Domination and Information Boundedness Principle for Aggregation

Michal Šabo
STU Bratislava, Radlinského 9,
81237 Bratislava, Slovakia
e-mail: michal.sabo@stuba.sk

Abstract

The information boundedness principle for rule based inference process requires that the knowledge obtained as a result of a rule should not have more information than that contained in the consequent of this rule. We formulate the information boundedness principle for aggregation and show that it can be expressed using the notion of dominance. We also investigate conditions under which this principle holds and give some open problems.

Keywords: Information boundedness principle, Aggregation, Specificity measure, Domination.

1 Introduction

The main aim of rule based fuzzy modelling inference process is to establish a value of the output for given input value. Such system usually involves the use of a rule base consisting of several fuzzy statements. The most used types of fuzzy statements are if-then rules with fuzzy predicates. Overall output can be obtained by aggregation of individual rule outputs and by a possible defuzzification. The individual rule output is often based on the use of fuzzy implications or fuzzy conjunctions. The generalization of these operators is a relevancy transformation operator (RET operator) [1, 4, 5]. So, the individual output is obtained from the relevancy of the used rule and the rule consequent. To have this process meaningful, we require: Knowledge obtained in the individual rule output should not have more information than that contained in the consequent of rule [4]. This principle is usually called the information boundedness principle (IBP) for a fuzzy rule. The IBP for RET operators is investigated in [1]. For purpose of this paper we need to define some measure of information. As we consider fuzzy subsets of a finite universe we shall use a specificity measure which is maximal for singletons. Of course, a pointwise aggregation of fuzzy sets with high specificity measures need not to be a fuzzy set with high specificity again; particularly the aggregation of singletons need not to be a singleton. Thus we can express IBP for aggregation as follows: *The specificity measure of aggregation of individual outputs should not exceed the aggregation of specificities of individual outputs.*

It is clear that the mentioned principle needs two aggregation operators in general, one for aggregation of fuzzy sets (individual outputs) and another one for aggregation of specifities. For simplicity, in this paper, we shall use the same aggregator in both cases.

The main goal of this contribution is to investigate mathematical aspects of the mentioned IBP for aggregation. We will not pay attention to philosophical background of this problem.

This contribution is organised as follows. We recall necessary basic notions in Section 2. We introduce IBP for aggregation operators in Section 3. Section 4 brings some new results and some open problems.

2 Preliminaries

In this paper we shall consider fuzzy subset of a finite universe X with the cardinality n. Then each fuzzy set can be represented by an n-tuple $(a_1, a_2, \ldots, a_n) \in [0, 1]^n$. Many information measures have been attached to finite fuzzy sets, e.g., entropy of Shannon, measure of imprecision etc. Motivated by Yager [4, 5, 6] we define a specificity measure as a mapping from the system of all fuzzy sets on X to the unit interval [1].

Definition 1 A mapping Sp: $[0, 1]^n \longrightarrow [0, 1]$ is called a specificity measure (specificity for short) if the following axioms are satisfied:

(S1) For any permutation $(p1, p2, \ldots, pn)$ of $(1, 2, ..., n)$ is

$$Sp(a_{p1}, a_{p2}, \ldots, a_{pn}) = Sp(a_1, a_2, \ldots, a_n).$$

(S2) $Sp(a_1, a_2, \ldots, a_n) = 1$ if and only if there exists the unique i such that $a_i = 1$ and $a_j = 0$ for $j \neq i$.

(S3) If $1 \geq b_1 > b_2 \geq a_2 \geq a_3 \geq \ldots \geq a_n$ then

$$Sp(b_1, a_2, \ldots, a_n) > Sp(b_2, a_2, \ldots, a_n).$$

(S4) If $1 \geq a_1 \geq a_2 \geq \ldots \geq a_n$ and $1 \geq a_1 \geq b_2 \geq \ldots \geq b_n$ and $a_i \geq b_i$ for $i = 2, 3, \ldots, n$, then

$$Sp(a_1, a_2, \ldots, a_n) \leq Sp(a_1, b_2, \ldots, b_n).$$

(S5) $Sp(0, 0, \ldots, 0) = 0$

Axioms (S3) and (S4) say that specificity is increasing in the greatest membership value and non increasing in the others. Axiom (S2) says that only singletons have maximal specificity.

Moreover we say that a specificity Sp is grounded if

$$Sp(b, b, \ldots, b) = 0$$

for all $b \in [0, 1]$ and we say that Sp is shift invariant if

$$Sp(a_1, a_2, \ldots, a_n) = Sp(a_1+b, a_2+b, \ldots, a_n+b)$$

for all $b \in [-\min_i(a_i), 1 - \max_i(a_i)]$.

