An abstract approach to fuzzy logics: implicational semilinear logics

Petr Cintula[1] Carles Noguera[2]

1. Institute of Computer Science, Czech Academy of Sciences
Prague, Czech Republic
2. Department of Mathematics and Computer Science, University of Siena
Siena, Italy
Email: cintula@cs.cas.cz, cnoguera@iiia.csic.es

Abstract— This paper presents a new abstract framework to deal in a uniform way with the increasing variety of fuzzy logics studied in the literature. By means of notions and techniques from Abstract Algebraic Logic, we perform a study of non-classical logics based on the kind of generalized implication connectives they possess. It yields the new *hierarchy of implicational logics*. In this framework the notion of *implicational semilinear logic* can be naturally introduced as a property of the implication, namely a logic L is an implicational semilinear logic iff it has an implication such that L is complete w.r.t. the matrices where the implication induces a linear order, a property which is typically satisfied by well-known systems of fuzzy logic. The hierarchy of implicational logics is then restricted to the semilinear case obtaining a classification of implicational semilinear logics that encompasses almost all the known examples of fuzzy logics and suggests new directions for research in the field.

Keywords— Abstract Algebraic Logic, Implicative logics, Leibniz Hierarchy, Mathematical Fuzzy Logic, Semilinear logics.

1 Introduction

Mathematical Fuzzy Logic is the subdiscipline of Mathematical Logic which studies the logical systems that, since the inception of the theory of fuzzy sets, have been proposed to deal with the reasoning with predicates that can be modelled by fuzzy sets. The first ones, coming from the many-valued logic tradition, were Łukasiewicz and Gödel-Dummett logics, both complete w.r.t. the semantics given by a continuous t-norm. Later, a third system with this feature was introduced: product logic. Starting from these three main examples, the area has followed a long process of increasing generalization that has led to wider and wider classes of fuzzy logics. The first step was taken by Hájek [14] when he proposed the Basic fuzzy Logic BL, which turn out to be complete w.r.t. the semantics of all continuous t-norms. Later on, to put it in Hájek's words, scholars kept *removing legs from the flea* by considering weaker notions of fuzzy logic: divisibility was removed in the logic MTL [8] which is complete w.r.t. the semantics of all left-continuous t-norms, negation was removed when considering fuzzy logics based on hoops [9], commutativity of t-norms was disregarded in [13], and t-norms were replaced by *uninorms* in [16]. On the other hand, logics with a higher expressive power were introduced by considering expanded real-valued algebras (with projection Δ, involution $\sim$, truth-constants, etc.), and in recent works fuzzy logics have started emancipating from the real-valued algebras as the only intended semantics by considering systems complete w.r.t. rational, finite or hyperreal linearly ordered algebras [5].

When dealing with this huge variety of fuzzy logics one may want to have some tools to prove general results that apply not only to a particular logic, but to a class of logics. To some extent this has been achieved by means of the notions of core and Δ-core fuzzy logics [15] and results for these classes can be already found in a number of papers. However, those classes contain roughly just expansions of MTL and $\mathrm{MTL}_\triangle$ logics, so they do not cover weaker systems such as those from [16]. This shows that general notions of fuzzy logics are very useful, but we need to look for a more abstract framework to cope with all known examples and with other new logics that may arise in the near future.

In doing so, one certainly needs some intuition about the class of objects he would like to mathematically determine, namely some intuition of what are the minimal properties that should be required for a logic to be fuzzy. The evolution outlined above shows that almost no property of these systems was essential as they were step-by-step disregarded. Nevertheless, there is one that has remained untouched so far: *completeness w.r.t. a semantics based on linearly ordered algebras*. It actually corresponds to the main thesis of [1] that defends that *fuzzy logics are the logics of chains*. Such a claim must be read as a methodological statement, pointing at a roughly defined class of logics, rather than a precise mathematical description of what fuzzy logics are, since there could be many different ways in which a logic might enjoy a complete semantics based on chains. The aim of the present paper is to use some notions and techniques from Abstract Algebraic Logic (AAL) to provide a new framework where we can develop in a natural way a particular technical notion corresponding to the intuition of fuzzy logics as the logics of chains. Namely, we will present the *hierarchy of implicational logics* as a new classification of non-classical logics extending the well-known *Leibniz hierarchy* and encompassing other important classes such as implicative logics [18] and weakly implicative logics [4]. Inside this new hierarchy we will build a very general class of fuzzy logics that we will call *implicational semilinear logics*.

The technical aspects of this paper are based on the submitted work [6] however it concentrates on presenting the justification of our new framework from the point of view of Mathematical Fuzzy Logic. Section 2 describes our general setting and Section 3 informally summarizes our arguments. The remaining sections present samples of technical arguments supporting our thesis. Finally, Appendix A recalls crucial preliminary notions from the theory of logical calculi.

