New axiomatizations of the Shapley interaction index for bi-capacities

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Abstract

Bi-capacities are a natural generalization of capacities (or fuzzy measures) in a context of decision making where underlying scales are bipolar. They are able to capture a wide variety of decision behaviours. After a short presentation of the basis structure, we introduce the Shapley value and the interaction index for capacities. Afterwards, the case of bi-capacities is studied with new axiomatizations of the interaction index.

Keywords: Bi-capacities, Shapley value, interaction index, partnership of criteria.

1 Introduction

Real-valued set functions are widely used in operations research [10], while capacities [2] have become a fundamental tool in decision making. There has been some attempts to define more general concepts, among which can be cited *bi-cooperative games* [1], in game theory, which generalize the idea of *ternary voting games* [3]. In the field of multicriteria decision making, there has been a recent proposal of more general functions, motivated by multicriteria decision making, leading to *bi-capacities*, which have been introduced by Grabisch and Labreuche [6]. Specifically, let us consider a set N of criteria and a set X of alternatives in a multicriteria decision making problem, where each alternative x is described by a vector of real valued score $(x_1, \ldots, x_n)$. A *decision maker* may provide a capacity ν defined over 2^N, where $\nu(A)$ for any $A \subseteq N$ is the score of every *binary alternative* $(1_A, 0_{A^c})$: all criteria of A have score 1 and others, 0. Then it is well known that the *Choquet integral* enables to compute an overall score of the alternative x by interpolation between binary alternatives. Motivated with perceptible limitations of such a model, the decision maker may score alternatives of X on a bipolar scale in this way: to each *bi-coalition* (A, B) of criteria — positive vs. negative ones — a *ternary alternative* $(1_A, -1_B, 0_{(A\cup B)^c})$ is associated: every criterion of A (the *positive part*) has a score equal to 1 (total satisfaction), every one in B (the *negative part*) has a score equal to -1 (total unsatisfaction) and the others have a score equal to 0 (neutrality). Scores are given to each ternary alternative, which defines a bi-capacity.

The concept of interaction index, can be seen as an extension of the notion of *value* or power index [13]. It is fundamental for it enables to measure the interaction phenomena modelled by a capacity on a set of criteria; such phenomena can be for instance substitution or complementarity effects between some criteria [7]. Our aim is to provide axiomatizations of the *Shapley interaction index* of a bi-capacity. Two of them are proposed: at first a *recursive axiom* is used by extension of the one of Grabisch and Roubens [9], and subsequently we work out the *reduced-partnership-consistency axiom* using the concept of partnership [4].

2 Capacities and bi-capacities

Throughout the paper, $N := \{1, \ldots, n\}$ denotes the finite referential set. Furthermore, cardinalities of subsets $S, T, \ldots$ are denoted by the corresponding lower case letters $s, t, \ldots$

We begin by recalling basic notion about capacities for finite sets [2]. A *cooperative game* $\nu : 2^N \to \mathbb{R}^+$ is a set function such that $\nu(\emptyset) = 0$, and ν is said to be a *capacity* if $A \subseteq B \subseteq N$ implies $\nu(A) \leq \nu(B)$ (monotonicity condition). If in addition $\nu(N) = 1$, the capacity is said to be *normalized*.

Let us denote $\mathcal{Q}(N) := \{(A, B) \in 2^N \times 2^N \mid A \cap B = \emptyset\}$.

Definition 1 *A function* $v : \mathcal{Q}(N) \to \mathbb{R}$ *is a* bi-capacity *if it satisfies:*

(i) $v(\emptyset, \emptyset) = 0$

(ii) $A \subseteq B$ *implies* $v(A, \cdot) \leq v(B, \cdot)$ *and* $v(\cdot, A) \geq v(\cdot, B)$.

In addition, v *is* normalized *if* $v(N, \emptyset) = 1 = -v(\emptyset, N)$.

In a multicriteria decision making framework, $v(A, B)$ represents the score of the ternary alternative $(1_A, -1_B, 0_{(A\cup B)^c})$. Note that the definition implies that $v(\cdot, \emptyset) \geq 0$ and $v(\emptyset, \cdot) \leq 0$. Actually, bi-capacities are particular *bi-cooperative games* [1], that is, functions defined over $\mathcal{Q}(N)$ with only condition *(i)* holding.