Example 1 Consider fuzzy sets over universe with two elements only, so $n = 2$. Put

(i) $Sp_1(x, y) = \max\{x, y\} - \min\{x, y\}$

(ii) $Sp_2(x, y) = \max\{x, y\} - 0.5 \min\{x, y\}$.

The specificity Sp_1 is grounded and shift invariant, but Sp_2 is also specificity but it is neither grounded nor shift invariant.

The specificity Sp_1 belongs to the family of so called linear specifities [1, 4].

Definition 2 Let $1 \geq v_2 \geq v_3 \geq \ldots \geq v_n$ be given constants such that $v_2 + v_3 + \ldots + v_n = 1$. Then a linear specificity Sp: $[0, 1]^n \longrightarrow [0, 1]$ is defined by:

$$Sp(a_1, a_2, \ldots, a_n) = a_{k1} - v_2\, a_{k2} - v_3\, a_{k3} - \ldots - v_n\, a_{kn}$$

where $(k1, k2, \ldots, kn)$ is a permutation of $(1, 2, ..., n)$ is such that $a_{k1} \geq a_{k2} \geq a_{k3} \geq \ldots \geq a_{kn}$.

Note that each linear specificity can be expressed using an OWA operator [4, 5] by

$$Sp(a_1, a_2, \ldots, a_n) = a_i -$$
$$\text{OWA}\ v_n, \ldots, v_3, v_2\ (a_1, a_2, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$$

with $a_i = \max(a_1, a_2, \ldots, a_n)$. For $n = 1$ we put $Sp(a_1) = a_1$.

Every linear specificity is grounded and shift invariant. Yager [4] showed that any linear specificity Sp satisfies

$$Sp_{l\min} \leq Sp \leq Sp_{l\max}$$

where

$$Sp_{l\min}(a_1, a_2, \ldots, a_n) = a_1 - a_2$$

and

$$Sp_{l\max}(a_1, a_2, \ldots, a_n) = a_1 - \frac{a_2 + a_3 + \ldots + a_n}{n-1}$$

for $1 \geq a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq 0$.

The notion of aggregation operator (aggregator for short) plays an important role in fuzzy inference processes, especially in the information fusion.

Definition 3 A mapping

$$Agg : \bigcup_{n \in N} [0,1]^n \to [0,1]$$

is called aggregator if for all naturals n

(*i*) $Agg(0, 0, \ldots, 0) = 0$, $Agg(1, 1, \ldots, 1) = 1$.

(*ii*) $Agg(x_1, x_2, \ldots x_n) \geq Agg(y_1, y_2, \ldots y_n)$ when $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$.

(*iii*) $Agg(x) = x$ for all $x \in [0, 1]$.

The arithmetic and weighted means, t-norms, t-conorms are examples of aggregators. In this paper we shall aggregate also fuzzy sets over the same universe pointwisely i. e.,

$$A = Agg\ (A_1, A_2, \ldots, A_k)$$

if for all $x \in X$

$$A(x) = Agg\ (A_1(x), A_2(x), \ldots, A_k(x)).$$

Remark that for given natural m the aggregator can be considered as m-ary operation

$$Agg: [0, 1]^m \longrightarrow [0, 1].$$

The notion of dominance [2, 3] plays an important role in many fields of mathematics. It can be defined for n-ary operations on posets. We restrict our definition on the unit interval.

Definition 4 Let U: $[0, 1]^m \longrightarrow [0, 1]$, V: $[0, 1]^n \longrightarrow [0, 1]$ be m-ary and n-ary operations respectively. We say that an operation U dominates an operation V or V is dominated by U, shortly $U \gg V$ or $V \ll U$, if for any matrix$(x_{i,j})$ of type $m \times n$ with elements from [0, 1] we have

$$U(V(x_{1,1}, \dots, x_{1,n}), \dots, V(x_{m,1}, \dots, x_{m,n})) \geq$$

$$V(U(x_{1,1}, \dots, x_{m,1}), \dots, U(x_{1,n}, \dots, x_{m,n})).$$

If U and V are binary operations, $U \gg V$ is equivalent to

$$U(V(x, y), V(u, v)) \geq V(U(x, u), U(y, v))$$

for all $x, y, u, v \in [0, 1]$.

