ON ITERATIVE BOOLEAN-LIKE LAWS OF FUZZY SETS

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Abstract

We study some boolean-like laws with iterative variables in Fuzzy Logic. We show that beyond the classical De Morgan triplets of connectives described by t-norms, t-conorms and strong negations there are interesting models with infinite solutions and surprising situations where there are none.

Keywords: t-norm, t-conorm, strong negation, De Morgan triplet, standard equation, boolean background, inconsistency, iterative variables.

1 Introduction

For a long time many models of Fuzzy Logic have been based upon an initial logical relation with boolean laws leading to functional equations in the structure $([0,1], S, T, N)$ where S is a t-conorm, T is a t-norm and N is a strong negation (see, e.g., [3], [6], [7], [10], [13]).When applying this procedure one founds an intriguing class of equations: those formulated in Fuzzy Logic where some variables appear several times because they come from boolean identities where no simplifications have been made (no application of idempotency or distributivity, absortion, etc.)

Our chief concern in this paper is to exhibit and solve various examples of such equations and show how different situations one may face.

In section 2 we recall the basic definitions of t-norms, t-conorms and strong negations as well as the representation theorem form them. In section 3 we remember that a single structure ([0,1],T, S, N) is not enough for all derived boolean laws. In section 4, we focus on standard equations in Fuzzy Logic and we give a classification in terms of boolean background, consistency, De Morgan types, dualities, etc. In this framework we study solve in section 5 various standard equations which include the so-called "*iterative variables*" and we show that with such equations one can find equations in all he classes described in section 4.

2 Basic definitions

Let us recall here the most basic definitions ([4], [13], [14]) and properties that we will be using in this paper.

Definition 2.1. A *t-norm* is a two-place function T from $[0,1]^2$ into $[0,1]$ such that the following conditions are satisfied for all x, x', y, y' and z in [0,1]:

(i) Associativity: $T(x, T(y,z)) = T(T(x,y), z)$;

(ii) Commutativity: $T(x,y) = T(y,x)$;

(iii) Monotonicity: $T(x,y) \leq T(x',y')$ whenever $x \leq x'$ and $y \leq y'$;

(iv) Unit element: $T(x,1) = T(1,x) = x$;

(v) Null element: $T(x,0) = T(0,x) = 0$.

Note that (v) follows from (iii) and (iv) and that with continuity conditions, (ii) follows from the other conditions.

The most celebrated t-norms are $\text{Min}(x,y) = \text{Minimum}\,\{x,y\}$, $\text{Prod}(x,y) = x \cdot y$, $W(x,y) =$

Max$(x + y - 1, 0)$ and in the discontinuous case $Z(x, y) = 0$ if $(x, y) \in [0, 1)^2$ and $Z(x, y) =$Min(x, y) when $x = 1$ or $y = 1$.

Definition 2.2. A *t-conorm* is a binary operation S on [0,1] such that $S^*(x, y) = 1 - S(1 - x, 1 - y)$ is a t-norm.

Definition 2.3. A **strong negation** N is a continuous strictly decreasing function from [0,1] onto [0,1] such that $N(0) = 1$, $N(1) = 0$ and $N(N(x)) = N(x)$ for all x in [0,1]. Thus the classical negation is $N_0(x) = 1 - x$.

Let us quote a representation theorem (see [1]) for continuous t-norms in its latest version [4]:

Theorem 2.1. Let T be a two-place function from $[0, 1]^2$ into $[0, 1]$ such that:

(i) $T(x, 0) = T(0, x) = 0$,

(ii) $T(1, 1) = 1$,

(iii) T is associative,

(iv) T is jointly continuous.

