Characterization of L-fuzzy semi-filters and semi-ideals

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Abstract— *Lattice-valued up-sets and down-sets of a poset are investigated under the cutworthy approach. It is proved that the collections of all lattice valued up-sets and down-sets of a given poset are complete lattices under fuzzy inclusion. Properties of these are investigated. For a given collection of crisp up-sets of a poset, necessary and sufficient conditions are given under which the family represents a collection of cuts of a lattice valued up-set. The corresponding conditions are also obtained for the analogous case of down-sets.*

Keywords— L-valued sets (L-fuzzy sets), L-valued up-sets, L-valued down-sets.

1 Introduction

Lattice valued sets and structures have been widely studied from Goguen's first paper [1] on this topic, and since Negoita and Ralescu published their book [2] back in 1975. They appear when the membership grades can be represented by elements of an arbitrary partially ordered set L, instead of just by numbers in the unit interval $[0, 1]$. Although a lot of studies have been done for fuzzy order structures (see, e.g., [3, 4, 5, 6] among many others), the relaxation of the requirements allow us to use lattice-valued sets in a wider context, since they can be more appropriate to model natural problems. It turns out that recently this approach to fuzziness attracts more and more interest, in particular when a structure has a residuated or similar lattice as a co-domain. Thus, in the last decade, lattice-valued mathematics have undergone a significant development. This have created bridges between them and algebraic theories, quantales and order-theoretic structures, various subdisciplines of topology and theoretical computer science (see, e.g., [7, 8, 9, 10, 11, 12, 13, 14]).

Due to the importance of up-sets (semi-filters) and down-sets (semi-ideals) in the classical representation theory, the main purpose of this paper is to study the generalization of these concepts for L-valued sets. Fuzzy semi-ideals (down-sets) and semi-filters (up-sets) were investigated in [15], refereing to a particular T-norm, therefore not in the framework of our cutworthy approach. Let us remember that a property generalized to a fuzzy structure is said to be cutworthy if the corresponding crisp property is preserved by the cut-structures. It is known that many properties of relations or algebraic structures are cutworthy if the co-domain of the investigated fuzzy structure is a complete lattice (e.g., properties of equivalence or ordering relations, then subalgebras of algebraic structures etc.). On the other hand, if this co-domain is a lattice with some additional operations like residuation, product etc., then this cutwortheness is not fulfilled. Since our aim is to connect a lattice valued structure with its cut-structures, the co-domain of all fuzzy structures here is a complete lattice, without additional operations.

In the present investigation, we start with a finite poset X and a complete lattice L. Then we introduce L-valued up-sets (semi-filters) and L-valued down-sets (semi-ideals) as isotone and anti-isotone mappings from X to L, respectively. It turns out that the cut-sets of these are precisely crisp up-sets and down-sets on X. Next we investigate collections of all L-valued up-sets and all L-valued down-sets on X. We prove that under the order induced by the one in L, these collections are complete lattices, fulfilling all identities satisfied by the lattice L. Moreover, we analyze the problem of existence of a representation of collections of crisp up-set on X by collections of cut-set of an L-valued up-set on X. We give necessary and sufficient conditions (fulfilled by the lattice L) under which such representation exists.

Our main motivation in this work was to present a mathematical characterization of these algebraic objets, namely semi-filters and semi-ideals, in the fuzzy environment. The justification to consider lattice valued fuzzy sets has been widely explained in the literature, since lattices are more richer structure and we can obtain non-comparable values of fuzzy sets. They can be applied, for instance, in image processing.

The work is structured in 4 sections. In Section 2 we recall the most relevant concepts concerning classical order sets, up-sets and down-sets, as well as lattice-valued sets. In Section 3 we introduce L-valued up-sets and down-sets and we relate these concepts. The first part of this section is devoted to a wide study of the properties and characterizations for L-valued up-sets. In the second part of the section, analogue problems are investigated for L-valued down-sets. In Section 4 we briefly address some conclusions and future work.

2 Preliminaries

In this section, some well-known definitions and preliminary results are recalled. They will be necessary in order to understand the new concepts introduced in this paper and their associated studies.