3 IBP for aggregators

Let m, n be naturals and B_1, B_2, …, B_m be fuzzy sets (individual fuzzy outputs) over the same finite universe X with cardinality n. Thus $B_1, B_2, \dots, B_m \in [0, 1]^n$. Let Sp: $[0, 1]^n \longrightarrow [0, 1]$ be a specificity measure. Then the aggregator Agg: $[0, 1]^m \longrightarrow [0, 1]$ fulfills IBP with respect to a given specificity Sp on the universe X if

$$Sp(B) \leq Agg(Sp(B_1), Sp(B_2), \dots, Sp(B_m))$$

for any $B_1, B_2, \dots, B_m \in [0, 1]^n$ and B (overall output) is given by

$$B = Agg(B_1, B_2, \dots, B_m)$$

Recall that we shall use the same aggregation operator for the aggregation of fuzzy sets and also for specifities. The next proposition allows verifying IBP by means of dominance.

Proposition 1 An aggregator Agg:$[0, 1]^m \longrightarrow [0, 1]$ fulfills IBP with respect to a given specificity Sp: $[0, 1]^n \longrightarrow [0, 1]$ on a finite universe X with cardinality n if and only if

$$Agg \gg Sp.$$

Proof: Agg:$[0, 1]^m \longrightarrow [0, 1]$fulfills IBP with respect to a given specificity Sp: $[0, 1]^n \longrightarrow [0, 1]$ if and only if for any $B_1, B_2, \dots, B_m \in [0, 1]^n$ and $B = Agg(B_1, B_2, \dots, B_m)$ we have

$$Sp(B) \leq Agg(Sp(B_1), Sp(B_2), \dots, Sp(B_m)).$$

Let

$$B_1 = (b_{1,1}, b_{1,2}, \dots, b_{1,n})$$

$$B_2 = (b_{2,1}, b_{2,2}, \dots, b_{2,n})$$

………………………..

$$B_m = (b_{m,1}, b_{m,2}, \dots, b_{m,n})$$

Then

$$B = Agg(B_1, B_2, \dots, B_m) =$$

$(Agg(b_{1,1}, b_{2,1}, \dots, b_{m,1}), Agg(b_{1,2}, b_{2,2}, \dots, b_{m,2}), Agg(b_{1,n}, b_{2,n}, \dots, b_{m,n}))$.

The last inequality is equivalent to

$$Sp(Agg(b_{1,1}, b_{2,1}, \dots, b_{m,1}), Agg(b_{1,2}, b_{2,2}, \dots, b_{m,2}), \mathrm{Agg}(b_{1,n}, b_{2,n}, \dots, b_{m,n})) \leq$$

$$Agg(Sp(b_{1,1}, b_{1,2}, \dots, b_{1,n}), Sp(b_{2,1}, b_{2,2}, \dots, b_{2,n}), \dots, Sp(b_{m,1}, b_{m,2}, \dots, b_{m,n}))$$

which means that $Agg \gg Sp$.

4 Results and open problems

We have introduced IBP for an aggregator (considered as the an m-ary operation) with respect to a given specificity measure for fuzzy sets on given n-membered universe. We shall try to characterize m-ary aggregators fulfilling IBP with respect to given specificity and formulate some open problems.

Proposition 2 Consider a linear specificity Sp: $[0, 1]^2 \longrightarrow [0, 1]$ and the arithmetic mean $Amean$: $[0, 1]^m \longrightarrow [0, 1]$ as the aggregator. Then for any natural m

$$Amean \gg Sp$$

Proof. Evidently for all $(x, y) \in [0, 1]^2$

$$Sp(x,y) = |x - y|$$

and for all $A_1 = (a_{11}, a_{12})$, $A_2 = (a_{2,1}, a_{2,2})$, …, $A_m = (a_{m,1}, a_{m,2})$ we have

$$Amean(Sp(a_{11}, a_{12}), Sp(a_{2,1}, a_{2,2}), Sp(a_{m,1}, a_{m,2}) = \frac{|a_{1,1} - a_{1,2}| + |a_{2,1} - a_{2,2}| + \dots + |a_{m,1} - a_{m,2}|}{m} \geq$$

$$\frac{|a_{1,1} + a_{2,1} + \dots + a_{m,1} - (a_{1,2} + a_{2,2} + \dots + a_{m,2})|}{m} =$$

$$Sp(Amean((a_{1,1}, \dots, a_{m,1}), Amean((a_{1,2} \dots, a_{m,2})$$

which proves that $Amean \gg Sp$.

In the next example we shall show that the modification of Proposition 2 is not more valid for $n = 3$; i.e., for any linear specificity Sp: $[0, 1]^3 \longrightarrow [0, 1]$.