Then T admits one of the following representations:

(a) $T(x, y) = \text{Min}(x, y)$;

(b) $T(x, y) = t^{(-1)}(t(x) + t(y))$, where t is a continuous and strictly decreasing function from $[0, 1]$ into $\mathbb{R}^+$, with $t(1) = 0$ and $t^{(-1)}$ is its pseudo-inverse of t;

(c) There exists a countable collection $\{[a_n, b_n]\}$ of non-overlapping, closed, non-degenerate subintervals of $[0, 1]$ and a collection of t-norms T_n each of them representable in the form (b) such that

$$T(x,y) = \begin{cases} a_n + (b_n - a_n) T_n\left(\frac{x-a_n}{b_n-a_n}, \frac{y-a_n}{b_n-a_n}\right), \\ \quad \text{if } (x,y) \in [a_n, b_n]^2 \text{ for some } n, \\ \text{Min}(x,y), \\ \quad \text{otherwise.} \end{cases}$$

The previous theorem yields a corresponding representation for all continuous t-conorms.

3 Boolean-like laws of fuzzy sets

Laws of fuzzy sets depend on the theory considered for $[0, 1]^X$, with X the universe of discourse, that is, on the connectives selected for intersection $(\cdot)$, union $(+)$ and complement $(')$. In the case of the standard theories $([0, 1]^X, T, S, N)$, all laws are equivalent to a corresponding law on the structure $([0, 1], T, S, N)$. To see it, it suffices to consider the constant function $\mu_r(x) = r$ for $r \in [0, 1]$ and all $x \in X$. For example, the law $\mu + \mu' = S \circ (\mu \times N \circ \mu) = \mu_1$ holds if and only if it holds $S(r, N(r)) = 1$ for all $r \in [0, 1]$. Also for example, if $T(r, N(r)) = 0$ holds for all $r \in [0, 1]$, then for all $\mu \in [0, 1]^X$ it is $T(\mu(x), N(\mu(x))) = 0$ for all $x \in I$, that is, the law $\mu \cdot \mu' = T \circ (\mu \times N \circ \mu) = \mu_0$ does hold in $([0, 1]^X, T, S, N)$. Because of this fact the study of the laws of the standard theories of fuzzy sets is reduced to their study on $([0, 1], T, S, N)$.

There is no one standard theory with all the laws of a boolean algebra. If $([0, 1], T, S, N)$ is a boolean algebra, then we would have the laws corresponding to $a + a = a$, and $a = a \cdot b + a \cdot b'$, for all a, b in the boolean algebra. That is, $S(r, r) = r$ and $r = S(T(r, s), T(r, N(s)))$, for all r, s in [0, 1], that imply the absurd $S =$Max, $S = W^*_\varphi$, where $W^*_\varphi(x, y) = \varphi^{-1}(W^*(\varphi(x), \varphi(y)))$.

Although for each basic law of a boolean algebra there are standard theories where it holds, what is not yet well known is which are the derived boolean laws that for no standard theory do hold. For example, if there are many theories where $a \cdot b + a \cdot b' = a$ is a law of fuzzy sets (see [2]), the boolean law

$$a \cdot b + a \cdot b' = a + a \cdot b'$$

easily derived from the former, is never a law in standard theories, since its validity would mean $S(T(r, s), T(r, N(s))) = S(r, T(r, N(s)))$, and

- $r = 1$ implies $S(1, N(1)) = 1$, or $S = W^*_\varphi$
- $r = 0$ implies $r = S(r, r)$, or $S =$Max.

A similar reasoning also shows that the derived boolean law $a \cdot b + a \cdot b' = a + a \cdot b$, is not a law on standard theories, that there is nor a standard theory where

$$\mu \cdot \sigma + \mu \cdot \sigma' = \mu + \mu \cdot \sigma,$$

for all μ, σ in $[0,1]^X$.

All that poses the problem of finding new theories of fuzzy sets where this kind of laws can hold. It is clear that standard theories of fuzzy sets are not enough to capture all ways of saying statements.