2.1 *Order, up-sets and down-sets*

Some necessary notions from the classical order theory are listed in the sequel, together with relevant properties. For more comprehensive presentation, see e.g., books [16, 17].

A **poset** is a nonempty set X equipped with an ordering relation $\leq$. A poset is usually denoted as an ordered pair $(X, \leq)$, or simply by the underlying set X. A sub-poset of $(X, \leq)$ is a poset on a subset Y of X in which the order is the one restricted from X, and usually denoted in the same way ($\leq$). An **up-set** (**semi-filter**) on a poset X is any sub-poset U, satisfying the following: for $x \in U, y \in X$, $x \leq y$

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implies $y \in U$. Dually, an **down-set** (**semi-ideal**) on X is any sub-poset D, satisfying: for $x \in D, y \in X$, $y \leq x$ implies $y \in D$. A **lattice** is a poset L in which for each pair of elements x, y there is a greatest lower bound (glb, infimum, meet) and a least upper bound (lub, supremum, join), denoted respectively by $x \wedge y$ and $x \vee y$. These are binary operations on L. A non-empty poset L is said to be a **complete** lattice if infimum and supremum exist for each subset of L. Complete lattice possesses the **top** (1) and the **bottom** element (0). On the other hand, a lattice L is **distributive**, if each operation is distributive with respect to the other. An example of distributive lattices is presented in the following lemma.

Lemma 1 *The collection of all up-sets (down-sets) of a poset X is a distributive lattice under inclusion.*

Given a lattice L, an element a in L is **completely meet-irreducible** if $a \neq 1$ and for every family $\{x_i \mid i \in I\}$ of elements from L, from $a = \bigwedge_{i \in I} x_i$ it follows that $a = x_i$ for some $i \in I$. When L is a lattice of finite length, a is just said to be **meet-irreducible**.

A **closure operator** on a lattice L is a function $C : L \to L$ such that, for all $p, q \in L$, it fulfills the three following requirements:

- $p \leq C(p)$,
- $p \leq q \to C(p) \leq C(q)$,
- $C(C(p)) = C(p)$.

If $p = C(p)$, then p is a **closed element** under the corresponding closure operator.

Lemma 2 *Let C be a closure operator on a lattice L. Then,*

1. *The subset of all closed elements of L is closed under meets in L.*
2. *The top element of L is a closed element under C.*

Lemma 3 *Let L be a complete lattice. Then,*

1. *For any closure operator C on L, the subset of all closed elements of L is a complete lattice under the order inherited from L.*
2. *If F is a subset of L closed under arbitrary meets, then the map such that to any element p in L associates the value $\bigwedge\{q \in F \mid p \leq q\}$ is a closure operator on L.*

By the definition of the supremum and the infimum in a complete lattice, we also have that $\bigwedge \emptyset = 1$ and $\bigvee \emptyset = 0$.

Lemma 4 *Let C be a closure operator on a complete lattice L. If we consider the relation $\sim$ on L defined by $x \sim y$ iff $C(x) = C(y)$, then*

1. *$\sim$ is an equivalence relation on L.*
2. *Each such equivalence class has the top element and this element is closed.*
3. *The set $L/\sim$ (denoted also by L/C) can be ordered: $[x]_\sim \leq [y]_\sim$ iff $C(x) \leq C(y)$ in L.*
4. *The poset $(L/\sim, \leq)$ is a lattice isomorphic with the poset of closed elements of L, under C.*

2.2 Lattice-valued sets

We present some notions from the theory of lattice-valued structures. More details about the relevant properties can be found in [18, 19].

Lattice-valued, L**-valued sets** or L**-fuzzy sets** are here considered to be mappings from a non-empty set X (domain) into a complete lattice L (co-domain) with the top and bottom elements 1 and 0, respectively. This concept can be seen as a generalization of the concept of fuzzy or valued set and it was introduced by Goguen (see [1]).

If $\alpha : X \to L$ is an L-fuzzy set on X then, for $p \in L$, the set

$$\alpha_p := \{x \in X \mid \alpha(x) \geq p\}$$

is called the p**-cut**, a **cut set** or simply a **cut** of α.