From its definition, $\mathcal{Q}(N)$ is isomorphic to the set of mappings from N to $\{-1, 0, 1\}$, hence $|\mathcal{Q}(N)| = 3^n$. Also, it is easy to see that $\mathcal{Q}(N)$ is a lattice, when equipped with the order:

$$(A, B) \sqsubseteq (C, D) \text{ if } A \subseteq C \text{ and } B \supseteq D.$$

Supremum and infimum are respectively

$$(A, B) \sqcup (C, D) = (A \cup C, B \cap D)$$
$$(A, B) \sqcap (C, D) = (A \cap C, B \cup D),$$

and top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$. We give in Fig. 1 the Hasse diagram of $(\mathcal{Q}(N), \sqsubseteq)$ for $n = 3$ (where top, bottom and the central point $(\emptyset, \emptyset)$ are represented by black circles).

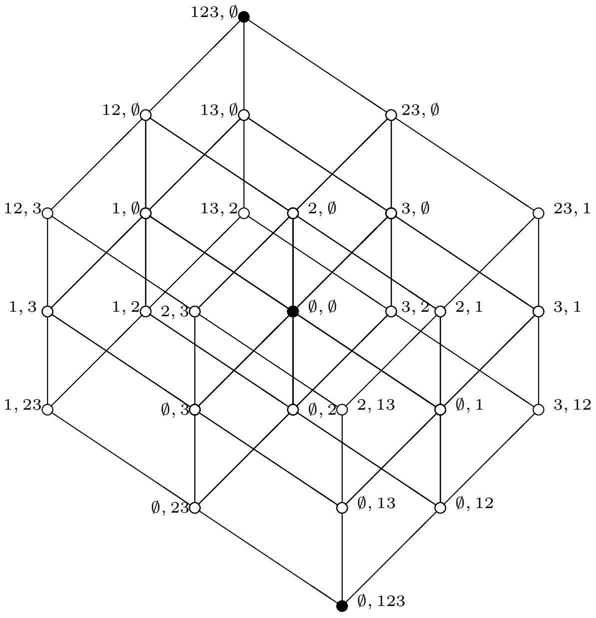

Figure 1: The lattice $\mathcal{Q}(N)$ for $n = 3$

Derivatives of bi-capacities play a central role in the definition of interaction [6] and are defined in this way: if v is a bi-capacity, and $i \in N$,

$$\Delta_{i,\emptyset} v(K, L) := v(K \cup i, L) - v(K, L), \quad \text{for any } (K, L) \in \mathcal{Q}(N \setminus i);$$
$$\Delta_{\emptyset,i} v(K, L) := v(K, L \setminus i) - v(K, L), \quad \text{for any } (K, L) \in \mathcal{Q}(N) \text{ with } i \in L.$$

Recursively, we define $\Delta_{S,T} v$ for any $(K, L) \in \mathcal{Q}(N \setminus S)$ with $L \subseteq T$, for any $i \in S$ and any $j \in T$, by

$$\begin{aligned}\Delta_{S,T} v(K, L) &:= \Delta_{i,\emptyset}(\Delta_{S\setminus i,T} v(K, L)) \\ &= \Delta_{\emptyset,j}(\Delta_{S,T\setminus j} v(K, L)),\end{aligned}$$

so that these values are always non-negative. This generalizes the notion of derivative for a capacity ν, that is $\Delta_i \nu(A) := \nu(A \cup i) - \nu(A)$ if $i \in N, A \subseteq N \setminus i$ and $\Delta_S \nu(A) := \Delta_i(\Delta_{S\setminus i} \nu(A))$ if $A \subseteq N \setminus S$.

Although we develop our results for bi-capacities, we emphasize the fact that all subsequent results remain valid for bi-cooperative games.

3 Previous work on interaction index for capacities

We recall in this section two main ways which have been conducted to axiomatize the interaction index for capacities. Since the following axioms extend the ones of the Shapley value, we may adopt the terminology of *Shapley interaction index.*

In this section, ν denotes a capacity on N. Let us recall its *Shapley value*: for any element $i \in N$,

$$\phi^\nu(i) := \sum_{S \subseteq N\setminus i} p_s^1(n)\,(\nu(S \cup i) - \nu(S)),$$

where the coefficients $p_s^1(n) := \frac{(n-s-1)!\,s!}{n!}$ define a probability distribution over $\{S \subseteq N \setminus i\}$.