4 On a classification of standard equations in Fuzzy Logic

Under the general frame of the theory of functional equations (Aczél, 1966) we want to fix our attention on a usual class of equations arising in Fuzzy Logic and distinguish among them several classes. So we start with the following:

Definition 4.1. A functional equation E will be said a *standard equation in Fuzzy Logic* if it involves (by means of compositions and pointwise operations), some given real functions $F_1, ..., F_k$ with domains of the form $[0,1]^{n_1}, ..., [0,1]^{n_k}$, some variables $x_1, ..., x_m$ in [0,1], some unknown functions which are a continuous t-norm T and/or a continuous t-conorm S and/or a strong negation N, and the functional equation is satisfied whenever all variables $x_1, ..., x_k$ are substituted by values in $\{0,1\}$ independently of the particular expressions of the unknowns.

For example, Frank's equation ([12])

$$T(x,y) + S(x,y) = x + y \qquad (*)$$

is a standard equation in Fuzzy Logic in two variables, and the idempotency equation ([4])

$$T(x,x) = x \qquad (**)$$

is another standard equation in a single variable.

These standard equations in Fuzzy Logic extend the usual identities in boolean algebras, i.e., given a material equality in a boolean algebra $(B, +, \cdot, ')$ one can formulate its analog in Fuzzy Logic by changing $+$ by a t-conorm, $\cdot$ by a t-norm, $'$ by a strong negation and the elements $a_1, ..., a_m$ of B appearing in the equality by the corresponding variables $x_1, ..., x_m$ in [0,1] (see e.g., [2], [6], [7]).

Definition 4.2. A standard equation in Fuzzy Logic whose unknowns are just a t-norm T and/or a t-conorm S and/or a strong negation N which has been obtained by a formal substitution as described above from a boolean identity will be said *to have a boolean background.*

The equation of idempotency (**) has a boolean background but Frank's equation (*) has not.

Concerning the possible solutions of standard equations we will distinguish several possibilities:

Definition 4.3. Let E be a standard equation in Fuzzy Logic. Then

(i) E will be said *inconsistent* if it has no solutions;

(ii) E will be said a *De Morgan equation* if the only solutions are among Min, Max, $1 - j$.

(iii) E will be said *a non De Morgan equation* if there are solutions T, S, N but (Min, Max, $1 - j$) is not a solution.

(iv) A *non De Morgan equation* will be said to exhibit *duality* if the solutions T and S are dual by means of N; otherwise we will mention its *non-duality.*

(v) A standard equation with (Min, Max, 1-j) as solutions as well as other triplets (T, S, N) will be said to be *heterogeneous.*

Thus equation (**) where the only solution is T =Min is a De Morgan equation but, e.g., $T(x, 1-x) = 0$ which has boolean background ($A \cap A' \neq \emptyset$) does not admit Min as solution but it has as solutions all t-norms whose zero-set includes the region $\{(x,y) \in [0,1]^2 \mid x + y \leq 1\}$, i.e., this a classical De Morgan equation. The distributivity equation $T(a, S(b,c)) = S(T(a,b), T(a,c))$ is an **heterogenous** equation since admits (Min, Max) as solutions as well as any couple of on-dual connectives of the form (T, Max).

5 On some standard equation in Fuzzy Logic with iterative variables

If one considers a boolean identity where some elements are repeated several times and, without simplifications, one considers the corresponding generalization: what happens with the solutions of the new equation? For instance $A \cap A \cap B = A \cap B$ yields $T(x, x, y) = T(x, y)$ but in this case since T is assumed to be a t-norm on gets (with $y = 1$) $T(x, x) = x$ so T=Min. Our aim here is to show that all kinds of possibilities may occur.

