Different types of convexity and concavity for copulas

E. Alvoni
Dipartimento MATEMATES
Università di Bologna
Viale Filopanti 5
I-40126 Bologna, Italy
elisabetta.alvoni@unibo.it

F. Durante
Department of Knowledge-Based
Mathematical Systems
Johannes Kepler University
A-4040 Linz, Austria
fabrizio.durante@jku.at

P.L. Papini
Dipartimento di Matematica
Università di Bologna
Piazza Porta S.Donato 5
I-40126 Bologna, Italy
papini@dm.unibo.it

C. Sempi
Dipartimento di Matematica
Ennio De Giorgi
Università del Salento
I-73100 Lecce, Italy
carlo.sempi@unile.it

Abstract

We present different notions of convexity and concavity for copulas and we study the relationships among them.

Keywords: Copulas, Schur-concavity, Quasi-concavity.

1 Introduction

A function $C\colon [0,1]^2 \to [0,1]$ is called *copula* if, for every x, x', y, y' in $[0,1]$, for $x \le x', y \le y'$,

$$C(x,0) = C(0,x) = 0,\ C(x,1) = C(1,x) = x, \quad (1)$$

$$C(x',y') - C(x,y') - C(x',y) + C(x,y) \ge 0. \quad (2)$$

In other words, a copula is a binary aggregation operator that has neutral element 1 and which satisfies inequality (2), called the *2–increasing property* [16]. In 1959, Abe Sklar introduced this concept in order to link a bivariate distribution function to its marginals [18]. Since then, copulas have played an important rôle not only in probability theory and statistics, but also in multi–criteria decision making and in fuzzy set theory. Classical examples of copulas are $\Pi(x,y) = xy$, $M(x,y) = \min(x,y)$ and $W(x,y) = \max(x+y-1,0)$. In particular, for each copula C, we have $W \le C \le M$ pointwise on $[0,1]^2$. The class of copulas shall be denoted by $\mathcal{C}$, which is a convex and compact (with respect to the L^∞ norm) subset in the class of all continuous functions from $[0,1]^2$ into $[0,1]$.

The following two construction of copulas will be considered in the sequel: ordinal sum and ϕ–transformations; they are recalled here.

Let $(C_i)_{i\in\mathcal{I}}$ be a family of copulas indexed by the (at most) countable set $\mathcal{I}$. Let $(]a_i,b_i[)_{i\in\mathcal{I}}$ be a family of pairwise disjoint subintervals of $[0,1]$ indexed by the same set $\mathcal{I}$. The *ordinal sum* of $(C_i)_{i\in\mathcal{I}}$ with respect to $(]a_i,b_i[)_{i\in\mathcal{I}}$ is the copula $C\colon [0,1]^2 \to [0,1]$ given by

$$C(x,y) = \begin{cases} a_i + (b_i - a_i) C_i\left(\frac{x-a_i}{b_i-a_i}, \frac{y-a_i}{b_i-a_i}\right) & \text{on }]a_i,b_i[^2, \\ M(x,y) & \text{otherwise.} \end{cases}$$

Usually, we denote $C = (\langle a_i, b_i, C_i\rangle)_{i\in\mathcal{I}}$.

Let C be a copula and let $\phi\colon [0,1] \to [0,1]$ be a continuous and concave bijection with $\phi(0) = 0$ and $\phi(1) = 1$. Then the *ϕ–transform* [9, 13] of C is the copula C_ϕ given by

$$C_\phi(x,y) = \phi^{-1}(C(\phi(x),\phi(y))). \quad (3)$$

A copula C is symmetric if $C(x,y) = C(y,x)$ for all x, y in $[0,1]$, and, in particular, the copulas W, Π and M are symmetric, but not every copula is symmetric. As showed in [11, 17], we can assign to each copula C a degree of non–symmetry, given by

$$\sigma_C = \sup\{|C(x,y) - C(y,x)|, x, y \in [0,1]\}.$$

It was proved in [11, 17] that $\sigma_C \le 1/3$ for every copula C, and the value $1/3$ is attained.