Example 1 *Let us consider the poset $(X, \leq)$ and the complete lattice $(L, \leq)$ represented in Figure 1.*

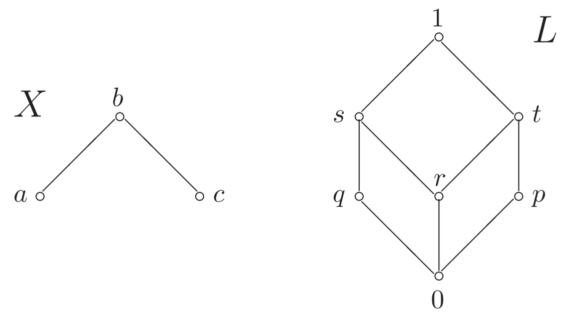

Figure 1: Hasse diagrams of X and L.

Some examples of L-valued sets are

X	a	b	c
α	r	t	p

X	a	b	c
ν	q	s	r

X	a	b	c
μ	r	0	p

X	a	b	c
λ	q	s	t

where we consider a tabular representation for the L-valued sets on X.

In the case of α, for instance, its p-cut is $\alpha_p = \{b, c\}$ and its t-cut is $\alpha_t = \{b\}$.

We can notice that the ordering in X does not have a role in this example. This ordering is used later in the paper.

Some characterizations of the collection of cuts of a lattice-valued set are presented in the next three results.

Proposition 1 *[18] Let $\mathcal{F}$ be a family of subsets of a nonempty set X which is closed under intersections and contains X. Let $\alpha : X \to \mathcal{F}$ be defined by*

$$\alpha(x) = \bigcap(p \in \mathcal{F} \mid x \in p).$$

Then, α is a $\mathcal{F}$-valued set on X with the codomain lattice $(\mathcal{F}, \supseteq)$ such that its family of p-cuts is $\mathcal{F}$ and for every $p \in \mathcal{F}$ it holds that $p = \alpha_p$.

Theorem 1 *[20] Let L be a fixed complete lattice. Necessary and sufficient conditions under which $\mathcal{F} \subseteq \mathcal{P}(X)$ is the collection of cut sets of an L-valued set with domain X are:*

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1. *$\mathcal{F}$ is closed under arbitrary intersections and contains X,*
2. *The dual poset of $\mathcal{F}$ under inclusion can be embedded into L, such that all infima and the top element are preserved under the embedding.*

Proposition 2 *[21] Let L be a lattice of finite length and let $\alpha : X \to L$ be an L-valued set. Then, all p-cuts of α are distinct if and only if all meet-irreducible elements of L belong to $\alpha(X)$.*

3 L-valued up-sets and down-sets

In classical order theory, the families of up-sets and down-sets, associated to any ordered set, play a central role in the representation theory. Once the L-valued sets are considered, a step forward would be to define and characterize up-sets and down-sets in the L-valued logic. That study is made in this section.

Classical up-sets and down-sets can be seen as sets which are "closed under going up" and "closed under going down", respectively. In fact, a subset is an up-set if and only if its characteristic function is order-preserving. Moreover, a subset is a down-set if and only if its characteristic functions is order-preserving, when the dual of the usual order is considered in $\{0, 1\}$. Thus, a natural generalization of these concepts for L-valued sets is presented in Definition 1.

Definition 1 *Let $(X, \leq)$ be a poset and let $(L, \leq)$[1] be a complete lattice in which 0 and 1 are the bottom and the top element, respectively.*

A function $\alpha : X \to L$ such that for all $x, y \in X$

$$x \leq y \implies \alpha(x) \leq \alpha(y)$$

*is an **L-valued up-set** or an **L-valued semi-filter** on X.*

Dually, a function $\mu : X \to L$ such that for all $x, y \in X$

$$x \leq y \implies \mu(y) \leq \mu(x)$$

*is an **L-valued down-set** or an **L-valued semi-ideal** on X.*

Example 2 *If we consider the L-valued sets introduced in Example 1, α and ν are L-valued up-sets, μ is an L-valued down-set and λ is neither L-valued up-set nor L-valued down-set.*

If $L = \{0, 1\}$, that is, if we are in a two-valued logic, it is immediate that any L-valued up-set is a classical up-set and the same happens for down-sets. Thus, these concepts generalize the classical ones.

