# Dominance of Ordinal Sums of $T_{\mathbf{L}}$ and $T_{\mathbf{P}}$

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## Abstract

Dominance is a relation on operations which are defined on a common poset. We treat the dominance relation on the set of ordinal sum t-norms which involve either exclusively the Łukasiewicz t-norm or exclusively the product t-norm as summand operations. We show that in both cases, the question of dominance can be reduced to a simple property of the idempotent elements of the dominating t-norm. We finally discuss the obtained results and possibilities of their generalization.

**Keywords:** Triangular norm, Ordinal sum, Idempotent element, Dominance.

## 1 Introduction

The concept of dominance has been introduced within the framework of probabilistic metric spaces as a property relevant for building Cartesian products of such spaces [14]. Later on the dominance of t-norms was studied in connection with the construction of fuzzy equivalence relations [3, 4, 15] and the construction of fuzzy orderings [2]. Further, the concept of dominance was extended to the more general class of aggregation operators [7, 9]. The dominance of aggregation operators emerges when investigating which aggregation procedures applied to the system of $T$-transitive fuzzy relations yield again a $T$-transitive fuzzy relation [7] or when seeking aggregation operators which preserve the extensionality of fuzzy sets with respect to given $T$-equivalence relations [8]. The definition of dominance we use was given in [14].

**Definition 1** *Let $(S, \leq)$ be a partially ordered set and let $f$ and $g$ be associative binary operations on $S$ with common identity $e \in S$. Then $f$* dominates *$g$, and we write $f \gg g$, if the inequality*

$$f(g(x, y), g(u, v)) \geq g(f(x, u), f(y, v)). \qquad \text{(D)}$$

*holds for all $x, y, u, v \in S$.*

Recall that a *triangular norm* (t-norm for short) [5, 14] is an associative, commutative, binary operation on the unit interval $[0, 1]$ which is non-decreasing in each argument and has neutral element 1. Most of our attention is confined to three prototypical t-norms – the minimum $T_{\mathbf{M}}(x, y) = \min\{x, y\}$, the product $T_{\mathbf{P}}(x, y) = xy$ and the Łukasiewicz t-norm $T_{\mathbf{L}}(x, y) = \max\{0, x+y-1\}$. Note that these t-norms are continuous. It is well known that $T_{\mathbf{M}}$ is the strongest t-norm, i.e., for any t-norm $T$ we have $T_{\mathbf{M}} \geq T$. Moreover, for any t-norm $T$ we have $T_{\mathbf{M}} \gg T$. Thanks to associativity and commutativity, each t-norm dominates itself; therefore dominance of t-norms is a *reflexive* relation. From its commutativity together with the fact that all t-norms have the common neutral element 1 it follows that dominance is a refinement of the standard point-wise order of t-norms, i.e., $T_1 \gg T_2$ implies that $T_1 \geq T_2$. Since *antisymmetry* of a binary relation is inherited by any refinement, the dominance of t-norms is an *antisymmetric* relation. Anyhow, the dominance of t-norms is *not transitive* [12] and this even if considered on continuous t-norms only. Plenty of counterexamples to the transitivity of dominance of continuous t-norms can be constructed by means of the results of the present paper. Before turning to the concrete contents we summarize facts and notions which will be essential later.

## 2 Ordinal Sums and Dominance

By an *order isomorphism* from $[a, b]$ to $[c, d]$ we mean any increasing bijection from the first interval to the second one. With the symbol $\psi_{[a,b]}$ we denote the unique *affine* order isomorphism from $[a, b]$ to the unit interval $[0, 1]$. If $O$ is a binary operation on the interval $[c, d]$ and $\varphi$ is an order isomorphism from $[a, b]$ to

$[c, d]$, the *$\varphi$-transform of $O$* is the binary operation

$$O_\varphi \colon [a, b]^2 \to [a, b] : (x, y) \mapsto \varphi^{-1}\big(O(\varphi(x), \varphi(y))\big).$$

All $\varphi$-transforms preserve dominance, i.e., $T_1 \gg T_2$ if and only if $(T_1)_\varphi \gg (T_2)_\varphi$ for any $\varphi$.

Recall that a continuous t-norm is said to be *Archimedean* if it is a $\varphi$-transform of $T_{\mathbf{L}}$ or of $T_{\mathbf{P}}$ for some order isomorphism $\varphi \colon [0, 1] \to [0, 1]$. For the brevity we write Archimedean t-norm instead of continuous Archimedean t-norm in the sequel.