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2 The hierarchy of implicational logics

Although implication in the vast majority of existing fuzzy logics is given by a single (primitive or derived) connective, we follow a long-established tradition of Abstract Algebraic Logic and consider that implication could be given by a (possibly parameterized) set of formulae. However the following convention will allow us to hide this feature of our approach, providing a high level of abstraction without any apparent increase of complexity. Let $\Rightarrow(p, q, \overrightarrow{r})$ be a set of $\mathcal{L}$-formulae in two variables and, possibly, with a sequence of parameters $\overrightarrow{r}$. Given formulae φ, ψ and a sequence of formulae $\overrightarrow{\alpha}$, $\Rightarrow(\varphi, \psi, \overrightarrow{\alpha})$ denotes the set obtained by substituting the variables in $\Rightarrow(p, q, \overrightarrow{r})$ by the corresponding formulae, and $\varphi \Rightarrow \psi$ denotes the set $\bigcup\{\Rightarrow(\varphi, \psi, \overrightarrow{\alpha}) \mid \overrightarrow{\alpha} \in \mathrm{Fm}_{\mathcal{L}}^{\leq\omega}\}$.

We generalize the following properties, typically satisfied by an implication, to sets of (parameterized) formulae. However a reader can always understand these conditions as if $\Rightarrow$ would be just a single binary connective.

Definition 1. *Let* L *be a logic and* $\Rightarrow(p, q, \overrightarrow{r}) \subseteq \mathrm{Fm}_{\mathcal{L}}$ *be a parameterized set of formulae. We say that* $\Rightarrow$ *is a* weak p-implication *in* L *if:*

(R) $\vdash_{\mathrm{L}} \varphi \Rightarrow \varphi$
(MP) $\varphi, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \psi$
(T) $\varphi \Rightarrow \psi, \psi \Rightarrow \chi \vdash_{\mathrm{L}} \varphi \Rightarrow \chi$
(sCng) $\varphi \Rightarrow \psi, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} c(\chi_1, \ldots, \chi_i, \varphi, \ldots, \chi_n)$
$\Rightarrow c(\chi_1, \ldots, \chi_i, \psi, \ldots, \chi_n)$
for each $\langle c, n\rangle \in \mathcal{L}$ *and each* $i < n$

We change the prefix 'weak' to 'algebraic' if there is a set $\mathcal{E}(p)$ *of equations in one variable such that*

(Alg) $p \dashv\vdash_{\mathrm{L}} E[\mathcal{E}(p)]$,

where $E(p, q, \overrightarrow{r}) = \Rightarrow(p, q, \overrightarrow{r}) \cup \Rightarrow(q, p, \overrightarrow{r})$
We change the prefix 'weak' to 'regular' if:

(Reg) $\varphi, \psi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$

We change the prefix 'weak' to 'Rasiowa' if:

(W) $\varphi \vdash_{\mathrm{L}} \psi \Rightarrow \varphi$

Finally, if $\Rightarrow$ *is parameter-free we drop the prefix 'p-'.*

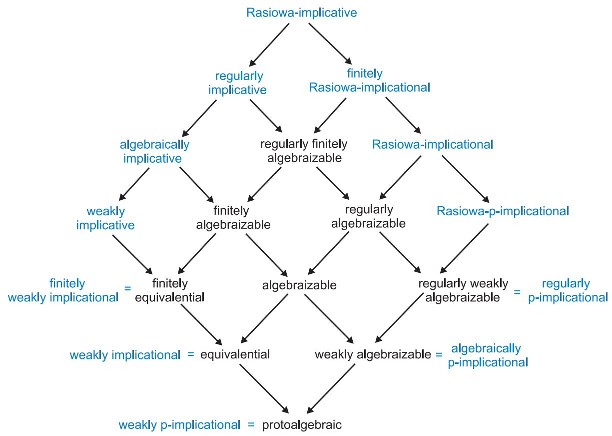
We can easily show the relative strength of defined notions: each Rasiowa p-implication is a regular p-implication and each regular p-implication is an algebraic p-implication.

Definition 2. *We say that a logic* L *is a* weakly/algebraically/regularly/Rasiowa- (p-)implicational logic *if there is a (parameterized) set of formulae* $\Rightarrow$ *which is a weak/algebraic/regular/Rasiowa (p-)implication in* L. *We add the prefix* 'finitely' *if* $\Rightarrow$ *is finite and we use the adjective* 'implicative' *instead of* 'implicational' *if* $\Rightarrow$ *is a* parameter-free singleton.

Each class of the well-known Leibniz hierarchy [7] coincides with some of our newly defined classes. In fact, our new taxonomy extends it, incorporates other already existing classes of logics,[1] and offers a more systematic way of classification: in one axis we go from p-implicational, implicational, finitely implicational to implicative (depending on the form of the implication set); in the second one we use prefixes 'weakly', 'algebraically', 'regularly', or 'Rasiowa-' (depending on extra properties fulfilled by that set). The translation table is:

Classes of Leibniz hierachy	*Our systematic names*
protoalgebraic	weakly p-implicational
(finitely) equivalential	(finitely) weakly impl.
weakly algebraizable	algebraically p-impl.
regularly weakly algebraizable	regularly p-implicational
(finitely) algebraizable	(finitely) algebraically impl.
(finitely) regularly algebraizable	(finitely) regularly impl.

Our new classification of logics, *the hierarchy of implicational logics* is depicted below (the arrows correspond to the class subsumption relation). We can show that almost all classes of logics in this hierarchy are mutually different; only the difference between Rasiowa-implicational and Rasiowa-p-implicational logics remains to be shown.