Recently, investigations on various notions of convexity for copulas have received much attention because of their potential applications: see, for example, [1, 16] and the recent papers [2, 4, 7, 8]. Here, we revisit different types of convexity and study their relationships through several examples. In particular, we shall note that, if a copula C satisfies some conditions of convexity or concavity, then the bounds for the degree of non–symmetry σ_C can be improved.

2 Global and directional convexity

We start with the classical notion of convexity.

Definition 1 *A copula C is called (globally) convex if, for all x_1, x_2, y_1, y_2 and λ in $[0,1]$,*

$$C(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \le \lambda C(x_1,x_2) + (1-\lambda)C(y_1,y_2). \quad (4)$$

A copula C is called (globally) concave if (4) holds with the reverse inequality sign.

In the class of copulas, convexity and concavity are strong properties in view of the following result.

Proposition 1 ([6]) *Let C be a copula.*

(a) *C is convex if, and only if, $C = W$;*

(b) *C is concave if, and only if, $C = M$.*

Weak versions of these properties are given by the following definitions.

Definition 2 *A copula C is called directionally convex if, for every z_0 in $[0,1]$, the functions $x \mapsto C(x, z_0)$ and $y \mapsto C(z_0, y)$ are convex, viz. for all x, y and λ in $[0,1]$,*

$$C(\lambda x + (1-\lambda)y, z_0) \leq \lambda C(x, z_0) + (1-\lambda)C(y, z_0),$$

$$C(z_0, \lambda x + (1-\lambda)y) \leq \lambda C(z_0, x) + (1-\lambda)C(z_0, y).$$

A copula C is called directionally concave if, for every z_0 in $[0,1]$, the functions $x \mapsto C(x, z_0)$ and $y \mapsto C(z_0, y)$ are concave.

Notice that W is directionally convex, M is directionally concave, Π is both directionally convex and concave. Such properties are useful in view of the following statistical interpretation.

Proposition 2 ([16]) *Let C be the copula associated with a pair (X, Y) of continuous random variables. Then:*

(a) *C is directionally convex if, and only if, Y is stochastically decreasing in X and X is stochastically decreasing in Y;*

(b) *C is directionally concave if, and only if, Y is stochastically increasing in X and X is stochastically increasing in Y.*

Example 1 *Let C_α be a member of the Farlie-Gumbel-Morgenstern family of copulas defined, for all α in $[-1,1]$ by*

$$C_\alpha(x, y) = xy(1 + \alpha(1-x)(1-y)).$$

Then C_α is directionally convex if $\alpha \in [-1, 0]$ and it is directionally concave if $\alpha \in [0, 1]$.

Directionally convex copulas are also called *P–increasing copulas* [5, 10] and they are used in order to ensure that the pointwise composition of two 2–increasing aggregation operators is also 2–increasing.

Proposition 3 *The class of all directionally concave (resp. convex) copulas is a convex and compact (with respect to the to the L^∞ norm) subset of $\mathcal{C}$.*

Note that an ordinal sum of directionally concave copula is also directionally concave, but a non-trivial ordinal sum of directionally convex copula is not directionally convex (because M is directionally concave). Moreover, the ϕ–transform of directionally concave (resp. convex) copula may not be directionally concave (resp. convex).

For directionally concave and convex copulas, we have the following bounds for the degree of non–symmetry.

Proposition 4 *Let C be a copula.*

(a) *If C is directionally concave, then $\sigma_C \leq 1/9$.*

(b) *If C is directionally convex, then $\sigma_C \leq 3 - \sqrt{2}$.*

Both bounds are sharp.

3 Schur–concave copulas

The notion of Schur–concavity was introduced in the context of majorization ordering [14] and it is here reformulated in the class of copulas.

Definition 3 *A copula C is called Schur–concave if, for all x, y and λ in $[0,1]$,*

$$C(x, y) \leq C(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y). \qquad (5)$$

A copula C is called Schur–convex if (5) holds with the reverse inequality sign.