The classical axioms introduced by Shapley [13] (see also Weber [14]) are the following

- Linearity: for any $i \in N$, $\phi(i)$ is linear on the set of capacities on N.

  $i \in N$ is said to be *dummy* for ν if $\forall S \subseteq N\setminus i$, $\nu(S \cup i) = \nu(S) + \nu(i)$.
- Dummy axiom: For any capacity ν and any $i \in N$ dummy for ν, $\phi^\nu(i) = \nu(i)$.
- Symmetry axiom: for any permutation σ on N, any capacity ν and any $i \in N$, $\phi^{\nu\circ\sigma^{-1}}(\sigma(i)) = \phi^\nu(i)$. This means that ϕ^ν must not depend on the labelling of the criteria.
- Efficiency axiom (E^c): for any capacity ν, $\sum_{i\in N} \phi^\nu(i) = \nu(N)$; that is to say the values of the criteria must be divided in proportion of the overall score $\nu(N)$.

By generalizing Murofushi and Soneda [12], Grabisch has defined the *interaction index* of capacities [5]. A first axiomatization have been proposed by Grabisch and Roubens and rests on a recursivity axiom [9]. For this, they introduce the following definitions:

Let K a non-empty subset of N and $B \subseteq N \setminus K$. The *restricted capacity* ν^K is the capacity ν restricted to 2^K. The *restriction of* ν *to* K *in the presence of* B is the capacity defined by

$$\nu^K_{\cup B}(S) := \nu(S \cup B) - \nu(B)$$

for any $S \subseteq K$. Lastly, the *reduced capacity* $\nu^{[K]}$ is the capacity defined on $N_{[K]} := (N\setminus K) \cup \{[K]\}$ by

$$\nu^{[K]}(A) := \nu(A^\star)$$

where $A^\star := \begin{cases} A \text{ if } [K] \notin A \\ (A \setminus [K]) \cup K \text{ else} \end{cases}$; $[K]$ actually indicates a single hypothetical player, which is the representative of the players in K.

Recursivity axiom 1 (R1^c): For any capacity ν, $\forall S \subseteq N,\ s > 1$, $\forall i \in S$,

$$I^\nu(S) = I^{\nu^{N\setminus i}_{\cup i}}(S \setminus i) - I^{\nu^{N\setminus i}}(S \setminus i).$$

Recursivity axiom 2 (R2^c): For any capacity ν, $\forall S \subseteq N,\ s > 1$,

$$I^\nu(S) = I^{\nu^{[S]}}([S]) - \sum_{\substack{K \subsetneq S \\ K \neq \emptyset}} I^{\nu^{N\setminus K}}(S \setminus K).$$

Theorem 1 (Grabisch, Roubens [9]) *Under linear axiom, dummy axiom, symmetry axiom, efficiency axiom (E^c) and ((R1^c) or (R2^c)), for any capacity* ν, $\forall S \subseteq N$, $S \neq \emptyset$,

$$I^\nu(S) = \sum_{T \subseteq N\setminus S} p_t^s(n)\,\Delta_S \nu(T),$$

where $p_t^s(n) := p_t^1(n-s+1) = \frac{(n-s-t)!\,t!}{(n-s+1)!}$.

Actually, the authors have shown that (R1^c) and (R2^c) are equivalent under the first axioms [9].

Now we present an axiomatization of Fujimoto, Kojadinovic and Marichal based on the concept of *partnership coalition* [4]; we use for this the following generalized axioms:

Linear axiom (L^c): For any $S \subseteq N$, $I(S)$ is linear on the set of capacities on N.

Dummy axiom (D^c): For any capacity ν and any $i \in N$ dummy for ν,

$$\begin{cases} I^\nu(i) = \nu(i) \\ \forall S \subseteq N \setminus i,\ S \neq \emptyset,\ I^\nu(S \cup i) = 0 \end{cases}.$$

Symmetry axiom (S^c): For any permutation σ on N, any capacity ν and any $S \subseteq N$,

$$I^{\nu\circ\sigma^{-1}}(\sigma(S)) = I^\nu(S).$$

For any $P \subseteq N$, P is said to be a *partnership* for ν if

$$\forall S \subsetneq P,\ \forall T \subseteq N \setminus P,\ \nu(S \cup T) = \nu(T).$$

In other words, as long as the elements of P are not present, the worth of any coalition outside P is left unchanged.