Let $I$ be an at most countable index set, let $\{T_i\}_{i \in I}$ be a system of arbitrary t-norms and let $\{[a_i, b_i]\}_{i \in I}$ be a system of subintervals of $[0, 1]$ with pairwise disjoint interiors. The *ordinal sum* determined by these two systems is a binary operation on the unit interval given as

$$T(x, y) = \begin{cases} (T_i)_{\psi_{[a_i, b_i]}}(x, y) & \text{if } x, y \in [a_i, b_i], \\ T_{\mathbf{M}}(x, y) & \text{otherwise.} \end{cases}$$

This operation is again a t-norm [5]; we denote it by $(\langle a_i, b_i, T_i \rangle)_{i \in I}$. To t-norms $T_i$ we refer as the *summands of $T$* and to intervals $[a_i, b_i]_{i \in I}$ as the *summand carriers*. In the sequel we will always assume that $I$ is an at most coutable index set and that the corresponding families of intervals have pairwisely disjoint interiors.

Observe that if for some t-norms $T_1, T_2$, with $T_2$ being the ordinal sum $(\langle a_i, b_i, T_{2,i} \rangle)_{i \in I}$ it holds that $T_1 \geq T_2$, then also $T_1$ is an ordinal sum with the same underlying summand carriers but with possibly different summands, i.e., $T_1 = (\langle a_i, b_i, T_{1,i} \rangle)_{i \in I}$. Since comparability is a necessarry condition for dominance, the same holds if assumed $T_1 \gg T_2$.

The dominance of ordinal sum t-norms can be examined summand by summand [10]:

**Theorem 2** Let $T_1 = (\langle a_i, b_i, T_{1,i} \rangle)_{i \in I}$ and $T_2 = (\langle a_i, b_i, T_{2,i} \rangle)_{i \in I}$ be two t-norms with the same structure of summand carriers. Then $T_1 \gg T_2$ if and only if $T_{1,i} \gg T_{2,i}$ holds for all $i \in I$.

Recall that an element $x \in [0, 1]$ is called an *idempotent element* of $T$ if $T(x, x) = x$. The set of all idempotent elements of $T$ will be denoted $\mathcal{I}_T$. As 0 and 1 are idempotent elements of every t-norm we refer to them as *trivial* idempotent elements. The structure of $\mathcal{I}_T$ is strongly tied to the dominance properties of $T$ [10]:

**Theorem 3** If $T_1 \gg T_2$ then $\mathcal{I}_{T_1}$ is closed with respect to $T_2$, i.e., if $x, y \in \mathcal{I}_{T_1}$ then also $T_2(x, y) \in \mathcal{I}_{T_1}$.

Let us concentrate now on the class $\mathcal{OS}_{T_s}$ of ordinal sums of type $(\langle a_i, b_i, T_s \rangle)_{i \in I}$ where all summands are equal to a fixed t-norm $T_s$. The elucidation of dominance on such a class can be simplified substantially making use of Theorem 2. Let $T_1, T_2 \in \mathcal{OS}_{T_s}$ with $T_1 \geq T_2$ which is a necessary condition for dominance. Observe that both these t-norms can be constructed as ordinal sum t-norms with the same summand carriers but different summands:

$$\begin{aligned} T_1 &= (\langle a_i, b_i, T_{1,i} \rangle)_{i \in I} \\ T_2 &= (\langle a_i, b_i, T_s \rangle)_{i \in I} \end{aligned}$$

where $T_{1,i} \in \mathcal{OS}_{T_s}$ for all $i \in I$. By Theorem 2 we have $T_1 \gg T_2$ if and only if $T_{1,i} \gg T_s$ for all $i \in I$. Thus, in order to solve the dominance on $\mathcal{OS}_{T_s}$ completely, it is sufficient to describe all t-norms $T \in \mathcal{OS}_{T_s}$ that dominate $T_s$. In the following two chapters we solve this question for the case $T_s = T_{\mathbf{L}}$ and $T_s = T_{\mathbf{P}}$.