The syntactical notion of weak p-implication that we have introduced has a natural semantical counterpart: a preorder in the models that becomes an order in reduced models.

Definition 3. *Let* $\Rightarrow$ *be a parameterized set of formulae and* $\mathbf{A} = \langle \mathcal{A}, F\rangle$ *a matrix. We define a binary relation* $\leq_{\mathbf{A}}^{\Rightarrow}$ *on* A*:*

$$a \leq_{\mathbf{A}}^{\Rightarrow} b \quad \text{iff} \quad a \Rightarrow^{\mathcal{A}} b \subseteq F.$$

Proposition 1. *Let* L *be a logic and* $\mathbf{A} \in \mathbf{MOD}(\mathrm{L})$. *Then a parameterized set* $\Rightarrow$ *is a weak p-implication in* L *iff* $\leq_{\mathbf{A}}^{\Rightarrow}$ *is a preorder and its symmetrization of* $\leq_{\mathbf{A}}^{\Rightarrow}$ *is the Leibniz congruence of* $\mathbf{A}$.

Clearly $\leq_{\mathbf{A}}^{\Rightarrow}$ is an order iff $\mathbf{A}$ is reduced. Thus (by virtue of Theorem 10) we can say that a L is complete w.r.t. the class of ordered matrices. Our main interest are the logics complete w.r.t. linearly ordered matrices in the following sense:

Definition 4. *Let* L *be a logic and* $\mathbf{A} = \mathbf{MOD}(\mathrm{L})$. *We say that* $\mathbf{A}$ *is a* linear model *w.r.t.* $\Rightarrow$ *if* $\leq_{\mathbf{A}}^{\Rightarrow}$ *is a linear order. The class of linear models of* L *is denoted by* $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Observe that the class of linear models is not intrinsically defined for a given logic as it depends on the chosen implication. However, we will see later that in a reasonably wide class of logics all *semilinear implications* define the same linear models. But even in a general case we can prove:

Theorem 1. *Let* L *be a protoalgebraic logic. Then, for any weak p-implication* $\Rightarrow$, $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L}) \subseteq \mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}}$.

[1] Rasiowa-implicative logics were already defined in 1974 by Rasiowa [18] and weakly implicative logics in 2006 by Cintula [4].

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3 Semilinear implications and logics

Given a logic L and a weak p-implication $\Rightarrow$, we say that $\Rightarrow$ is a weak *semilinear* p-implication if the logic is complete w.r.t. the class of its corresponding linear models. Formally:

Definition 5. *Let* L *be a logic and* $\Rightarrow$ *a weak p-implication. We say that* $\Rightarrow$ *is a weak* semilinear *p-implication if*

$$\vdash_{\mathrm{L}} \;=\; \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}.$$

Later we define *implicational semilinear logics* as those possessing some weak semilinear p-implication. Obviously, they will be fuzzy logics in the sense of [1]. However, we choose the term ‘semilinear’ instead of ‘fuzzy’ in spite of the fact that a first step towards the general definition we are offering here had been done by the first author in [4], when he defined the class of *weakly implicative fuzzy logics* (in our new framework: logics with a weak semilinear implication given a single binary connective). We have realized that the attempt of [4] of using the term ‘fuzzy’ to formally define a class of logics was rather futile as such word is heavily charged with many conflicting potential meanings.

Therefore, we have opted now for the new neutral name ‘semilinear’. The term was first used by Olson and Raftery in [17] in the context of residuated lattices; it refers to the Universal Algebra tradition of calling a class of algebras ‘semi$X$’ whenever its subdirectly irreducible members have the property X (how this is related to our case will be apparent after Theorem 3). Despite using a new neutral name our intention remains the same: to formally delimit the class of fuzzy logics inside some existing abstract class of formal non-classical logics (originally, among weakly implicative ones now among the protoalgebraic ones). Of course it should include almost all the prominent examples of fuzzy logics known so far and exclude non-classical logics which are usually not recognized as fuzzy logics in the Logic community.

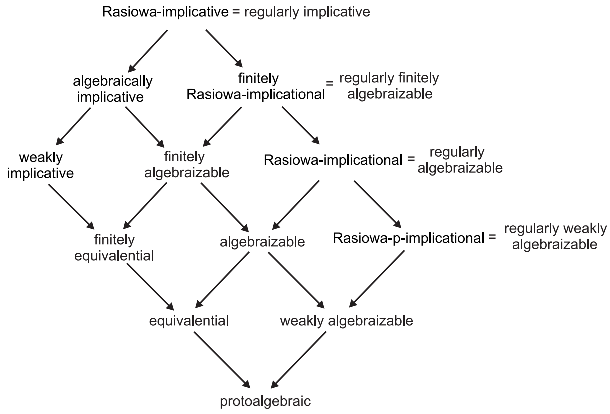
However let us stress that we *do not* expect to capture in a mathematical definition *the whole* intuitive notion of *arbitrary* fuzzy logic. Even if we would agree that linearity of semantics is crucial for a formal logic to be *fuzzy* there could be several other ways in which a logic might have a complete semantics somehow based on chains (see e.g. [2] or some recent work on modal fuzzy logics).