> **Reduced-partnership-consistency axiom (RPC^c):** For any capacity ν and $P \subseteq N$ partnership for ν,
>
> $$I^{\nu}(P) = I^{\nu^{[P]}}([P]).$$

Theorem 2 (Fujimoto, Kojadinovic, Marichal, [4]) *Under (L^c), (D^c), (S^c), (E^c) and (RPC^c), for any capacity ν, $\forall S \subseteq N$, $S \neq \emptyset$,*

$$I^{\nu}(S) = \sum_{T \subseteq N \setminus S} p_t^s(n)\, \Delta_S \nu(T),$$

As in Theorem 1, I^{ν} is again the Shapley interaction index of ν.

4 Axiomatization of the interaction for bi-capacities

In the sequel, v is a bi-capacity. Since criterion i has two possible situations (either being in the positive part or in the negative part of the bi-coalition), the effects of which being not necessarily symmetric on v, we should define a value $\Phi_{i,\emptyset}$ representing the contribution of i "joining the positive part" and a value $\Phi_{\emptyset,i}$ representing the contribution of i "leaving the negative part". Indeed, Labreuche and Grabisch have already axiomatized a Shapley value for bi-capacities [11], which is done by introducing axioms similar to the original ones of Shapley that we recalled above:

> **Linearity (L):** For any $i \in N$, $\Phi_{i,\emptyset}$ and $\Phi_{\emptyset,i}$ are linear on the set of bi-capacities on N.

$i \in N$ is said to be *left-null* (resp. *right-null*) for v if $\forall (K,L) \in \mathcal{Q}(N \setminus i)$,

$$v(K \cup i, L)\ \big(\text{resp. } v(K, L \cup i)\big) = v(K,L).$$

> **Left-null axiom (LN):** For any bi-capacity v and any $i \in N$ left-null for v, $\Phi^v_{i,\emptyset} = 0$.

> **Right-null axiom (RN):** For any bi-capacity v and any $i \in N$ right-null for v, $\Phi^v_{\emptyset,i} = 0$.

> **Invariance axiom (I):** For any two bi-capacities v, w, and any $i \in N$ such that $\forall (K,L) \in \mathcal{Q}(N \setminus i)$
> $$\begin{cases} v(K \cup i, L) = w(K,L) \\ v(K,L) = w(K, L \cup i) \end{cases},$$
> then $\Phi^v_{i,\emptyset} = \Phi^w_{\emptyset,i}$.

This axiom which, has no equivalent in the case of capacities, says that when a game w behaves symmetrically with v, then the Shapley values are the same.

> **Symmetry axiom (S):** For any permutation σ on N, any bi-capacity v and any $i \in N$,
>
> $$\Phi^{v \circ \sigma^{-1}}_{\sigma(i),\emptyset} = \Phi^v_{i,\emptyset} \text{ and } \Phi^{v \circ \sigma^{-1}}_{\emptyset,\sigma(i)} = \Phi^v_{\emptyset,i}.$$

> **Efficiency axiom (E):** For any bi-capacity v,
>
> $$\sum_{i \in N} (\phi^v_{i,\emptyset} + \phi^v_{\emptyset,i}) = v(N,\emptyset) - v(\emptyset,N).$$

Theorem 3 (Labreuche, Grabisch [11]) *Under (L), (LN), (RN), (I), (S) and (E), for any bi-capacity v, $\forall i \in N$,*

$$\Phi_{i,\emptyset} = \sum_{S \subseteq N \setminus i} p_s^1(n)\, [v(S \cup i, N \setminus (S \cup i)) - v(S, N \setminus (S \cup i))],$$

$$\Phi_{\emptyset,i} = \sum_{S \subseteq N \setminus i} p_s^1(n)\, [v(S, N \setminus (S \cup i)) - v(S, N \setminus S)].$$

Now, since Grabisch and Labreuche have also defined an interaction index I^v over $\mathcal{Q}(N)$ for bi-capacities [8], it is necessary to give satisfactory properties to characterize it.