## 3 Dominance on $\mathcal{OS}_{T_{\mathbf{L}}}$

The principal result of this section is a generalization and strenghtening of methods developed in [12] (see also [11] for a nice survey). The main message is, that Theorem 3 can be strenghthened provided all t-norms are from $\mathcal{OS}_{T_{\mathbf{L}}}$.

**Theorem 4** Consider a triangular norm $T \in \mathcal{OS}_{T_{\mathbf{L}}}$. Such $T$ dominates $T_{\mathbf{L}}$ if and only if $\mathcal{I}_T$ is closed with respect to $T_{\mathbf{L}}$.

Thus the question of dominance between two t-norms from the class $\mathcal{OS}_{T_{\mathbf{L}}}$ can be reduced to the verification of a very simple algebraic property of some subset of the unit interval. To demostrate the power of this characterisation, let us first consider some simple examples.

Let us define $T_\lambda = (\langle \lambda, 1, T_{\mathbf{L}} \rangle)$ with $\lambda \in [0, 1]$; such t-norms form a one-parametrical family. Clearly, $T_{\lambda_1} \geq T_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$. In this situation $T_1$ can be expressed as the ordinal sum $(\langle \lambda_2, 1, T_{\lambda^*} \rangle)$ where $\lambda^* = (\lambda_1 - \lambda_2)/(1 - \lambda_2)$. Clearly $\lambda^* \in [0, 1]$. By Theorem 2 t-norm $T_{\lambda_1}$ dominates $T_{\lambda_2}$ if and only if $T_{\lambda^*} \gg T_{\mathbf{L}}$. Further, by Theorem 4 the latter holds if and only if the set $\mathcal{I}_{T_{\lambda^*}} = [0, \lambda^*] \cup \{1\}$ is closed with respect to $T_{\mathbf{L}}$. Since this is the case for any $\lambda^* \in [0, 1]$ we have that within this family $T_{\lambda_1} \gg T_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$. As a consequence, this family is completely ordered by the dominance relation (see [11] for the detailed treatement).

Now, let us redefine $T_\lambda = (\langle 0, \lambda, T_{\mathbf{L}} \rangle)$ with $\lambda \in [0, 1]$; this is a one-parametrical family often refered to as the Mayor-Torrens family [6]. We have $T_{\lambda_1} \geq T_{\lambda_2}$ if and only if $\lambda_1 \leq \lambda_2$. In this situation $T_1$ can be expressed as the ordinal sum $(\langle 0, \lambda_2, T_{\lambda^*}\rangle)$ where $\lambda^* = \lambda_1/\lambda_2$. Again, we have $\lambda^* \in [0,1]$. By Theorem 2 the t-norm $T_{\lambda_1}$ dominates $T_{\lambda_2}$ if and only if $T_{\lambda^*} \gg T_{\mathbf{L}}$. And by Theorem 4 the latter holds if and only if the set $\mathcal{I}_{T_{\lambda^*}} = \{0\} \cup [\lambda^*, 1]$ is closed with respect to $T_{\mathbf{L}}$. A very simple analysis reveals that this property is never satisfied unles $\lambda^* = 0$ or $\lambda^* = 1$. These two cases correspond to the situations $\lambda_1 = 0$ or $\lambda_1 = \lambda_2$ respectively. Observe that both these situations encode the trivial cases of dominance only; while the first one corresponds to the fact that $T_{\mathbf{M}}$ dominates any member of the family, the second one indicates that the dominance is a reflexive relation. As a consequence there are no other cases of dominance within this family. Therefore this family is ordered by the dominance relation although, in contrast to the previous example, the order is not linear (again, for a detailed treatment see [11]).

Finally, let us put $T_1 = (\langle \frac{1}{2}, 1, T_{\mathbf{L}}\rangle)$, $T_2 = (\langle 0, \frac{1}{2}, T_{\mathbf{L}}\rangle, \langle \frac{1}{2}, 1, T_{\mathbf{L}}\rangle)$ and $T_3 = T_{\mathbf{L}}$. Making use of Theorem 2 and Theorem 4 one can show easily that $T_1 \gg T_2$, $T_2 \gg T_3$ while $T_1 \not\gg T_3$ [12]. It follows in turn, that dominance is not transitive on the class $\mathcal{OS}_{T_{\mathbf{L}}}$. Plenty of other counterexamples to the transitivity of dominance of continous triangular norms can be constructed with the aid of the just presented method [11].