5.1 A standard equation in Fuzzy Logic with iterative variables and boolean background which is inconsistent

In a boolean algebra of sets $(P(X), \cup, \cap, ')$ we have

$$(A \cup A) \cap (A \cap A)' = \emptyset,$$

for all A in $P(X)$. This leads to the equation

$$T(S(a, a), N(T(a, a))) = 0. \qquad (1)$$

Obviously (1) is a standard equation in Fuzzy Logic (note that when $a \in \{0, 1\}$ then (1) holds) and it has a boolean background. The surprising result is the following

Theorem 5.1. Equation (1) has no solutions, i.e., (1) is inconsistent

Proof. Assume that there exists a continuous triplet (T, S, N) satisfying (1). Then we will distinguish two cases according to the possible slopes of the zero set of T and the representation theorem for continuous t-norms.

Case 1. If $z(T) = \{0\} \times [0, 1] \cup [0, 1] \times \{0\}$, i.e., if T is either Min or T is a strict t-norm or T is and ordinal sum not vanishing in any square of the form $[0, k]^2$, then from (1) with $a \in (0, 1)$ we deduce that either $S(a, a) = 0$ or $T(a, a) = 1$; but both possibilities yield the contradictions:

$$0 < a \leq S(a, a) = 0 \quad \text{or} \quad 1 = T(a, a) \leq a < 1.$$

Case 2. If there is a value k in (0,1) such that $[0, k]^2 \subset z(T)$, i.e., T is a non-strict Archimedean t-norm or T is an ordinal sum vanishing on $[0, k]^2$, then we would derive from (1) the contradiction

$$\begin{aligned} 0 &= T(S(k, k), N(T(k, k))) = T(S(k, k), N(0)) \\ &= S(k, k) \geq k > 0. \end{aligned}$$

□

Note. If one formules a Pexider version of (1) of the form

$$T_1(S(a, a), N(T_2(a, a))) = 0, \qquad (2)$$

then *(2) has infinite solutions*, e.g., consider T_1 to be any non-strict Archimedean t-norm generated by t_1, let T_2 be any t-norm not vanishing in its diagonal so (2) yields $t_1(S(a, a)) + t_1(N(T_2(a, a))) \geq 1$ whence $S(a, a) \leq t_1^{-1}[1 - t_1(N(T_2(a, a)))]$ and there are also infinite t-conorms S such that their diagonal satisfy last inequality.

5.2 A standard equation in Fuzzy Logic with iterative variables and a boolean background which is a non De Morgan equation

The boolean relation in $(P(X), \cup, \cap, ')$ given by

$$A \cup B = (A \cap B) \cup [(A \cup B) \cap (A \cap B)^c]$$

admits the generalization

$$\text{Max}(a, b) =$$
$$= S\left(\text{Min}(a, b), T\left(\text{Max}(a, b), N(\text{Min}, (a, b))\right)\right) \qquad (3)$$

a standard equation with a boolean background but such that T =Min, S =Max and $N = 1 - j$ are not solutions (substituting T =Min, S =Max, $N = 1 - j$, $a = 1/2$ and $b = 3/4$ into (3) yields a contradiction. But equation (3) presents a unique set of non-classical solutions.

Theorem 5.2. Equation (3) holds for a continuous t-norm T, a continuous t-conorm S and a strong negation N if and only if there exists a continuous strictly increasing function $s : [0, 1] \to [0, 1]$ with $s(0) = 0$, $s(1) = 1$ such that

$$\begin{aligned} S(x, y) &= s^{(-1)}(s(x) + s(y)), \\ T(x, y) &= s^{-1}(W(s(x), s(y))), \end{aligned}$$

$$N(x) \quad = \quad s^{-1}(1-s(x)). \tag{4}$$

Proof. Assume that (3) holds. The substitution $b = 1$ yields $S(a, N(a)) = 1$, for all a in [0,1] so S is a non-strict Archimedean t-conorm representable in the form (9) with s as described above. The substitution $a = b$ into (3) implies $a = S(a, T(a, N(a)))$ whence $T(a, N(a)) = 0$, for all a, i.e., T is a non-strict Archimedean t-norm representable in the form $T(x, y) = t^{[-1]}(t(x) + t(y))$ with t continuous strictly decreasing and such that $t(0) = 1$, $t(1) = 0$. Therefore for all $a < b$ in [0,1) the condition $b = S(a, T(b, N(a))) = s^{-1}(s(a) + s(T(b, N(a)))$ implies

