreasing functions, they satisfy the boundary conditions $N(0) = 1$ and $N(1) = 0$, and they have a unique fixed point in $]0,1[$, that we will denote x_N, verifying $0 < x_N < 1$ and $N(x_N) = x_N$.

3 On the satisfaction of the Non-Contradiction Law

Once equipped with aggregation operators and strong negations, the Non-Contradiction law for aggregation operators may be defined as follows:

Definition 3.1 Let A be a binary aggregation operator and let N be a strong negation. It is said that A *satisfies the Non-Contradiction law with respect to (w.r.t.)* N when it is $A(x, N(x)) = 0$ for all $x \in [0,1]$.

Note that operators satisfying NC w.r.t. any strong negation are worth-mentioning:

Definition 3.2 Let A be a binary aggregation operator. It is said that A *satisfies the Non-Contradiction law* when it verifies NC w.r.t. any strong negation N.

Regarding the satisfaction of the NC principle, we will first state two very simple but useful conditions that any aggregation operator must verify in order to satisfy this law w.r.t. a given strong negation:

Proposition 3.1 *Let A be a binary aggregation operator and let N be a strong negation with fixed point x_N. If A satisfies NC w.r.t. N, then A must verify the two following conditions:*

1. $A(x_N, x_N) = 0$
2. *Zero is an annihilator element for A, i.e., $A(x,0) = A(0,x) = 0$ holds for any $x \in [0,1]$.*

Proof. Both conditions are easily obtained from the definition of NC, the fact that x_N is the fixed point of N and the monotonicity of A. ∎

The above proposition clearly shows that not every category of aggregation operators is able to satisfy the NC law:

Corollary 3.1 *Let A be a binary aggregation operator and let N be a strong negation. If A is either an averaging operator or a disjunctive operator, then it does not satisfy NC w.r.t. N.*

Proof. Indeed, neither averaging operators (due to their idempotency) nor disjunctive operators (since they verify $A(x,x) \geq x$ for any $x \in [0,1]$) satisfy the first condition of the last proposition. ∎

Therefore, aggregation operators satisfying NC for some N may only be found among conjunctive and hybrid operators verifying the two conditions given in Proposition 3.1 (note that the second one is always true in the case of conjunctive operators). In fact, the following characterization is easily obtained:

Proposition 3.2 *Let A be a binary aggregation operator and let N be a strong negation. A satisfies NC w.r.t. N if and only if for any $x,y \in [0,1]$ it is:*

$$A(x,y) = \begin{cases} 0, & \text{if } y \leq N(x) \\ B(x,y), & \text{otherwise} \end{cases}$$

where B is a binary non-decreasing operator verifying $B(1,1) = 1$.

Proof. If A is an aggregation operator satisfying NC w.r.t. N, the equality $A(x, N(x)) = 0$ for any $x \in [0,1]$ implies, by monotonicity, $A(x,y) = 0$ for any $y \leq N(x)$; when it is $y > N(x)$, A must verify both the boundary condition $A(1,1) = 1$ and the monotonicity property of any aggregation operator. The converse is obvious. ∎

Clearly, the above characterization provides either conjunctive operators (when B is chosen such that $B \leq Min$) or hybrid operators (otherwise). It is also easy to see -thanks to the monotonicity property- that if an aggregation operator satisfies the NC principle w.r.t. some strong negation N, then it satisfies it for an infinite set of strong negations, and that any weaker aggregation operator does also satisfy the principle:

Proposition 3.3 *Let A be a binary aggregation operator satisfying NC w.r.t. some strong negation N. Then:*

1. *A satisfies NC w.r.t. any strong negation N_1 such that $N_1 \leq N$.*
2. *Any binary aggregation operator B verifying $B \leq A$ satisfies NC w.r.t. N.*

Regarding the satisfaction of the NC law w.r.t. *any* strong negation, the following necessary condition may be stated:

Proposition 3.4 *Let A be a binary aggregation operator. If A satisfies NC w.r.t. any strong negation, then $A(x,y) = 0$ for all $(x,y) \in [0,1[^2$.*