## 4 Dominance on $\mathcal{OS}_{T_{\mathbf{P}}}$

We say that a t-norm $T$ is *ordinally irreducible* if the only way how to represent it as an ordinal sum is $(\langle 0, 1, T\rangle)$. We consider some ordinal sum t-norm $T = (\langle a_1, b_i, T_i\rangle)_{i\in I}$ and assume that all its summands are ordinally irreducible, i.e., the actual representation of $T$ as an ordinal sum is the finest possible. Then we define the so called *axis* of $T$ as a set

$$\mathcal{AX}_T = \{(x,x) \,|\, x \in [0,1]\} \cup \left( \bigcup_{i\in I} [a_i, b_i]^2 \right).$$

Note that due to the chosen representation of $T$, its axis $\mathcal{AX}_T$ is uniquely defined.

The function $f\colon [0,1] \to [0,1]$ is said to be *superhomogenous* [1] if the mapping $x \mapsto f(x)/x$ is non-increasing on the interval $]0,1]$. A t-norm is said to be superhomogenous if all its horizontal sections are superhomogenous. Note that $T_{\mathbf{P}}$ and $T_{\mathbf{M}}$ are examples of superhomogenous t-norms while $T_{\mathbf{L}}$ is not.

**Theorem 5** Let $T$ be a superhomogeneous ordinal sum t-norm. If

$$T(T_{\mathbf{P}}(x,y), T_{\mathbf{P}}(u,v)) \geq T_{\mathbf{P}}(T(x,u), T(y,v))$$

holds for any $(y,v) \in \mathcal{AX}_T$, then $T \gg T_{\mathbf{P}}$.

In particular, if all summands are superhomogeneous, the resulting ordinal sum is so; therefore, Theorem 5 applies to all $T \in \mathcal{OS}_{T_{\mathbf{P}}}$. Moreover in the case of $\mathcal{OS}_{T_{\mathbf{P}}}$ one can characterize the dominance in a style analogical to Theorem 4.

**Theorem 6** Consider a triangular norm $T \in \mathcal{OS}_{T_{\mathbf{P}}}$. Such $T$ dominates $T_{\mathbf{P}}$ if and only if $\mathcal{I}_T$ is closed with respect to $T_{\mathbf{P}}$.

Thus the question of dominance within the class $\mathcal{OS}_{T_{\mathbf{P}}}$ can be reduced to a verification of a very simple algebraic property of the set of idempotent elements of one of the t-norms involved. Again, we list a few of examples.

Similarly as in the previous section, we can define the one-parametric family $T_\lambda = (\langle \lambda, 1, T_{\mathbf{P}}\rangle)$ for $\lambda \in [0,1]$. Again, $T_{\lambda_1} \geq T_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$. In this case we can also write $T_1 = (\langle \lambda_2, 1, T_{\lambda^*}\rangle)$ where $\lambda^* = (\lambda_1 - \lambda_2)/(1-\lambda_2)$. By Theorem 2 $T_{\lambda_1} \gg T_{\lambda_2}$ if and only if $T_{\lambda^*} \gg T_{\mathbf{P}}$. Obviously $\lambda^* \in [0,1]$, so $\mathcal{I}_{T_{\lambda^*}} = [0, \lambda^*] \cup \{1\}$; this set is closed with respect to multiplication regardless of $\lambda^*$. By Theorem 6 $T_{\lambda_1} \gg T_{\lambda_2}$ if and only if $\lambda_1 \geq \lambda_2$. Observe that the treatment of this one-parametric family was formally the same as the treatment of the analogic family in the previous section. One can also mimic the investigation of the second family from the previous section, thus obtaining an analogical result for dominance of triangular norms $T_\lambda = (\langle 0, \lambda, T_{\mathbf{P}}\rangle)$.

In order to construct less trivial t-norms $T \in \mathcal{OS}_{T_{\mathbf{P}}}$ dominating $T_{\mathbf{P}}$ let us fix arbitrary $p, q \in ]0,1[$ and define the set $M = \{p^n q^m \,|\, n, m \in \mathbb{N}\} \cup \{1\}$. Obviously, $M$ is a countable set closed with respect to multiplication. Moreover, for each $a \in M$ there exists a unique element $a' \in M$ with $a' < a$ and such that there exist no $c \in M$ with $a' < c < a$. Now define $T = (\langle a', a, T_{\mathbf{P}}\rangle)_{a\in M}$. For this t-norm we have $\mathcal{I}_T = M \cup \{0\}$. Evidently, also $\mathcal{I}_T$ is closed with respect to multiplication; from Theorem 6 follows immediately that $T \gg T_{\mathbf{P}}$.