The duality between these two concepts is showed in the following proposition, by means of complement operators [22].

Proposition 3 *Let $(X, \leq)$ be a poset and let $(L, \leq)$ be a complete lattice in which 0 and 1 are the bottom and the top element, respectively. If there exists a complement operator c on L, that is, a map $c : L \to L$ fulfilling*

1. *Boundary values: $c(0) = 1$ and $c(1) = 0$;*
2. *Non-increasing: $p \leq q \Rightarrow c(q) \leq c(p), \forall p, q \in L$;*
3. *Involution: $c(c(p)) = p, \forall p \in L$;*

then an L-valued set $\alpha : X \to L$ is an L-valued up-set if and only if, its c-complement α^c is an L-valued down-set, where α^c denotes the L-valued set on X defined by $\alpha^c(x) = c(\alpha(x))$.

3.1 Characterization of the family of L-valued up-sets

In this section we present a wide study of the concept of L-valued up-sets. This study is repeated later, in next section, for L-valued down-sets, due to the duality between these two classes of L-valued sets. Let us start with a characterization of the concept of L-valued up-set by means of α-cuts.

Theorem 2 *Let $(X, \leq)$ be a poset, let $(L, \leq)$ be a complete lattice and let α an L-valued set on X. The following two statements are equivalent:*

- *α is an L-valued up-set on X.*
- *The p-cut α_p of α is crisp up-set (semi-filter) on X, for any $p \in L$.*

The proof of this theorem is a immediate consequence of the definitions of p-cut and L-valued up-set.

Example 3 *Let us consider again the L-valued up-set α introduced in Examples 1 and 2. The family of its cuts is formed by the elements*

$$\alpha_0 = X, \alpha_p = \{b, c\}, \alpha_r = \{a, b\}, \alpha_t = \{b\}$$

$$\text{and } \alpha_q = \alpha_s = \alpha_1 = \emptyset.$$

As a consequence of the previous theorem, we know that all these crisp sets are up-sets. Conversely, since all the cuts are crisp up sets on X the previous theorem says that α is an L-valued up-set which is easy to check.

From now on, we denote by $\mathcal{U}_L(X)$ the subset of the L-valued power set of X formed by all L-valued up-sets on the poset X, that is,

$$\mathcal{U}_L(X) = \{\alpha : X \to L \mid \alpha \text{ is an } L\text{-valued up-set}\}.$$

This poset can be ordered naturally using the order induced by the one from the lattice L. Thus, for any $\alpha, \beta \in \mathcal{U}_L(X)$, we say that $\alpha \leq \beta$ if and only if for each $x \in X$ $\alpha(x) \leq \beta(x)$.

Moreover, meet and join exist for any subset of $\mathcal{U}_L(X)$, as it is stated in the following theorem.

Theorem 3 *The poset $(\mathcal{U}_L(X), \leq)$ is a complete lattice.*

In fact, the poset $(\mathcal{U}_L(X), \leq)$ is a complete sublattice of the lattice $(L^X, \leq)$. More specifically, it is a $\{0, 1\}$-complete sublattice with $0(x) = 0$ and $1(x) = 1$, for all $x \in X$.

Due to the definition of the order in $\mathcal{U}_L(X)$, we have the following obvious consequence.

Proposition 4 *If a lattice identity holds in L, then the same identity is satisfied in the lattice $(\mathcal{U}_L(X), \leq)$.*

[1] For simplicity, we consider the same notation for the order in X and in L. Of course these orders do not have to coincide. However, possible ambiguity is removed by the context, so that in any moment we are able to identify order represented by $\leq$.

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From Proposition 4, if L is e.g., a distributive lattice, then we also have that the lattice $(\mathcal{U}_L(X), \leq)$ is distributive.

Now, we prove a result which allows us to know when, starting from a given collection of crisp up-sets, it is possible to built an L-valued up-set.