Proof. If $x, y \neq 0, 1$, it is always possible to find a strong negation N such that $y = N(x)$, and therefore $A(x,y) = A(x, N(x)) = 0$. If $x = 0$ or $y = 0$, Proposition 3.1 has already shown that 0 is necessarily an annihilator element for A. ∎

Note that the above condition implies, in particular, that operators satisfying NC w.r.t. any N have necessarily a diagonal section which is null except for the point $x = 1$, i.e., they verify $A(x,x) = 0$ for any $x \in [0,1[$. Moreover, such operators may be characterized as follows:

Proposition 3.5 *Let A be a binary aggregation operator. A satisfies NC w.r.t. any strong negation if and only if for any $x,y \in [0,1]$ it is:*

$$A(x,y) = \begin{cases} 0, & \text{if } (x,y) \in [0,1[^2 \\ f_1(x), & \text{if } y = 1 \\ f_2(y), & \text{if } x = 1 \end{cases}$$

where $f_1, f_2 : [0,1] \to [0,1]$ are non-decreasing functions verifying the boundary conditions $f_1(0) = f_2(0) = 0$ and $f_1(1) = f_2(1) = 1$.

Proof. If A satisfies the NC principle w.r.t. any strong negation, Proposition 3.4 shows that it must be $A(x,y) = 0$ whenever $(x,y) \in [0,1[^2$. If $y = 1$, it is necessarily $A(0,1) = 0$ (since

$1 = N(0)$ for any N), $A(1,1) = 1$ (boundary condition of any aggregation operator) and $A(x_1, 1) \leq A(x_2, 1)$ whenever $x_1 \leq x_2$ (monotonicity of aggregation operators), i.e., $A(x, 1) = f_1(x)$. Similar considerations lead to the function f_2. The converse is obvious. ∎

The above characterization shows that aggregation operators satisfying NC w.r.t. any strong negation may be either conjunctive operators (when both f_1 and f_2 are contractive functions, i.e., they verify $f_1, f_2 \leq Id$) or hybrid operators (otherwise), but that, in the later case, they necessarily verify $A \leq Max$.

4 Some Examples

As we have just seen in the previous section, aggregation operators satisfying the NC principle, either w.r.t. some specific strong negation or w.r.t. any of them, can never be averaging or disjunctive operators, and may therefore only be found among either conjunctive or hybrid operators.

4.1 Conjunctive Aggregation Operators

Given a strong negation N, any operator A constructed as in Proposition 3.2, but with the restriction $B \leq Min$ (i.e., choosing B as any conjunctive aggregation operator), is clearly a conjunctive aggregation operator satisfying NC w.r.t. N. Of course, the strongest aggregation operator in this class is obtained when taking $B = Min$, and in this case the resulting operator, given by

$$A(x, y) = \begin{cases} 0, & \text{if } y \leq N(x) \\ Min(x, y), & \text{otherwise} \end{cases}$$

is a t-norm (commutative, associative and non-decreasing operator with neutral element 1), known, when $N = 1 - Id$, as the nilpotent minimum ([8]). Note that in order to get t-norms satisfying NC, it is necessary for B to behave as a t-norm (otherwise the overall operator A will loose some of the t-norms' properties), but this is not sufficient, since the associativity property may be lost (this happens, for example, if B is taken as the product t-norm). Given a strong negation N and a t-norm T, aggregation operators as the ones given in Proposition 3.2 but taking $B = T$ have been studied in the context of residuated lattices and theories based on left-continuous t-norms (see [6] and [7] for an overview and detailed references on this matter); they are denoted by $T_{(N)}$ and called the N-annihilation of T. Therefore, the set of t-norms satisfying NC w.r.t. N coincides with the set of N-annihilations which end up in a t-norm. To that respect, the following results provide wide families of left-continuous t-norms satisfying NC:

- [5] characterizes those $T_{(N)}$ that end up in a t-norm when T is taken as a continuous t-norm.
- [3] generalizes the previous result by characterizing those $T_{(N)}$ that are t-norms when T is taken as a left-continuous t-norm.