## 5 Possible generalizations

It is interesting that both Theorems 4 and 6 are formally of the same structure captured by the following schema.

**Schema 7** Consider a triangular norm $T \in \mathcal{OS}_{T_{\mathrm{s}}}$. Such $T$ dominates $T_{\mathrm{s}}$ if and only if $\mathcal{I}_T$ is closed with respect to $T_{\mathrm{s}}$.

The main message of the previous two chapters could be summarized as that Schema 7 holds if $T_{\mathrm{s}} = T_{\mathbf{L}}$

or $T_s = T_{\mathbf{P}}$. Immediatelly a natural question arises — which other t-norms satisfy this schema ? In the sequel we provide an example of an Archimedean t-norm which violates Schema 7.

The requirement that a subset of the unit interval has to be closed with respect to $T_{\mathbf{L}}$ or $T_{\mathbf{P}}$ is rather restrictive. Among different consequences of this condition, the one summarized in the following theorem is of a particular importance for our following considerations.

**Theorem 8** If some $M \subseteq [0,1]$ fulfills $\{0,1\} \subseteq M$ and is closed with respect to some Archimedean t-norm, then either $M = [0,1]$ or the point 1 is isolated in $M$, i.e., there exists some $a < 1$ such that $]a, 1[ \cap M$ is the empty set.

This observation taken into account together with Theorem 3 yields interesting consequences for dominance of ordinal sum t-norms. In particular, if the t-norm $T = (\langle a_i, b_i, T_i \rangle)_{i \in I}$ dominates some Archimedean t-norm, then either $T$ is simply the trivial ordinal sum equal to $T_{\mathbf{M}}$ ($\mathcal{I}_T = [0,1]$) or the right endpoint of some summand carrier coincides with 1 (meaning that $b_i = 1$ for some index $i$). If we restrict to nontrivial cases only, then one summand carrier of $T$ is of the type $[a, 1]$. We will refer to this summand carrier as the *top summand carrier* and to the corresponding summand as the *top summand*. In the special case when the dominated t-norm is $T_{\mathbf{L}}$ we can say even more.

**Theorem 9** If the ordinal sum t-norm $T$ different from $T_{\mathbf{M}}$ dominates $T_{\mathbf{L}}$, then so does its top summand.

Now, let $T_s = (T_{\mathbf{L}})_\varphi$ for some order isomorphism $\varphi\colon [0,1] \to [0,1]$. Clearly, $T_s$ is Archimedean. Assume that the t-norm $T \in \mathcal{OS}_{T_s}$ different from $T_{\mathbf{M}}$ dominates $T_s$. In other words $T$ satisfies the conclusion of Schema 7. Denote by $[a, 1]$ the top summand carrier of $T$. Since $\varphi$-transforms preserve dominance, we have $T_{\varphi^{-1}} \gg T_{\mathbf{L}}$. By Theorem 9 the top summand of $T_{\varphi^{-1}}$ has to dominate $T_{\mathbf{L}}$ as well. Simple computation reveals that this top summand is $(T_{\mathbf{L}})_{\phi_a}$ where

$$\phi_a = \varphi \circ \psi_{[a,1]} \circ \varphi^{-1} \circ \psi^{-1}_{[\varphi(a),1]}.$$

So we have the following necessary condition.

**Theorem 10** Let $\varphi\colon [0,1] \to [0,1]$ be an order isomorphism. If the t-norm $T = (T_{\mathbf{L}})_\varphi$ satisfies Schema 7 then

$$(T_{\mathbf{L}})_{\phi_a} \gg T_{\mathbf{L}}$$

for all $a \in\, ]0,1[$.

Put now $\varphi\colon x \mapsto x^2$. Corresponding $T_s = (T_{\mathbf{L}})_\varphi$ is the so called Schweizer-Sklar t-norm [13]. Now for the choice $a = \frac{1}{2}$ one can show easily that the corresponding $T_{\phi_a}$ does not dominate $T_{\mathbf{L}}$. Thus $T_s$ is an Archimedean t-norm violating Schema 7.

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