We formally define classes of implicational semilinear logics based on the form of semilinear implication they possess.

Definition 6. *We say that* L *is a* weakly/algebraically/ Rasiowa- (p-)implicational semilinear logic *if there is a (parameterized) set of formulae* $\Rightarrow$ *such that it is a weak/algebraic/Rasiowa semilinear (p-)implication in* L. *We add the* ‘finitely’ *if the set* $\Rightarrow$ *is finite and we use* ‘implicative’ *instead of* ‘implicational’ *if* $\Rightarrow$ *is a parameter-free singleton.*

We have not defined the class of *regularly* (p-)implicational (implicative) semilinear logics, because (as we will see in Corollary 2) we would obtain that each regularly p-implicational semilinear logic is a Rasiowa-p-implicational semilinear logic (and analogously for the other three Rasiowa-classes in the hierarchy of implicational logics). See all the classes and their inclusions in the next figure.

We can prove the mutual difference of many classes, but three differences remain to be seen: Rasiowa-implicational



semilinear logics $\neq$ Rasiowa-p-implicational semilinear logics, algebraizable semilinear logics $\neq$ weakly algebraizable semilinear logics, and equivalential semilinear logics $\neq$ protoalgebraic semilinear logics.

Proposition 2. *Let* $\mathcal{X}$ *be any class in the hierarchy of implicational logics. Then, there is an* $\mathcal{X}$ *logic which is not an* $\mathcal{X}$ *semilinear logic.*

The three main logics based on continuous t-norms (Łukasiewicz, Gödel-Dummett, and Product logics) as well as the logic of all continuous t-norms BL are clearly Rasiowa-implicative semilinear logics. The same can be said in general as regards to left-continuous t-norm-based logics such as MTL and its t-norm based axiomatic extensions, and even for all *axiomatic extensions of MTL* (even those which are not complete w.r.t. a semantics of t-norms) as all of them are complete w.r.t. a subvariety of MTL-algebras generated by its linearly ordered members. Two incomparable superclasses of this one have been considered in the literature. On one hand, we have the so-called *core fuzzy logics* introduced in [15] as finitary logics expanding MTL or $\mathrm{MTL}_\triangle$, satisfying (sCng) for $\to$, and one of the following forms of Deduction Theorem: (i) $T, \varphi \vdash_{\mathrm{MTL}_\triangle} \psi$ iff $T \vdash_{\mathrm{MTL}_\triangle} \Delta\varphi \to \psi$, for expansions of $\mathrm{MTL}_\triangle$, or (ii) $T, \varphi \vdash_{\mathrm{MTL}} \psi$ iff there is $n \in \mathbb{N}$ such that $T \vdash_{\mathrm{MTL}} \varphi^n \to \psi$, for expansions of MTL. On the other hand, we can consider the class of all *semilinear finitary extensions of MTL*. Their equivalent quasivariety semantics are the subquasivarieties of MTL-algebras generated by chains. Since there are such quasivarieties that are not varieties, we have that this class is strictly bigger than that of axiomatic extensions of MTL. Both incomparable classes are included in the class of *semilinear expansions of MTL*, and finally this one is included in the Rasiowa-implicative semilinear logics. In the recent paper [16] the fuzzy logic UL based on uninorms instead of t-norms has been studied. It is an algebraizable logic without weakening, so it belongs to the class of algebraically implicative semilinear logics. We can consider the same structure of classes as above without weakening by replacing MTL for UL. See the resulting hierarchy of classes of semilinear logics in the next figure. We realize that all of them lie on the top of our classification, above Rasiowa-implicative or algebraically implicative semilinear logics. But if, by means of our definition of semilinear implication presented in this paper, we have succeeded in capturing an interesting way by which a logic can be fuzzy this means that fuzzy logics are a much wider class than those studied so far. Thus,

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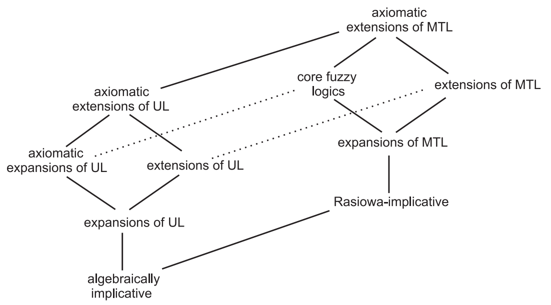
future research in the field will probably bring new significant examples of fuzzy logics throughout the whole hierarchy of implicational semilinear logics.



4 Characterizations of semilinearity

This section provides some useful mathematical characterization of semilinear implications. First we define:

Definition 7. *Let* $\mathbf{A} = \langle \mathcal{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$*. The filter* F *is called* $\Rightarrow$-linear *if* $\leq^{\Rightarrow}_{\mathbf{A}}$ *is a total preorder.*

We say that L *has the* Linear Extension Property (LEP) *w.r.t.* $\Rightarrow$ *if for every theory* $T \in Th(\mathrm{L})$ *and every formula* $\varphi \in \mathrm{Fm}_{\mathcal{L}} \setminus T$*, there is a* $\Rightarrow$*-linear theory* $T' \supseteq T$ *s.t.* $\varphi \notin T'$.