Regarding continuity, recall also that the only continuous t-norms satisfying NC w.r.t. to some strong negation N are those which are isomorphic to the Łukasiewicz t-norm , i.e., operators of the form $W_\varphi(x, y) = \varphi^{-1}(Max(0, \varphi(x) + \varphi(y) - 1))$ where $\varphi : [0, 1] \to [0, 1]$ is a continuous and strictly increasing function verifying $\varphi(0) = 0$ and $\varphi(1) = 1$, and $N \leq N_\varphi = \varphi^{-1} \circ N \circ \varphi$. Note in addition that there are also non-left-continuous t-norms satisfying this law, as, for instance, the one known as the drastic product, which is given by:

$$Z(x, y) = \begin{cases} 0, & \text{if } (x, y) \in [0, 1[^2 \\ Min(x, y), & \text{otherwise} \end{cases}$$

When looking for conjunctive operators satisfying NC w.r.t. any strong negation, the characterization given in Proposition 3.5 provides a wide family of operator with this property, since it suffices to choose two contractive functions f_1 and f_2 to obtain one of them. The strongest operator in this class is the one obtained when choosing $f_1 = f_2 = Id$, i.e., the drastic product t-norm Z. Since this t-norm is the weakest one, it is, as a consequence, the only t-norm satisfying NC w.r.t. any strong negation. Of course, any weaker operator satisfies NC w.r.t. any N.

4.2 Hybrid Aggregation Operators

Given a strong negation N, any operator A constructed as in Proposition 3.2, with the restriction

$B \not\leq Min$, is clearly a hybrid aggregation operator satisfying NC for N. Therefore, wide families of these operators may be easily constructed by choosing B among either averaging, hybrid or disjunctive aggregation operators. Obviously, the strongest aggregation operator in this class is the one defined, for any $x, y \in [0,1]$, as:

$$A(x,y) = \begin{cases} 0, & \text{if } y \leq N(x) \\ 1, & \text{otherwise} \end{cases}$$

Regarding the most important known classes of hybrid aggregation operators ([2]), note that neither uninorms nor nullnorms will satisfy the NC principle, since none of them have the structure given in Proposition 3.2. The same happens, for example, with commutative N-symmetric sums (i.e., N-self-dual aggregation operators), because they verify $A(x, N(x)) = x_N$ for any $x \in]0,1[$ and any N. Examples of hybrid aggregation operators satisfying the NC principle w.r.t. some strong negation may be found, for instance, in the class of *quasi-linear T-S operators* ([9]), that is, operators which, in the binary case, are of the form $QL_{T,S,\lambda,f}(x,y) = f^{-1}((1-\lambda)f(T(x,y)) + \lambda f(S(x,y)))$, where T is a t-norm, S is a t-conorm, $\lambda \in [0,1]$ and $f : [0,1] \to [-\infty, \infty]$ is a continuous and strictly monotone function such that $\{f(0), f(1)\} \neq \{-\infty, +\infty\}$. Indeed, it is possible to characterize the operators in this class satisfying NC w.r.t. some strong negation N:

Proposition 4.1 *Let $QL_{T,S,\lambda,f}$ be a binary quasi-linear T-S operator and let N be a strong negation. Then $QL_{T,S,\lambda,f}$ satisfies the NC law w.r.t. N if and only if it is $\lambda = 0$ or ($\lambda \neq 1$ and $f(0) = \pm\infty$), and T satisfies NC w.r.t. N.*

Proof. If $QL_{T,S,\lambda,f}$ satisfies NC, then, according to Proposition 3.1, it must have zero as annihilator, and this happens (see [9]) if and only if it is $\lambda = 0$ (i.e., the operator coincides with T) or $\lambda \neq 1$ and $f(0) = \pm\infty$. In such circumstances, $QL_{T,S,\lambda,f}(x, N(x)) = 0$ implies $(1-\lambda)f(T(x,N(x))) + \lambda f(S(x,N(x))) = \pm\infty$, which, since $S(x, N(x)) \geq Max(x, N(x)) \neq 0$, may only happen if $T(x, N(x)) = 0$, i.e., T satisfies NC w.r.t. N. The converse is obvious. ∎

This, along with the results on t-norms which have been recalled before, provides a wide class of hybrid aggregation operators satisfying NC. This class includes, when choosing $f = log$, the so-called exponential convex T-S operators ([2]), given by $E_{T,S,\lambda}(x,y) = T(x,y)^{1-\lambda} \cdot S(x,y)^{\lambda}$, whenever it is $\lambda \neq 1$ and the t-norm T satisfies NC w.r.t. N.