In the first place, as the interaction index for capacities can be obtained from the Shapley value by a recursion formula, we give here a similar approach to build $I^v_{S,T}$ from $\Phi^v_{i,\emptyset} =: I^v_{i,\emptyset}$ and $\Phi^v_{\emptyset,i} =: I^v_{\emptyset,i}$. Practically, $I^v_{S,T}$ denotes the interaction index when S is added to the positive part, and T is withdrawn from the negative part (i.e. the elements of T become neutral).

For any non-empty subset K, the *restricted* bi-capacity v^K is the restriction of v to $\mathcal{Q}(K)$. Besides, $v_+^{N\setminus i}$ and $v_-^{N\setminus i}$ are particular restricted bi-capacities defined by

$$\begin{aligned} v_+^{N\setminus i}(A,B) &:= v(A\cup i, B) - v(i,\emptyset) \\ v_-^{N\setminus i}(A,B) &:= v(A, B\cup i) - v(\emptyset, i), \end{aligned}$$

for any $(A,B) \in \mathcal{Q}(N\setminus i)$. We respectively call $v_+^{N\setminus i}$ and $v_-^{N\setminus i}$ the *restrictions* of v in *positive* and *negative presence* of i. Note that the substractions of $v(i,\emptyset)$ and $v(\emptyset,i)$ are necessary to constraint the nullity in $(\emptyset,\emptyset)$. The following axiom generalizes (R1^c).

> **Recursivity axiom (R):** For any bi-capacity v, $\forall (S,T) \in \mathcal{Q}(N)$, $s+t \geq 2$;
>
> $$\forall i \in S, \quad I^v_{S,T} = I^{v_+^{N\setminus i}}_{S\setminus i,T} - I^{v^{N\setminus i}}_{S\setminus i,T}, \text{ if } s \geq 1,$$
>
> $$\forall i \in T, \quad I^v_{S,T} = I^{v^{N\setminus i}}_{S,T\setminus i} - I^{v_-^{N\setminus i}}_{S,T\setminus i}, \text{ if } t \geq 1.$$

Theorem 4 *Under (L), (LN), (RN), (I), (S), (E) and (R), for any bi-capacity v, for any bi-coalition (S,T), $(S,T) \neq (\emptyset,\emptyset)$,*

$$I^v_{S,T} = \sum_{K\subseteq N\setminus(S\cup T)} p_k^{s+t}(n)\,\Delta_{S,T} v(K, N\setminus(K\cup S)).$$

Let us remark that a such result has also been derived from a generalization of (R2^c) (see [8]).

In the second place, one can take inspiration from the Fujimoto, Kojadinovic and Marichal's work [4] in working out an equivalent axiom of the above (RPC) axiom for capacities. Let us start by defining the concepts of partnership and reduced bi-capacity.

For any $P \subseteq N$, P is said a *partnership* for v if

$\forall (S,T) \in \mathcal{Q}(N\setminus P)$, $\forall P_+, P_- \subsetneq P$ such that $P_+ \cap P_- = \emptyset$, we have

$$v(S\cup P_+, T\cup P_-) = v(S,T).$$

The meaning is the same that for capacities, that is, if all elements of P are not joined together then they have a null effect on the worth of v.

For any non-empty subset K, the *reduced* bi-capacity $v^{[K]}$ is the bi-capacity defined on $N_{[K]} := (N\setminus K)\cup\{[K]\}$ by

$$v^{[K]}(S,T) := v(S^\star, T^\star),$$

where $A^\star := \begin{cases} A \text{ if } [K] \notin A \\ (A\setminus[K])\cup K \text{ else} \end{cases}$, and $[K]$ is still comparable to a single macro player.

> **Reduced-partnership-consistency axiom (RPC):** For any bi-capacity v and any partnership $P \subseteq N$ for v,
>
> $$I^v_{P,\emptyset} = I^{v^{[P]}}_{[P],\emptyset}.$$

A first remark is that one could replace this axiom with its symmetric, that is, $I^v_{\emptyset,P} = I^{v^{[P]}}_{\emptyset,[P]}$, when P is still a partnership for v, one or the other being sufficient. On the other hand, from this axiom and the above ones (N), (LN), (RN), (I), (S) and (E), it is impossible to compute every $I^v_{S,T}$ whenever $T \neq \emptyset$. Consequently, we do it by generalizing these axioms:

> **Generalized linearity (GL):** For any $(S,T) \in \mathcal{Q}(N)$, $I_{S,T}$ is linear on the set of bi-capacities on N.