Example 2 Consider fuzzy sets over a universe with three elements and the minimal linear specificity Sp: $[0, 1]^3 \longrightarrow [0, 1]$ given by

$$Sp\ (a_1, a_2, a_3) = a_1 - a_2$$

where $1 \geq a_1 \geq a_2 \geq a_3 \geq 0$ and the arithmetic mean *Amean* as an aggregator. Put

$$A = (1, 0, 0.8),\ B = (0, 1, 0.8)$$

Then $Sp(A) = Sp(B) = 0.2$, $Amean(A, B) = (0.5, 0.5, 0.8)$ and $0.2 = Amean(Sp(A), Sp(B)) < Sp(Amean(A, B) = 0.3$. The aggregator *Amean* does not dominate the specificity *Sp*; IBP is not fulfilled.

Proposition 2 can be generalized for a wider class of specifities.

Proposition 3 Consider a specificity Sp: $[0, 1]^2 \longrightarrow [0, 1]$ given by

$$Sp(x, y) = \max\{x, y\} - \alpha \min\{x, y\}$$

for all $(x, y) \in [0, 1]^2$ and for a fixed $\alpha \in [0, 1]$ and the arithmetic mean as an aggregator *Amean* : $[0, 1]^m \longrightarrow [0, 1]$. Then for any natural m holds

$$Amean \gg Sp.$$

Proof. For all $A_1 = (a_{1,1}, a_{1,2})$, $A_2 = (a_{2,1}, a_{2,2})$, …, $A_m = (a_{m,1}, a_{m,2})$ we have

$$Sp(Amean(A_1, A_2, \ldots A_m)) =$$

$$Sp\left(\frac{a_{1,1} + a_{2,1} + \ldots + a_{m,1}}{m}, \frac{a_{1,2} + a_{2,2} + \ldots + a_{m,2}}{m}\right) =$$

$$\max\left\{\frac{a_{1,1} + a_{2,1} + \ldots + a_{m,1}}{m}, \frac{a_{1,2} + a_{2,2} + \ldots + a_{m,1}}{m}\right\} -$$

$$\alpha \min\left\{\frac{a_{1,1} + a_{2,1} + \ldots + a_{m,1}}{m}, \frac{a_{1,2} + a_{2,2} + \ldots + a_{m,2}}{m}\right\} \leq$$

$$\frac{\max\{a_{1,1} + a_{1,2}\} + \max\{a_{2,1} + a_{2,2}\} + \max\{a_{m,1} + a_{m,2}\}}{m} -$$

$$\alpha \frac{\min\{a_{1,1} + a_{1,2}\} + \min\{a_{2,1} + a_{2,2}\} + \min\{a_{m,1} + a_{m,2}\}}{m}$$

which proves the claim.

The following proposition shows that there exists an aggregator which fulfills IBP for arbitrary cardinality of finite universe with respect to all linear specificities. Remark that the maximum operator is often used as an aggregator in fuzzy modeling inference processes.

Proposition 4 Consider a linear specificity Sp: $[0, 1]^n \longrightarrow [0, 1]$ and the aggregator Agg:$[0, 1]^m \longrightarrow [0, 1]$ given by

$$Agg\ (b_1, b_2, \ldots, b_m) = \max\{b_1, b_2, \ldots, b_m\}.$$

Then for all naturals n, m

$$Agg \gg Sp.$$

Proof: The proof is trivial if $m = 1$ or $n = 1$
Consider $n, m \in \{2, 3, \ldots\}$,

$$B_1 = (b_{1,1}, b_{1,2}, \ldots, b_{1,n})$$

$$B_2 = (b_{2,1}, b_{2,2}, \ldots, b_{2,n})$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$B_m = (b_{m,1}, b_{m,2}, \ldots, b_{m,n})$$

and

$$B = \max\{B_1, B_2, \ldots, B_m\} = (b_1, b_1, \ldots, b_n) =$$

$(\max(b_{1,1}, b_{2,1}, \ldots, b_{m,1}), \max(b_{1,2}, b_{2,2}, \ldots, b_{m,2}), \max(b_{1,n}, b_{2,n}, \ldots, b_{m,n}))$.
Then

$$Sp(Agg\ (B_1, B_2, \ldots, B_m)) =$$

$$Sp(\max\ (B_1, B_2, \ldots, B_m)) = b_i -$$

$$\text{OWA}\ v_n, \ldots, v_3, v_2\ (b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n),$$

where $b_i = \max\{b_1, b_2, \ldots, b_n\}$ is the maximal element of the matrix $(b_{i,j})$ (maximal membership value in all considered fuzzy sets $B_1, B_2, \ldots, B_m$). Then $b_i = b_{k,i}$ for some $k \in \{1, 2, \ldots, m\}$ and