The following result allows to investigate only Schur–concave copulas.

Proposition 5 ([8]) *W is the only Schur–convex copula.*

Every Schur–concave copula C is symmetric (and hence $\sigma_C = 0$), but the converse implication is not true.

Example 2 *Let C be the copula defined by*

$$C(x,y) = \begin{cases} \frac{xy}{2} & \text{on } \left[0, \frac{1}{2}\right]^2, \\ \frac{x\,(3y-1)}{2} & \text{on } \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right], \\ \frac{y\,(3x-1)}{2} & \text{on } \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right], \\ \frac{xy+x+y-1}{2} & \text{on } \left[\frac{1}{2}, 1\right]^2. \end{cases}$$

Then C is symmetric, but

$$C\left(\frac{6}{10}, \frac{4}{10}\right) = \frac{32}{200} < \frac{33}{200} = C\left(\frac{7}{10}, \frac{3}{10}\right),$$

and, hence, C is not Schur–concave.

Remark 1 *If $z = C(s,t)$ is the surface associated with a Schur–concave copula C, then the intersections of the surface with all the vertical planes of the form $s+t=2x$, for all $x \in [0,1]$ and $s \in [0,x]$, are curves that are decreasing from (x,x) to $(2x,0)$, if $x \le 1/2$, and from (x,x) to $(2x-1,1)$, otherwise.*

For a copula, the notion of Schur–concavity can be expressed in terms of its derivatives.

Proposition 6 ([8]) *Let C be a continuously differentiable copula. Then C is Schur–concave on $[0,1]^2$ if, and only if,*

(a) *C is symmetric;*

(b) *for all $x,y \in [0,1]$, $x \ge y$, $\frac{\partial C(x,y)}{\partial x} \le \frac{\partial C(x,y)}{\partial y}$.*

As a consequence, it is easily proved that the copula Π is Schur–concave. Moreover, the following copulas are Schur–concave:

- associative copulas (in particular, M, W and Π);
- Fréchet copulas, $C_{\alpha,\beta} = \alpha M + (1-\alpha-\beta)\Pi + \beta W$ for α and β in $[0,1]$, $\alpha+\beta=1$;
- Farlie–Gumbel–Morgenstern copulas, $C_\alpha(x,y) = xy + \alpha xy\,(1-x)\,(1-y)$ for α in $[-1,1]$.

We denote by $\mathcal{C}_{SC}$ the class of all Schur–concave copulas.

Proposition 7 ([8]) *The set $\mathcal{C}_{SC}$ is a convex and compact (with respect to the to the L^∞ norm) subset of $\mathcal{C}$.*

Moreover, the class $\mathcal{C}_{SC}$ is closed with respect to ordinal sums and bijective concave transformations.

Proposition 8 ([8]) *The ordinal sum of Schur–concave copulas is Schur–concave.*

Proposition 9 *Let $\phi\colon [0,1] \to [0,1]$ be a continuous and concave bijection with $\phi(0)=0$ and $\phi(1)=1$. If C is Schur–concave, then the ϕ–transform C_ϕ given by (3) is also Schur–concave.*

Proof. Let x, y and λ be in $[0,1]$. Because ϕ is concave, we have

$$\begin{aligned}\phi(\lambda x + (1-\lambda)y) &\ge \lambda\phi(x) + (1-\lambda)\phi(y),\\ \phi((1-\lambda)x + \lambda y) &\ge (1-\lambda)\phi(x) + \lambda\phi(y).\end{aligned}$$

Moreover, since C is Schur–concave, we have

$$C(\lambda\phi(x) + (1-\lambda)\phi(y), (1-\lambda)\phi(x) + \lambda\phi(y)) \ge C(\phi(x), \phi(y)).$$

But C is increasing in each variable so that

$$C_\phi(\lambda x + (1-\lambda)y, (1-\lambda)x + \lambda y) \ge C_\phi(x,y),$$

which is the desired assertion. □

4 Weak Schur–concave copulas

Recently, E.P. Klement, R. Mesiar and E. Pap [12] raised some problems on binary aggregation operators. Specifically, in Problem 5 they suggested to study the inequality

$$C\left(\max(x-a,0), \min(x+a,1)\right) \le C(x,x), \qquad (6)$$

for all $x \in [0,1]$ and for all $a \in \,]0,1/2[$. In [4], this problem was investigated for the class of copulas and the connection between inequality (6) and the notion of Schur–concavity was stressed.