Notice that a matrix $\langle \mathcal{A}, F\rangle \in \mathbf{MOD}(\mathrm{L})$ is in $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ iff it is reduced and F is $\Rightarrow$-linear. Clearly the (LEP) says that the $\Rightarrow$-linear theories form a basis of the closure system $Th(\mathrm{L})$. Next theorem shows that an analogous statement holds for other than Lindenbaum matrices, as one of the so-called 'transfer principles' from AAL.

Theorem 2. *Let* L *be a finitary logic with* (LEP) *w.r.t.* $\Rightarrow$ *and* $\mathcal{A} \in \mathbf{ALG}^{*}(\mathrm{L})$*. Then* $\Rightarrow$*-linear filters form a basis of* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$.

Next we generalize the 'Prelinearity property' from [4]. However, here we prefer the new name 'Semilinearity Property' following our new terminology.

Definition 8. *We say that* L *has the* Semilinearity Property (SLP) *w.r.t.* $\Rightarrow$ *if the following meta rule is valid:*

$$\frac{\Gamma, \varphi \Rightarrow \psi \vdash_{\mathrm{L}} \chi \qquad \Gamma, \psi \Rightarrow \varphi \vdash_{\mathrm{L}} \chi}{\Gamma \vdash_{\mathrm{L}} \chi}$$

Theorem 3 (Characterization of semilinear implications). *The following are equivalent:*

1. $\Rightarrow$ *is semilinear in* L,
2. L *has the* (LEP) *w.r.t.* $\Rightarrow$.

Furthermore, if L *is finitary we can add:*

3. L *has the* (SLP) *w.r.t.* $\Rightarrow$,
4. $\mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Moreover, if $\Rightarrow$ *is finite we can add:*

5. $\mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

The previous theorem has several important corollaries. Using Theorem 1 we obtain that, at least in a reasonably wide class of logics, being the class of linear models w.r.t. any finite semilinear implication is an intrinsic property of a logic.

Corollary 1. *Let* L *be a finitary protoalgebraic logic and* $\Rightarrow$ *a finite weak semilinear p-implication. Then* $\mathbf{MOD}^{*}(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Corollary 2. *Each regular semilinear p-implication is a Rasiowa p-implication.*

Corollary 3. *Let* $\Rightarrow$ *a weak semilinear p-implication in* L*. Then,* $\Rightarrow$ *is semilinear in all axiomatic extensions of* L.

This last corollary will be particulary useful for showing that some logic has *no* semilinear implication. It is quite easy to show that an implication in some logic is not semilinear, consider e.g. the normal implication of the intuitionistic logic; the well-know fact that the linear Heyting algebras do not generate the variety of Heyting algebras does the job. However using part 5. of the characterization theorem we can show much more: there is *no weak semilinear p-implication definable* in the intuitionistic logic, i.e. not only the standard nice Rasiowa implication given by a single formula is not semilinear but even using an infinite set with parameters we could never obtain an implication whose linearly ordered Heyting algebras would generate the variety of Heyting algebras.

Proposition 3. *Let* L *be the logic of a quasivariety of pointed residuated lattices containing the variety of Heyting algebras. Then,* L *is not weakly semilinear p-implicational logic.*

Many well-known logics fall under the scope of the previous proposition: Full Lambek logic (possibly extended with structural rules), multiplicative-additive fragment of (Affine) Intuitionistic Linear logic, Relevance logic R, etc.

Corollary 4. *Let* L *be a logic and* $\Rightarrow$ *a weak p-implication. Then, there is the weakest logic extending* L *where* $\Rightarrow$ *is semilinear. Let us denote this logic as* $\mathrm{L}^{\ell}_{\Rightarrow}$.

In Section 6 we will show how to axiomatize $\mathrm{L}^{\ell}_{\Rightarrow}$. However, to determine a complete semantics is simple:

Proposition 4. *Let* $\Rightarrow$ *be a weak p-implication in* L*. Then,* $\mathrm{L}^{\ell}_{\Rightarrow} = \models_{\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})}$ *and* $\mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L}^{\ell}_{\Rightarrow}) = \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

5 Disjunctions

In order to provide additional characterizations of semilinearity (and to fill the gap between our abstract setting and real-life logics) we need to study a generalized notion of disjunction. As in the case of implication, given a parameterized set of formulae $\nabla(p, q, \overrightarrow{r})$ and formulae φ and ψ we define $\varphi\nabla\psi$.

Definition 9. *Given a logic* L *and a parameterized set of formulae* ∇*, we define the following properties:*

(PD)	$\varphi \vdash_{\mathrm{L}} \varphi\nabla\psi$ *and* $\psi \vdash_{\mathrm{L}} \varphi\nabla\psi$
(C)	$\varphi\nabla\psi \vdash_{\mathrm{L}} \psi\nabla\varphi$
(I)	$\varphi\nabla\varphi \vdash_{\mathrm{L}} \varphi$
(A)	$\varphi\nabla(\psi\nabla\chi) \dashv\vdash_{\mathrm{L}} (\varphi\nabla\psi)\nabla\chi$
(PCP)	*If* $\Gamma, \varphi \vdash_{\mathrm{L}} \chi$ *and* $\Gamma, \psi \vdash_{\mathrm{L}} \chi$*, then* $\Gamma, \varphi\nabla\psi \vdash_{\mathrm{L}} \chi$.