Theorem 4 *Let $(X, \leq)$ be a poset, let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of some up-sets of a poset X, and let $(L, \leq)$ be a complete lattice. Then, there is an L-valued up-set $\alpha : X \to L$ such that its family of cuts is equal to $\mathcal{F}$ if and only if the following two conditions hold:*

1. *$\mathcal{F}$ is closed under intersections and contains X;*
2. *there is a closure operator C on L, such that the poset $(L/C, \leq)$ is order isomorphic to $(\mathcal{F}, \supseteq)$.*

In the proof of this theorem we have to prove the equivalence among these conditions and the conditions imposed in Theorem 1.

Example 4 *Let us consider again the poset X, the complete lattice L and the L-valued sets α, ν introduced in Example 1. Let $\mathcal{F}$ be the family of up-sets of X defined by*

$$\mathcal{F} = \{X, \{a, b\}, \{b, c\}, \{b\}, \emptyset\}.$$

This family fulfills Conditions 1 and 2 in Theorem 4 and therefore, there exists an L-valued up-set whose family of cuts is equal to $\mathcal{F}$. This L-valued up-set is precisely the L-valued set α. Of course, it does not need to be unique. For instance, also ν is an L-valued up-set with $\mathcal{F}$ as its family of cuts.

In the previous theorem, we have characterized families of up-sets which are the cut-sets of an L-valued up-set. Now, we are going to characterize the L-valued up-set such that this family is formed by all the (crisp) up-sets.

Proposition 5 *Let $(X, \leq)$ be a poset, let $(L, \leq)$ be a complete lattice and let $\alpha : X \to L$ be an L-valued up-set. The following two statements are equivalent:*

- *The family of cut sets of α is formed by all the crisp up-sets of poset $(X, \leq)$.*
- *For every family $\{x_i \mid i \in I\}$ of elements from X and every $x \in X$, it holds that*

$$\alpha(x) \geq \bigwedge_{i \in I} \alpha(x_i) \Rightarrow \exists i \in I \mid x \geq x_i.$$

A main step in the proof of the previous proposition is to consider that the $\bigcup_{i \in I} \uparrow x_i$ is a crisp up-set of α, where $\{x_i \mid i \in I\}$ is the family of elements from X such that $\alpha(x) \geq \bigwedge_{i \in I} \alpha(x_i)$ for $x \in X$ and $x \ngeq x_i$ for all $i \in I$.

As a consequence of the previous proposition, we obtain two necessary conditions for a representation by means of all the crisp up-sets. They are presented in the two following corollaries.

Corollary 1 *Let $(X, \leq)$ be a poset, let $(L, \leq)$ be a complete lattice and let $\alpha : X \to L$ be an L-valued up-set. If the family of cut sets of α is formed by all the crisp up-sets of poset $(X, \leq)$, then α is an order-embedding.*

In the following result it is necessary to consider the subposet L^α of L associated to any L-valued set α on X and generated by taking all infima of all the subsets of $\alpha(X)$ (including infimum of the empty set - the top element of L). Thus, given an L-valued set α on X,

$$L^\alpha = \{p \in L \mid p = \wedge B \text{ with } B \subseteq \alpha(X)\}.$$

Let us note that it is immediate that L^α is a complete lattice.

Corollary 2 *Let $(X, \leq)$ be a poset, let $(L, \leq)$ be a complete lattice and let $\alpha : X \to L$ be an L-valued up-set. If the family of cut sets of α is formed by all the crisp up-sets of poset $(X, \leq)$, then $\alpha(x)$ is a completely meet-irreducible element in the lattice L^α for every $x \in X$ with $\alpha(x) \neq 1$.*

In order to illustrate Proposition 5 and Corollaries 1 and 2, we show an example of an L-valued set such that its family of cuts is formed by all the crisp up-sets of its domain.

Example 5 *Let us consider again the L-valued up-set α introduced in Example 1. In Example 4 we have obtained its family of cut sets. It is immediate that this is the family of all up-sets of X. Thus, by Corollaries 1 and 2, we know that α is an order-embedding and $\alpha(x)$ is a completely meet-irreducible element in the lattice $L^\alpha = \{0, p, r, t, 1\}$ for every $x \in X$.*

Now, we suppose that L is a lattice of finite length. Then, combining Proposition 2 and Proposition 5, we obtain the following result.