In the spirit of these investigations, in [7] the following weakened form of Schur–concavity is introduced.

Definition 4 *A copula C is called weakly Schur-concave (WSC, for short) if, for all $x \in [0,1]$ and for all $a \in \,]0,1/2[$, both*

$$C\left(\max(x-a,0), \min(x+a,1)\right) \le C(x,x) \qquad (7)$$

and

$$C\left(\min(x+a,1), \max(x-a,0)\right) \le C(x,x), \qquad (8)$$

hold.

If (7) and (8) are satisfied by C with reverse inequality sign, then C is said to be weakly Schur-convex.

Proposition 10 ([7]) *If a copula C is weakly Schur–convex, then $C(t,t) = \max(2t-1,0)$ for every t in $[0,1]$.*

Therefore, by using [3, 15], if C is weakly Schur–convex, then C has the following representation

$$C(x,y) = \begin{cases} \frac{C_1(2x,2y-1)}{2} & \text{on } \left[0,\frac{1}{2}\right] \times \left[\frac{1}{2},1\right],\\ \frac{C_2(2x-1,2y)}{2} & \text{on } \left[\frac{1}{2},1\right] \times \left[0,\frac{1}{2}\right],\\ W(x,y) & \text{otherwise,}\end{cases}$$

for suitable copulas C_1 and C_2.

Remark 2 *Geometrically speaking, a copula C is weakly Schur–concave if the maximum of C along the line*

$$d_k = \{(x,y) \in [0,1]^2 \mid x+y = 2k\},$$

for every $k \in [0,1]$, is attained at the point (k,k). Moreover, if a copula C is differentiable, then the property WSC implies that the partial derivatives of C at the points of the main diagonal of $[0,1]^2$ are equal.

Important examples of WSC copulas are W, Π and M. If C is a Schur–concave copula, then it is easily proved that C is also WSC. But, on the other hand, if a copula C is WSC, then it does not need to be Schur–concave.

Example 3 *Let C be the copula defined by*

$$C(x,y) = \begin{cases} \frac{M(3x,3y-2)}{3} & \text{on } \left[0,\frac{1}{3}\right] \times \left[\frac{2}{3},1\right], \\ \frac{M(3x-1,3y-1)}{3} & \text{on } \left[\frac{1}{3},\frac{2}{3}\right] \times \left[\frac{1}{3},\frac{2}{3}\right], \\ \frac{M(3x-2,3y)}{3} & \text{on } \left[\frac{2}{3},1\right] \times \left[0,\frac{1}{3}\right], \\ W(x,y) & \text{otherwise.} \end{cases}$$

Then C is WSC, but C is not Schur–concave. In fact, given the points $\left(\frac{2}{10},\frac{7}{10}\right)$ and $\left(\frac{3}{10},\frac{6}{10}\right)$, we have

$$C\left(\frac{3}{10},\frac{6}{10}\right) = 0 < \frac{1}{30} = C\left(\frac{2}{10},\frac{7}{10}\right),$$

which implies that C is not Schur–concave.

We denote by $\mathcal{C}_{WSC}$ the class of all WSC copulas.

Proposition 11 ([7]) *The set $\mathcal{C}_{WSC}$ is a convex and compact (with respect to the to the L^∞ norm) subset of $\mathcal{C}$.*

Moreover, the class $\mathcal{C}_{WSC}$ is closed with respect to ordinal sums and bijective concave transformations.