They correspond to well known usual properties of disjunction connectives. (C), (I) and (A) are respectively commutativity, idempotency and associativity, which are typically also satisfied by conjunction connectives. In contrast, the (PCP) is typically satisfied only by disjunction connectives. In [6] we also study a weaker variant of (PCP) for $\Gamma = \emptyset$.

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Definition 10. *Given a logic* L *and a parameterized set of formulae* $\nabla(p, q, \overrightarrow{r})$ *and a set of properties* $\sigma \subseteq \{(\mathrm{C}), (\mathrm{I}), (\mathrm{A})\}$, *we say that* ∇ *is a* σ-p-protodisjunction *in* L *if* (PD) *and the properties of* σ *are satisfied. Furthermore a p-protodisjunction* ∇ *is a* p-disjunction *if it satisfies* (PCP). *Finally, if* ∇ *has no parameters we drop the prefix 'p-'.*

All these defined notions are mutually distinct and any p-disjunction is in fact a $\{(\mathrm{C}), (\mathrm{I}), (\mathrm{A})\}$-p-protodisjunction. Moreover, the notion of p-disjunction is intrinsic as any two p-disjunctions are mutually interderivable.

Definition 11. *We call a logic* (p-)disjunctional *if it has a (p-)disjunction. Furthermore, we call a logic* disjunctive *if it has a disjunction given by a single parameter-free formula.*

The classes of disjunctive and disjunctional logics are mutually different. E.g. the implication fragment of Gödel logic is not disjunctive but the set $\{(p \to q) \to q, (q \to p) \to p\}$ is its disjunction.

Definition 12. *Let* L *be a logic,* ∇ *a parameterized set of formulae,* $\mathcal{A} \in \mathbf{ALG}^*(\mathrm{L})$, *and* $F \in \mathcal{F}i_{\mathrm{L}}(\mathcal{A})$. F *is called* ∇-prime *if for every* $a, b \in A$, $a\nabla^{\mathcal{A}}b \subseteq F$ *iff* $a \in F$ *or* $b \in F$.

The *prime extension property* (PEP) is defined as the (LEP) by substituting the notion of $\Rightarrow$-linear filter for that of ∇-prime filter. Then we can prove:

Theorem 4. *Let* L *be a finitary logic and* ∇ *a p-protodisjunction. Then* L *has the* (PEP) *iff* ∇ *is p-disjunction.*

Theorem 5. *Let* L *be a finitary p-disjunctional logic. Then* ∇-*prime filters form a basis of* $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ *for any* $\mathcal{A} \in \mathbf{ALG}^*(\mathrm{L})$.

Now we provide a syntactical characterization of (PCP). Let us by R^{∇} (for an $\mathcal{L}$-consecution $R = \Gamma \rhd \varphi$) denote the set $\{\Gamma\nabla\chi \rhd \delta \mid \chi \in \mathrm{Fm}_{\mathcal{L}}$ and $\delta \in \varphi\nabla\chi\}$.

Theorem 6. *Let* L *be a finitary logic with a presentation* $\mathcal{AS}$ *and* ∇ *a* $\{(\mathrm{C}), (\mathrm{I})\}$-*p-protodisjunction. Then, the following are equivalent:*

1. ∇ *is a p-disjunction,*
2. $R^{\nabla} \subseteq \mathrm{L}$ *for each (finitary)* $R \in \mathrm{L}$,
3. $R^{\nabla} \subseteq \mathrm{L}$ *for each* $R \in \mathcal{AS}$.

Corollary 5. *Let* ∇ *be a p-disjunction in a finitary logic* L_1 *and* L_2 *an expansion of* L_1 *by a set of consecutions* $\mathcal{C}$. *Then:*

- ∇ *is a p-disjunction in* L_2 *if* $R^{\nabla} \subseteq \mathrm{L}_2$ *for each* $R \in \mathcal{C}$.
- *If all the consecutions from* $\mathcal{C}$ *are finitary, then* $R^{\nabla} \subseteq \mathrm{L}_2$ *for each* $R \in \mathcal{C}$ *iff* ∇ *is a p-disjunction in* L_2.
- *If all the consecutions from* $\mathcal{C}$ *are axioms, then* ∇ *is a p-disjunction.*

Definition 13. *Let* L *be a logic and* ∇ *a parameterized set of formulae. We denote by* L^{∇} *the least logic extending* L *where* ∇ *is a p-disjunction.*

Theorem 7. *Let* L *be a finitary logic with a finitary presentation* $\mathcal{AS}$ *and* ∇ *a* $\{(\mathrm{C}), (\mathrm{I}), (\mathrm{A})\}$-*p-protodisjunction. Then,* L^{∇} *is axiomatized by* $\mathcal{AS} \cup \bigcup\{R^{\nabla} \mid R \in \mathcal{AS}\}$.