> **Generalized left-null axiom (GLN):** For any bi-capacity v and any $i \in N$ left-null for v,
>
> $$I^v_{S\cup i,T} = 0, \quad \forall (S,T) \in \mathcal{Q}(N\setminus i).$$

> **Generalized right-null axiom (GRN):** For any bi-capacity v and any $i \in N$ right-null for v,
>
> $$I^v_{S,T\cup i} = 0, \quad \forall (S,T) \in \mathcal{Q}(N\setminus i).$$

> **Generalized invariance axiom (GI):** For any two bi-capacities v, w and any $i \in N$ such that $\forall (K,L) \in \mathcal{Q}(N\setminus i)$, $\begin{cases} v(K\cup i, L) = w(K,L) \\ v(K,L) = w(K, L\cup i) \end{cases}$, we have
>
> $$I^v_{S\cup i,T} = I^w_{S,T\cup i}, \quad \forall (S,T) \in \mathcal{Q}(N\setminus i).$$

Generalized symmetry axiom (GS): For any permutation σ on N, any bi-capacity v and any $(S,T) \in \mathcal{Q}(N)$,

$$I^{v\circ\sigma^{-1}}_{\sigma(S),\sigma(T)} = I^v_{S,T}.$$

Finally, we give the following result:

Theorem 5 *Under (GL), (GLN), (GRN), (GI), (GS) and (E), (R) and (RPC) are equivalent, thus for any bi-capacity v, for any bi-coalition (S,T), $(S,T) \neq (\emptyset,\emptyset)$,*

$$I^v_{S,T} = \sum_{K\subseteq N\setminus(S\cup T)} p^{s+t}_k(n)\,\Delta_{S,T}v(K, N\setminus(K\cup S)).$$

5 Conclusion

Axiomatic characterizations of the interaction index of bi-capacities have been proposed. The presented description is based on generalizations of the recursivity axiom and the reduced-partnership-consinstency axiom. According to the choice of the one or the other, more or less powerful linearity, invariance and symmetry are required.

References

[1] J.M. Bilbao, J.R. Fernández, A. Jiménez Losada, and E. Lebrón. Bicooperative games. In J.M. Bilbao, editor, *Cooperative games on combinatorial structures*. Kluwer Acad. Publ., 2000.

[2] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[3] D. Felsenthal and M. Machover. Ternary voting games. *Int. J. of Game Theory*, 26:335–351, 1997.

[4] K. Fujimoto, I. Kojadinovic, and J.L. Marichal. Axiomatic characterizations of probabilistic and cardinal-probabilistic interaction indices. *Games and Economic Behavior*, to appear.

[5] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[6] M. Grabisch and Ch. Labreuche. Bi-capacities for decision making on bipolar scales. In *EUROFUSE Workshop on Informations Systems*, pages 185–190, Varenna, Italy, September 2002.

[7] M. Grabisch and Ch. Labreuche. Interaction between attributes in a general setting for knowledge discovery. In *4th Int. JIM Conf. (Journées de l'Informatique Messine) on Knowledge Discovery and Discrete Mathematics*, pages 215–222, Metz, France, September 2003.

[8] M. Grabisch and Ch. Labreuche. Bi-capacities. Part 1: definition, Möbius transform and interaction. *Fuzzy Sets and Systems*, 151:211–236, 2005.

[9] M. Grabisch and M. Roubens. An axiomatic approach to the concept of interaction among players in cooperative games. *Int. Journal of game theory*, 28:547–565, 1999.

[10] P.L. Hammer and S. Rudeanu. *Boolean Methods in Operations Research and Related Areas*. Springer, 1968.

[11] Ch. Labreuche and M. Grabisch. A value for bi-cooperative games. *Games and Economic Behavior*, submitted.

[12] T. Murofushi and S. Soneda. Techniques for reading fuzzy measures (III): interaction index. In *9th Fuzzy System Symposium*, pages 693–696, Sapporo, Japan, May 1993. In Japanese.

[13] L.S. Shapley. A value for n-person games. In H.W. Kuhn and A.W. Tucker, editors, *Contributions to the Theory of Games, Vol. II*, number 28 in Annals of Mathematics Studies, pages 307–317. Princeton University Press, 1953.

[14] R.J. Weber. Probabilistic values for games. In A.E. Roth, editor, *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, pages 101–119. Cambridge University Press, 1988.