Proposition 12 ([7]) *The ordinal sum of WSC copulas is WSC.*

Proposition 13 ([7]) *Let $\phi\colon [0,1] \to [0,1]$ be a continuous and concave bijection with $\phi(0) = 0$ and $\phi(1) = 1$. If C is WSC, then the ϕ–transform of C given by (3) is also WSC.*

A WSC copula does not need to be symmetric. For instance, consider the copula

$$C(x,y) = \begin{cases} xy, & x \le y; \\ \frac{(x+y)^2}{4}, & y < x \le 2\sqrt{y} - y; \\ y, & x \ge 2\sqrt{y} - y. \end{cases}$$

For WSC copulas, we have the following bound for the degree of non–symmetry.

Proposition 14 ([7]) *Let C be a WSC copula. Then $\sigma_C \le 1/4$.*

Notice that a WSC copula C such that $\sigma_C = 1/4$ is defined in the following way:

$$C(x,y) = \begin{cases} \max(x+y-1,0), & x \ge \frac{3}{4} \text{ or } x+y \ge \frac{3}{2}, \\ \max\left(y - \frac{1}{4}, 0\right), & y \le x \le \frac{3}{4}, \\ x, & x \le y - \frac{1}{2}, \\ \max\left(\frac{x+y}{2} - \frac{1}{4}, 0\right), & \text{otherwise.} \end{cases}$$

5 Quasi–concave copulas

The notion of quasi–concavity was introduced in the context of optimization theory and it is here reformulated in the class of copulas.

Definition 5 *A copula C is called quasi–concave if, for all x_1, x_2, y_1, y_2 and λ in $[0,1]$,*

$$C(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \ge \min(C(x_1,x_2), C(y_1,y_2)). \quad (9)$$

A copula C is said to be quasi–convex if, for all x_1, x_2, y_1, y_2 and λ in $[0,1]$,

$$C(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \le \max(C(x_1,x_2), C(y_1,y_2)). \quad (10)$$

The following result allows us to investigate only quasi–concave copulas.

Proposition 15 *W is the only quasi–convex copula.*

Notice that every associative copula is quasi–concave, and in particular W, Π and M are quasi–concave.

Remark 3 *Geometrically speaking, a copula C is quasi–concave if, given a segment $\overline{PQ}$ in $[0,1]^2$, the value of C at any internal point of this segment is not smaller than either the value of C at P or the value of C at Q.*

We denote by $\mathcal{C}_{QC}$ the class of all quasi–concave copulas.

Proposition 16 ([2]) *The set $\mathcal{C}_{QC}$ is a convex and compact (with respect to the to the L^∞ norm) subset of $\mathcal{C}$.*

Moreover, the class $\mathcal{C}_{QC}$ is closed with respect to ordinal sums and bijective concave transformations.

Proposition 17 *The ordinal sum of quasi–concave copulas is quasi–concave.*

Proof. The proof can be easily reproduced by using the geometrical interpretation of quasi–concavity given above. □

Proposition 18 *Let $\phi\colon [0,1] \to [0,1]$ be a continuous and concave bijection with $\phi(0) = 0$ and $\phi(1) = 1$. If C is quasi–concave, then the ϕ–transform of C given by (3) is also quasi–concave.*

Proof. Let x_1, x_2, y_1, y_2 and λ be in $[0,1]$. Because ϕ is concave, we have

$$\begin{aligned} \phi(\lambda x_1 + (1-\lambda)y_1) &\geq \lambda\phi(x_1) + (1-\lambda)\phi(y_1), \\ \phi(\lambda x_2 + (1-\lambda)y_2) &\geq \lambda\phi(x_2) + (1-\lambda)\phi(y_2). \end{aligned}$$

Moreover, since C is increasing in each variable and quasi–concave, we have

$$\begin{aligned} & C(\phi(\lambda x_1 + (1-\lambda)y_1), \phi(\lambda x_2 + (1-\lambda)y_2)) \\ \geq\ & C(\lambda\phi(x_1) + (1-\lambda)\phi(y_1), \lambda\phi(x_2) + (1-\lambda)\phi(y_2)) \\ \geq\ & \min(C(\phi(x_1), \phi(x_2)), C(\phi(y_1), \phi(y_2))). \end{aligned}$$