6 Disjunctions and semilinearity

In this section we consider the interesting relationships between the several kinds of disjunctions and implications we have defined and their corresponding properties. First, we introduce two natural syntactical conditions: a version of Modus Ponens with disjunction (DMP): $\varphi \Rightarrow \psi, \varphi\nabla\psi \vdash_{\mathrm{L}} \psi$ and $\varphi \Rightarrow \psi, \psi\nabla\varphi \vdash_{\mathrm{L}} \psi$, and a generalization of the prelinearity axiom used in fuzzy logics (P): $\vdash_{\mathrm{L}} (\varphi \Rightarrow \psi)\nabla(\psi \Rightarrow \varphi)$.

Theorem 8. *Let* L *be a logic,* ∇ *a p-protodisjunction, and* $\Rightarrow$ *a weak p-implication.*

- *If* L *fulfills* (DMP), *we have:*
  1. *each* $\Rightarrow$-*linear theory is* ∇-*prime,*
  2. *if* $\Rightarrow$ *has the* (LEP), *then* ∇ *has the* (PEP),
  3. *if* $\Rightarrow$ *has the* (SLP), *then* ∇ *has the* (PCP).
- *If* L *fulfills* (P), *we have:*
  4. *each* ∇-*prime theory is* $\Rightarrow$-*linear,*
  5. *if* ∇ *has the* (PEP), *then* $\Rightarrow$ *has the* (LEP).
- *If* L *fulfills* (P) *and either it is finitary or* $\Rightarrow$ *is finite and parameter-free, we have:*
  6. *if* ∇ *has the* (PCP), *then* $\Rightarrow$ *has the* (SLP).

This theorem together with known relations of the properties (SLP), (PCP), (PEP), (LEP) and semilinearity (Theorems 3 and 4) allows us to formulate numerous corollaries about their mutual relationships.

Corollary 6. *If* L *is finitary,* ∇ *is a p-protodisjunction, and* $\Rightarrow$ *a weak p-implication, the following are equivalent:*

1. L *satisfies* (DMP) *and* $\Rightarrow$ *is semilinear.*
2. L *satisfies* (DMP) *and* $\Rightarrow$ *has the* (SLP).
3. L *satisfies* (DMP) *and* $\Rightarrow$ *has the* (LEP).
4. L *satisfies* (P) *and* ∇ *has the* (PEP).
5. L *satisfies* (P) *and* ∇ *has the* (PCP).

Thus e.g. a weak p-implication in a finitary p-disjunctional logic L is semilinear iff L satisfies (P). In p-disjunctional logic we can strengthen two important results from the previous section. First, we can remove the precondition of finiteness of implication in Part 5. of Theorem 3.

Corollary 7. *Let* L *be a finitary p-disjunctional logic and* $\Rightarrow$ *a weak p-implication. Then the following are equivalent:*

1. $\Rightarrow$ *is semilinear in* L,
2. $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} \subseteq \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$.

Furthermore, in any finitary p-disjunctional protoalgebraic logic it holds that: $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}} = \mathbf{MOD}^{\ell}_{\Rightarrow}(\mathrm{L})$ *for any semilinear p-implication* $\Rightarrow$.

Theorem 9. *If* L *is finitary,* ∇ *is a* $\{(\mathrm{C}),(\mathrm{I}),(\mathrm{A})\}$-*p-protodisjunction,* $\Rightarrow$ *is a weak p-implication and* L *satisfies* (DMP), *then* $\mathrm{L}^{\ell}_{\Rightarrow}$ *is the extension of* L^{∇} *by* (P).

Corollary 8. *Let* L *be a finitary p-disjunctional logic and* $\Rightarrow$ *a weak p-implication. Then,* $\mathrm{L}^{\ell}_{\Rightarrow}$ *is extension of* L *by* (P).

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A Bits of the theory of logical calculi

We recall some basic definitions and results of Abstract Algebraic Logic.[2] The notion of *propositional language* $\mathcal{L}$ is defined in the usual way (a set of connectives with finite arity). By $\mathbf{Fm}_{\mathcal{L}}$ we denote the free term algebra over a denumerable set of variables in the language $\mathcal{L}$, by $\mathrm{Fm}_{\mathcal{L}}$ we denote its universe and we call its elements *$\mathcal{L}$-formulae*.

A *$\mathcal{L}$-consecution* is a pair $\Gamma \rhd \varphi$, where $\Gamma \subseteq \mathrm{Fm}_{\mathcal{L}}$ and $\varphi \in \mathrm{Fm}_{\mathcal{L}}$. A consecution $\Gamma \rhd \varphi$ is *finitary* if Γ is finite. For a set of consecutions L we write $\Gamma \vdash_{\mathrm{L}} \varphi$ rather than $\Gamma \rhd \varphi \in \mathrm{L}$. A *propositional logic* is a pair $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$ where $\vdash_{\mathrm{L}}$ is a structural consequence relation.