Theorem 5 *Let $(X, \leq)$ be a poset, let $(L, \leq)$ be a complete lattice of finite length and let $\alpha : X \to L$ be an L-valued up-set such that*

1. *α is an bijection to set $M(L)$ of all meet-irreducible elements of lattice L.*
2. *For every family $\{x_i \mid i \in I\}$ of elements from X, if $\alpha(x) \geq \bigwedge_{i \in I} \alpha(x_i)$ for $x \in X$, then $x \geq x_i$, for some $i \in I$.*

Then, the lattice L is isomorphic with the family of all up-sets of the poset X and therefore L is distributive.

Example 6 *Let us consider again the L-valued up-set α introduced in Example 1. Since $\alpha_s = \alpha_q = \emptyset$, it is not true that all its p-cuts are distinct (see Proposition 2) and therefore Condition (1) in the previous theorem does not hold. We can also notice that s is a meet irreducible element in lattice L and it is not a value of the function α. Therefore, we cannot use this example in order to illustrate Theorem 5.*

Let us consider a different poset $(X, \leq)$, a different complete lattice $(L, \leq)$ and a different L-valued up-set α, which are represented in Figure 2.

Given the L-valued set $\alpha : X \to L$, the p-cuts of α are all the up-sets of X:

$$\alpha_0 = \{0, a, b, c, 1\},\ \alpha_n = \{a, b, c, 1\},\ \alpha_p = \{a, b, 1\},$$

$$\alpha_q = \{a, c, 1\},\ \alpha_r = \{b, c, 1\},\ \alpha_s = \{a, 1\}\ \alpha_t = \{b, 1\},$$

$$\alpha_u = \{c, 1\},\ \alpha_v = \{1\},\ \alpha_1 = \emptyset,$$

which are all distinct.

Observe that using the mapping $E(l) = \alpha_l$ for all $l \in L$, that is, if we change l by α_l, the result is the set of all the upsets ordered by $\supseteq$.

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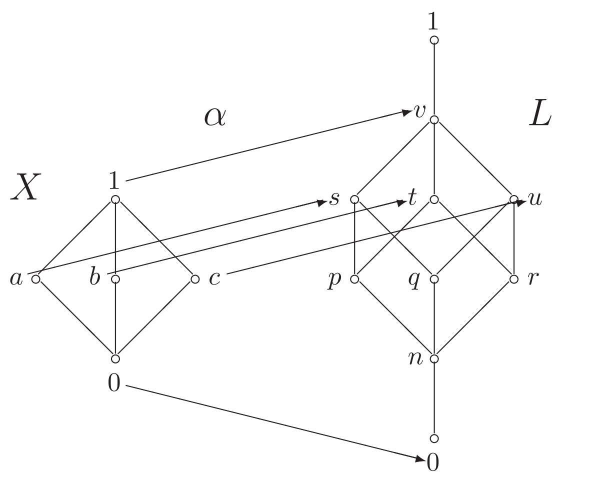

Figure 2: Hasse diagrams of X and L and L-valued set α.

Let us conclude this section with a version of the well-known Birkhoff Representation Theorem: on the one hand, by this theorem, we know that any finite distributive lattice L can be represented by the set of all up-sets on the set of all meet-irreducible elements of L. Now by Theorem 5, we can define an L-valued up-set whose cut sets are order isomorphic to L. Therefore, we can obtain a representation of any distributive lattice as a family of cut-sets of an L-valued up-set. This L-valued up-set is the embedding of $M(L)$ in L, that is, $\alpha : M(L) \to L$ with $\alpha(p) = p, \forall p \in M(L)$.

3.2 Characterization of the family of L-valued down-sets

All the previous studies made for L-valued up-sets could be repeated for L-valued down-sets. By the duality between these two concepts (see, for instance, Proposition 3), the obtained results are totally analogous.