On the other hand, any aggregation operator built as in Proposition 3.5 by means of non-contractive functions f_1, f_2 is a hybrid operator that satisfies NC w.r.t. any strong negation. The strongest one is obtained when choosing f_1 and f_2 such that $f_1(x) = f_2(x) = 1$ for any $x \neq 0$. The class of quasi-linear T-S aggregation operators does also include some operators satisfying NC w.r.t. any strong negation. Indeed, it may be proved (similarly to Proposition 4.1) that an **operator $QL_{T,S,\lambda,f}$ satisfies NC w.r.t. any strong negation if and only if it is $\lambda = 0$ or ($\lambda \neq 1$ and $f(0) = \pm\infty$), and T satisfies NC w.r.t. any strong negation**. Note that, according to the results given in previous section, the later happens if and only if T is the drastic product t-norm. This means, in particular, that when $\lambda \neq 1$, the exponential convex combination of the drastic product Z with any t-conorm S, i.e., the operator $E_{Z,S,\lambda}(x,y) = Z(x,y)^{1-\lambda} \cdot S(x,y)^{\lambda}$, is a hybrid operator satisfying NC w.r.t. any N.

5 Conclusions and future work

This paper has characterized the classes of binary aggregation operators satisfying the Non-Contradiction law, either w.r.t. to a given strong negation or w.r.t. to any strong negation. It has been shown that both averaging operators and disjunctive operators have to be excluded, but that it is possible to find large families of operators with these properties among conjunctive or hybrid operators. In the case of conjunctive operators, it appears that t-norms satisfying NC w.r.t. a given N are exactly those t-norms built as the N-annihilation of other t-norms, and wide families of the later may be found in the literature ([7]). Regarding conjunctive operators satisfying NC w.r.t. any strong negation, there is just one t-norm in this class, namely the drastic product. When dealing with hybrid operators, even if there are wide families of them satisfying the NC law, it appears that the majority of known operators in this category (such as nullnorms, uninorms or symmetric sums) do not satisfy it. Nevertheless, examples of such operators may be found, for instance, among quasi-linear T-S operators.

Note also that the above results can be easily used to characterize the satisfaction of the Excluded-Middle (EM) law, $p \vee \neg p = 1$, since, as it is well-known, both laws are closely related via duality. In the case of aggregation operators, the EM law may be translated into $A(x, N(x)) = 1$ for any $x \in [0, 1]$, and, when using strong negations, the two laws still have a clear relationship: indeed, if N is a strong negation, it is easy to prove that an aggregation operator A satisfies EM w.r.t. to N if and only if its dual operator, $A_N = N \circ A \circ N \times N$, satisfies NC w.r.t. N.

Regarding the continuation of the present work, we are currently exploring the results obtained when considering a different interpretation of the Non-Contradiction law. Indeed, the way in which the NC law has been interpreted in this paper is not the only possible one (see [11]). The fact "$p \wedge \neg p$ is impossible" has at least two interpretations: the one used in this paper, $p \wedge \neg p = 0$, based on the concept of falsity, or the one used in ancient Aristotelian logic, thought in terms of self-contradiction, that leads to the inequality $p \wedge \neg p \leq \neg(p \wedge \neg p)$. Even if both interpretations coincide in some particular structures (such as orthocomplemented lattices), they differ in many others, where the later is clearly weaker than the former, thus leading to different results. In the case of binary aggregation operators, this new interpretation of the NC law provides the inequality $A(x, N(x)) \leq N(A(x, N(x)))$, or, equivalently, $A(x, N(x)) \leq x_N$, for any $x \in [0, 1]$. The solutions of this inequality include wider families of operators, among which one may find, for example, instances of averaging operators.

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