A logic L is *finitary* if for every $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}_{\mathcal{L}}$ such that $\Gamma \vdash_{\mathrm{L}} \varphi$ there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \vdash_{\mathrm{L}} \varphi$. We write $\Gamma \vdash_{\mathrm{L}} \Delta$ when $\Gamma \vdash_{\mathrm{L}} \varphi$ for every $\varphi \in \Delta$. A *theory* of a logic L is a set of formulae T such that if $T \vdash_{\mathrm{L}} \varphi$ then $\varphi \in T$. By $Th(\mathrm{L})$ we denote the set of all theories of L.

Given a *finitary* logic $\mathrm{L} = \langle \mathcal{L}, \vdash_{\mathrm{L}} \rangle$, we say that a set $\mathcal{AS}$ of $\mathcal{L}$-consecutions whose left member is finite is a *presentation* of L if the relation $\vdash_{\mathrm{L}}$ coincides with the provability relation given by $\mathcal{AS}$ as a Hilbert-style axiomatic system.

Given a language $\mathcal{L}$, an *$\mathcal{L}$-matrix* is a pair $\mathbf{A} = \langle \mathcal{A}, F \rangle$ where $\mathcal{A}$ is an $\mathcal{L}$-algebra and F is a subset of A called the *filter* of $\mathbf{A}$. A homomorphism from $\mathbf{Fm}_{\mathcal{L}}$ to $\mathcal{A}$ is called an $\mathbf{A}$-*evaluation*. The semantical consequence w.r.t. a class of matrices $\mathbb{K}$ is defined as: $\Gamma \models_{\mathbb{K}} \varphi$ iff for each $\mathbf{A} \in \mathbb{K}$ and each $\mathbf{A}$-evaluation e we obtain $e(\varphi) \in F$ whenever $e[\Gamma] \subseteq F$. Clearly, $\langle \mathcal{L}, \models_{\mathbb{K}} \rangle$ is a logic. We say that a matrix $\mathbf{A}$ is a *model* of L if $\vdash_{\mathrm{L}} \subseteq \models_{\mathbf{A}}$ and write $\mathbf{A} \in \mathbf{MOD}(\mathrm{L})$.

Given an $\mathcal{L}$-algebra $\mathcal{A}$, a subset $F \subseteq A$ is an L-*filter* if $\langle \mathcal{A}, F \rangle \in \mathbf{MOD}(\mathrm{L})$. Let $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ be the set of all L-filters over $\mathcal{A}$. Observe that for every set of formulae T, we have $T \in Th(\mathrm{L})$ iff $\langle \mathbf{Fm}_{\mathcal{L}}, T \rangle \in \mathbf{MOD}(\mathrm{L})$; these models are called the *Lindenbaum matrices* for L.

It is straightforward to check that $\mathcal{F}i_{\mathrm{L}}(\mathcal{A})$ is closed under arbitrary intersections and hence it is a closure system. Recall that a family $\mathcal{B} \subseteq \mathcal{C}$ is a *basis* of a closure system $\mathcal{C}$ if for every $X \in \mathcal{C}$ there is a $\mathcal{D} \subseteq \mathcal{C}$ such that $X = \bigcap \mathcal{D}$ (which can be equivalent formulated as: for every $Y \in \mathcal{C}$ and every $a \in A \setminus Y$ there is $Z \in \mathcal{B}$ such that $Y \subseteq Z$ and $a \notin Z$).

Given a matrix $\mathbf{A} = \langle \mathcal{A}, F \rangle$, a binary relation $\Omega_{\mathcal{A}}(F)$ is defined as $\langle a, b \rangle \in \Omega_{\mathcal{A}}(F)$ if, and only if, for every sequence of parameters $\overrightarrow{z}$, $\mathcal{L}$-formula $\varphi(x, \overrightarrow{z})$, and $\overrightarrow{c} \in A^{<\omega}$ we have $\varphi^{\mathcal{A}}(a, \overrightarrow{c}) \in F$ iff $\varphi^{\mathcal{A}}(b, \overrightarrow{c}) \in F$. Inspired by the famous Leibniz's identity of indiscernibles principle, $\Omega_{\mathcal{A}}(F)$ is called the *Leibniz congruence* of $\langle \mathcal{A}, F \rangle$.

A matrix is said to be *reduced* if its Leibniz congruence is the identity relation. Given a logic L, the class of its reduced models is denoted by $\mathbf{MOD}^*(\mathrm{L})$, and the class of algebraic reducts of $\mathbf{MOD}^*(\mathrm{L})$ is denoted by $\mathbf{ALG}^*(\mathrm{L})$. They are enough to provide a complete semantics for the logic:

Theorem 10. *Let* L *be a logic. Then* $\Gamma \vdash_{\mathrm{L}} \varphi$ *if, and only if,* $\Gamma \models_{\mathbf{MOD}^*(\mathrm{L})} \varphi$*, for every set of formulae* $\Gamma \cup \{\varphi\}$.

A matrix $\mathbf{A} \in \mathbf{MOD}^*(\mathrm{L})$ is called *(finitely) subdirectly irreducible* if it is not a non-trivial (finite) subdirect product of reduced matrices. The corresponding classes of matrices are denoted as $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RSI}}$ and $\mathbf{MOD}^*(\mathrm{L})_{\mathrm{RFSI}}$ respectively.

[2] For a comprehensive survey see [7, 10, 11]. Any necessary background on Universal Algebra can be found e.g. in [3].

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