Theorem 6 *Let $(X, \leq)$ be a poset and let $(L, \leq)$ be a complete lattice. A function $\mu : X \to L$ is an L-valued down-set on X if and only if any p-cut μ_p of μ is a crisp down-set (semi-ideal) on X, for every $p \in L$.*

Theorem 7 *Let $(X, \leq)$ be a poset and let $(L, \leq)$ be a complete lattice. If we consider the set $\mathcal{D}_L(X)$ formed by the collection of all L-valued down-sets on X and the induced order on $\mathcal{D}_L(X)$ is defined for any $\mu, \nu \in \mathcal{D}_L(X)$ as follows*

$$\mu \leq \nu \text{ if and only if } \mu(x) \leq \nu(x), \forall x \in X,$$

then the poset $(\mathcal{D}_L(X), \leq)$ is a complete lattice.

Proposition 6 *Let $(X, \leq)$ be a poset and let $(L, \leq)$ be a complete lattice. If a lattice identity holds in L, then the same identity is satisfied in the lattice $(\mathcal{D}_L(X), \leq)$. Thus, for a distributive lattice L, the lattice $(\mathcal{D}_L(X), \leq)$ is also distributive.*

Theorem 8 *Let $(X, \leq)$ be a poset, let $\mathcal{F} \subseteq \mathcal{P}(X)$ be a family of some down-sets of a poset X, and let $(L, \leq)$ be a complete lattice. Then, there is an L-valued down-set $\mu : X \to L$ such that its family of cuts is equal to $\mathcal{F}$ if and only if the following two conditions hold:*

1. *$\mathcal{F}$ is closed under intersections and contains X;*
2. *there is a closure operator C on L, such that the poset $(L/C, \leq)$ is order isomorphic to $(\mathcal{F}, \supseteq)$.*

Proposition 7 *Let $(X, \leq)$ be a poset, let $(L, \leq)$ be a complete lattice and let $\mu : X \to L$ be an L-valued down-set. The family of cut sets of μ is formed by all the crisp down-sets of poset $(X, \leq)$ if and only if for every family $\{x_i \mid i \in I\}$ of elements from X the condition $\mu(x) \geq \bigwedge_{i \in I} \mu(x_i)$ for $x \in X$ implies that $x \leq x_i$ for some $i \in I$.*

Corollary 3 *Let $(X, \leq)$ be a poset, let $(L, \leq)$ be a complete lattice and let $\mu : X \to L$ be an L-valued down-set. If the family of cut sets of μ is formed by all the crisp down-sets of poset $(X, \leq)$, then the two following statements hold:*

- *μ is an order-embedding.*
- *For any $x \in X$, $\mu(x)$ is a completely meet-irreducible element in the lattice L^{μ}.*

Theorem 9 *Let $(X, \leq)$ be a poset, let $(L, \leq)$ be a complete lattice of finite length and let $\mu : X \to L$ be an L-valued down-set such that*

1. *μ is an bijection to set $M(L)$ of all meet-irreducible elements of lattice L.*
2. *For every family $\{x_i \mid i \in I\}$ of elements from X, if $\mu(x) \geq \bigwedge_{i \in I} \mu(x_i)$ for $x \in X$, then $x \leq x_i$, for some $i \in I$.*

Then, the lattice L is distributive and L is isomorphic with the family of all down-sets of the poset X.

4 Conclusion

In this work, we have carried out an in-depth study of some particular classes of L-valued sets: up-sets and down-sets. We have started with the L-valued up-sets and we have proven that this is a cutworthy property. Moreover, we have established necessary and sufficient conditions under which for a given family of crisp up-sets, there exists an L-valued set such that its collection of cuts coincides with the family of crisp up-sets. In particular, we have done a deeper study for the L-valued up-sets such that its family of cuts is formed by all the crisp up-sets. In the case of lattices of finite length, we have arrived to the Birkhoff Representation Theorem. All these studies have been repeated for the case of L-valued down-sets and analogous results have been obtained.

In future works we would like to obtain some necessary and sufficient conditions not only for the existence of an L-valued up-set (resp. down-set) for a given family of up-sets (resp. down-sets), but for the uniqueness of such L-valued up-set (resp. down-set). The properties under which uniqueness is guaranteed were already studied for general fuzzy sets in [23]. Now, we try to adapt these results in two senses: L-valued sets instead of fuzzy sets (valued in the interval $[0, 1]$) and up-sets (resp. down-sets) instead of any sets.

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