$$Sp(B_k) = b_i -$$

$$\text{OWA}\ v_n, \ldots, v_3, v_2\ (b_{k,1}, b_{k,2}, \ldots, b_{k,i-1}, b_{k,i-1}, \ldots, b_{k,n})$$

and

$$Agg(Sp(B_1), Sp(B_2), \ldots, Sp(B_m)) =$$

$$\max(Sp(B_1), Sp(B_2), \ldots, Sp(B_m)) \geq Sp(B_k)$$

Replacing $b_{k,1}, b_{k,2}, \ldots, b_{k,i-1}, b_{k,i+1}, \ldots, b_{k,n}$ by non smaller values $b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n$ we have

$$Agg(Sp(B_1), Sp(B_2), \ldots, Sp(B_m)) \geq Sp(B_k) \geq b_i -$$

$$\text{OWA}\ v_n, \ldots, v_3, v_2\ (b_1, b_2, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n) =$$

$$Sp(Agg\ (B_1, B_2, \ldots, B_m))$$

for any $B_1, B_2, \ldots, B_m \in [0, 1]^n$. Thus

$$Agg = \max \gg Sp.$$

Contrary to Proposition 4, the following one shows that there exists an aggregator which does not fulfill IBP with respect to any specificity.

Proposition 5 Consider n, $m \in \{2, 3, \ldots\}$ and the aggregator Agg:$[0, 1]^m \longrightarrow [0, 1]$ given by

$$Agg\,(b_1,\ b_2,\ \ldots,\ b_m) = \min\{b_1,\ b_2,\ \ldots,\ b_m\}.$$

Then Agg does not fulfill IBP with respect to any specificity Sp: $[0, 1]^n \longrightarrow [0, 1]$.

Proof: Let

$$B_1 = (1, 0, \ldots, 0)$$

$$B_2 = B_3 = \ldots = B_m = (1, ½, ½, \ldots, ½).$$

Because $Sp(B_1) = 1$, $Sp(B_2) < 1$ and

$$Agg\,(B_1, B_2, \ldots,\ B_m) = B_1$$

we have

$$Agg(Sp(B_1\ ), Sp(B_2), \ldots, Sp(B_\mathrm{m})) =$$

$$\min(Sp(B_1\ ), Sp(B_2), \ldots, Sp(B_\mathrm{m})) < 1$$

and

$$Sp(Agg\,(B_1, B_2, \ldots,\ B_m)) = Sp(B_1\ ) = 1.$$

Agg does not dominate any specificity.

The problem of dominance $Agg \gg Sp$ evokes many questions. We formulate some of them:

Global open problem:

- to characterize aggregators Agg: $[0, 1]^m \longrightarrow [0, 1]$ and specificity measures Sp: $[0, 1]^n \longrightarrow [0, 1]$ fulfilling IBP for some (or all) m, $n \in \{2, 3, \ldots\}$.

Particular open problems

- to characterize aggregators Agg:$[0, 1]^m \longrightarrow [0, 1]$ which fulfills IBP with respect to linear specificity Sp: $[0, 1]^n \longrightarrow [0, 1]$

- to characterize the specificity Sp : $[0, 1]^n \longrightarrow [0, 1]$ for which the arithmetic mean $Amean$:$[0, 1]^m \longrightarrow [0, 1]$ fulfills IBP with respect to Sp.

- to prove or reject:

  - $Agg \gg Sp_1 \geq Sp_2 \Rightarrow Agg \gg Sp_2$

  - $Agg_1 \geq Agg_2 \gg Sp \Rightarrow Agg_1 \gg Sp$

Acknowledgements

This work was supported by grant VEGA 1/2005/05.

References

[1] P. Sarkoci and M. Šabo: Information boundedness principle in fuzzy inference process, *Kybernetika* 38 (2002), No. 3, 327-338.

[2] P. Sarkoci: Domination in the Families of Frank and Hamacher t-norms, *Kybernetika* 41 (2005), 349-360.

[3] R.M.Tardiff: On a Generalised Minkovski Inequality and Its Relation to Dominates for t-norms, Aequationes Math. 27(1984), 308-316.

[4] R. R. Yager: Global requirements for implication operators in fuzzy modus ponens, *Fuzzy Sets and Systems* 106 (1999), 3-10.

[5] R. R. Yager: Uninorms in fuzzy modeling, *Fuzzy Sets and Systems* 122 (2001), 167-175.

[6] R. R. Yager: Ordinal measure of specificity, *International Journal of General Systems* 17 (1990), 57-72.