But ϕ is strictly increasing so that

$$\begin{aligned} C_\phi(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2) \\ \geq \min(C_\phi(x_1, x_2), C_\phi(y_1, y_2)), \end{aligned}$$

which is the desired assertion. □

A quasi–concave copula does not need to be symmetric (so it is not necessarily Schur–concave): see, for instance, [16, section 3.2.1]. Moreover, we can improve the estimation of the degree of non–symmetry in the class of quasi–concave copulas.

Proposition 19 ([2]) *Let C be a quasi–concave copula. Then $\sigma_C \leq 1/5$, and the value $1/5$ is attained.*

In the symmetric case, we obtain the following result.

Proposition 20 ([2]) *For a quasi–concave copula C, the following statements are equivalent:*

(a) *C is symmetric,*

(b) *C is WSC,*

(c) *C is Schur–concave.*

Notice that a symmetric and Schur–concave copula does not need to be quasi–concave ([16, Example 3.2.8]). Example 2 in [4], instead, describes a symmetric and WSC copula that is neither Schur–concave nor, as a consequence, quasi–concave.

Proposition 21 ([2]) *A directionally concave copula is quasi–concave.*

The converse implication of the above proposition is not true, even when C is symmetric.

References

[1] C. Alsina, M.J. Frank and B. Schweizer, Associative Functions: Triangular Norms and Copulas, World Scientific, Hackensack, 2006.

[2] E. Alvoni, P.L. Papini, Quasi-concave copulas, asymmetry, transformations, Comment. Math. Univ. Carolinae, in press.

[3] B. De Baets and H. De Meyer, Copulas and the pairwise probabilistic comparison of ordered lists, in: Proceedings of the 10th International Conference IPMU, Perugia, 2004, pp. 1091–1098.

[4] F. Durante, Solution of an open problem for associative copulas, Fuzzy Sets and Systems 152 (2005) 411–415.

[5] F. Durante, R. Mesiar, P.L. Papini, C. Sempi, 2–increasing binary aggregation operators, Inform. Sci. 177 (2007) 111–129.

[6] F. Durante, J.J. Quesada-Molina, C. Sempi, Semicopulas: characterizations and applicability, Kybernetika 42 (2006) 287–302.

[7] F. Durante, P.L. Papini, A weakening of Schur–concavity for copulas, Fuzzy Sets and Systems, in press.

[8] F. Durante, C. Sempi, Copulæ and Schur–concavity, Int. Math. J. 3 (2003) 893–905.

[9] F. Durante and C. Sempi, Copula and semicopula transforms, Int. J. Math. Math. Sci. 2005 (2005), 645–655.

[10] F. Durante, C. Sempi, On the characterization of a class of binary operations on bivariate distribution functions, Publ. Math. Debrecen 69 (2006) 47–63.

[11] E.P. Klement, R. Mesiar, How non-symmetric can a copula be?, Comment. Math. Univ. Carolinae 47 (2006) 141–148.

[12] E.P. Klement, R. Mesiar, E. Pap, Problems on triangular norms and related operators, Fuzzy Sets and Systems 145 (2004) 471–479.

[13] E.P. Klement, R. Mesiar and E. Pap, Archimax copulas and invariance under transformations, C.R. Math. Acad. Sci. Paris 340 (2005) 755–758.

[14] A. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.

[15] R. Mesiar, J. Szolgay, W–ordinal sums of copulas and quasi–copulas, in: Proceedings of the MAGIA 2004 Conference, Kočovce, Slovak Republic, October 2004, pp. 78-83.

[16] R. B. Nelsen, An Introduction to Copulas, Springer, New York, 2006.

[17] R. B. Nelsen, Extremes of nonexchangeability, Statist. Papers 48 (2007) 329–336.

[18] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8 (1959) 229–231.