$$s(b) = s(a) + (s \circ t^{[-1]})(t(b) + t(N(a))),$$

i.e.,

$$s(b) - s(a) = (s \circ t^{-1})(t(b) + t(N(a)))$$

and introducing the variable $s(a) = x < s(b) = y$ we obtain

$$y - x = (s \circ t^{-1})((t \circ s^{-1})(y) + (t \circ s^{-1})(s \circ N \circ s^{-1})(x).$$

then $f = t \circ s^{-1}$ is a continuous strictly decreasing function from [0,1] into itself, $f(0) = 1$, $f(1) = 0$, $n = s \circ N \circ s^{-1}$ is a strong negation and we need to have

$$y - x = f^{-1}(f(y) + f(n(x))).$$

Introducing $y - x = z$ in (0,1) we will have for all x, z in (0,1) such that $x + z \in (0, 1)$:

$$f(z) = f(x + z) + f(n(x))$$

and by virtue of the regularity conditions of f when z tends to 0 we obtain $1 = f(x) + f(n(x))$, i.e., $n(x) = f^{-1}(1 - f(x))$ and

$$f(z) = f(x + z) + 1 - f(x),$$

i.e.,

$$f(x + z) - 1 = f(x) - 1 + f(z) - 1,$$

for all x, z in (0,1) with $x + z \in (0, 1)$, i.e., $f(u) - 1$ satisfies the conditional Cauchy equation above whence ([1]) $f(u) - 1 = A \cdot u$ and $0 = f(1) = 1 + A$, so $A = -1$, $f(u) = 1 - u$, and $t(x) = 1 - s(x)$. Thus (10) and (11) follows at once. The converse is immediate.

Thus (3) is a non De Morgan equations whose solutions given by (4) form a De Morgan triplet since T and S are duals by means of N.

5.3 A standard equation in Fuzzy Logic with iterative variables and a boolean background which is a De Morgan equation

Let us consider the boolean relations of sets

$$(A \cup A \cup B) \cup (A \cap B \cap B) = A \cup B \tag{5}$$

which yields the standard equation

$$S(S(a, a, b), S^*(a, b, b)) = S(a, b),$$

then $b = 0$ yields $S(a, a) = a$, i.e., S =Max and therefore S^* =Min, i.e., we obtain only the classical De Morgan connectives.

Note that if we consider the more general model

$$S(S(a, a, b), T(a, b, b)) = S(a, b) \tag{6}$$

then, as above, $b = 0$ yields S =Max so (6) holds for any t-norm. With this kind of equations if the variables are not repeated then solution may look quite different. For example, if we rewrite (5) in the form

$$(A \cup B) \cup (A \cap B) = A \cup B$$

and we consider the generalization

$$S(S(a, b), T(a, b)) = S(a, b)$$

then ([10]) we have found all possible solutions which include very large families of non-Archimedean solutions as well as interesting families of ordinal sums. Only some cases present duality.

6 Concluding Remark

Along this paper it clearly appears the insufficiency of standard theories of fuzzy sets $([0, 1]^X, T, S, N)$ to capture all the forms of saying statements in boolean algebras where, for example, the amount of laws reduces to 16 the number of formulas with two variables. What seems interesting at this respect is to consider new Pexider-type theories of fuzzy sets

$$([0,1]^X; T_1, ..., T_p; S_1, ..., S_q; N_1, ..., N_n),$$

where **and**, **or** and **not** can be represented in several forms reflecting different uses of such connectives and allowing again to reduce the study of laws to Pexider functional equations in $([0,1]; T_1, ..., T_p; S_1, ..., S_q; N_1, ..., N_n)$.

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