Extending the Choquet integral

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Abstract

In decision under uncertainty, the Choquet integral yields the expectation of a random variable with respect to a fuzzy measure (or non-additive probability or capacity). In general, for the discrete setting, this technique allows to integrate functions taking values on a finite n-set with respect to a (fuzzy) measure taking values on subsets of such a set. Yet, the integrand may well be treated as an additive function taking values on subsets itself: the value associated with each subset is simply the sum of the values associated with all the atoms (or 1-cardinal subsets) in that subset. The Choquet technique is here extended to the case where the integrand, just like the measure, is a non-additive function taking values on subsets itself. The resulting aggregation operator is an extension of the Choquet integral: the former coincides with the latter whenever the integrand is additive. Four such extensions are provided, two of which are obtained by means of the Möbius inversion of the integrand and the (fuzzy) measure with respect to which integration is performed. In all cases, the resulting integral is an extension of the measure: it coincides with this latter on the vertices of the n-dimensional unit hypercube. Yet, one of these extensions also inherits another main feature of the (traditional) Choquet integral: if the fuzzy measure is convex, then this extended Choquet integral equals its minimum over all probabilities in the core of the measure. The general technique applies to both monotone and antitone integrands, and when the integrand is real-valued (i.e., taking both positive and negative values) it allows for both a symmetric and an asymmetric form. Two conceivable applications are provided. One furnishes an expectation of diversity in a random sample of a known population. Here the integrand is a *diversity function*, which is monotone by construction. The other application furnishes the certainty equivalent for a problem of decision making under uncertainty where the decision maker has some belief about what information (in the form of an event containing the 'true' state) will be available before taking action. Here the integrand is antitone.

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1 Introduction

The (discrete) Choquet integral is an aggregation operator very useful in DUU (decision under uncertainty) and MCDM (multicriteria decision making). In the former case, a DM (decision maker) has to take action in response to an unknown state of Nature. Preferences take the form of a utility function associating a real number to each pair consisting of a state and an action. The set of utility values attained on all pairs consisting of a state and a fixed action is a random variable. This leads to rank actions according to the associated expectation, requiring, in turn, some beliefs about what state will occur. If beliefs take the form of a probability or additive measure, then actions get ranked according to the EU (expected utility) model. On the other hand, if beliefs take the form of a capacity or fuzzy measure, then actions get ranked according to the CEU (Choquet expected utility) model (see [3] and [10]). In both cases, for each action, the DM performs an aggregation, based on beliefs, of all the utility values such an action may yield. An important fact inspiring the present paper is that the CEU model constitutes, in fact, an extension of the EU one. In particular, fuzzy measures comprehend additive ones as special cases, and whenever the measure is additive the Choquet expectation coincides with the traditional one.

In MCDM, the DM has to choose within a given (finite) set of alternatives according to their score on different evaluation criteria. In this case, for each alternative, the DM has to aggregate the scores that such an alternative attains on each criterion. In particular, if criteria display no interaction, then aggregation can be performed most naturally through a weighted average. More precisely, for the DM each criterion has its own importance or (positive) weight. Such weights may well be normalized so to add up to unity. Accordingly, each alternative can be evaluated according to the weighted average of the scores it attains on criteria. On the other hand, the situation where criteria display interaction is formalized by specifying a weight for each subset of criteria. In this case, in order to keep into account such an interaction, aggregation can no longer be performed through a weighted average. Conversely, the Choquet integral becomes the needed aggregation operator, as the collection of subsets' weights constitutes a fuzzy measure. Here again, this latter aggregation technique appears as an extension: as soon as the fuzzy measure is additive, in which case criteria display no interaction, the Choquet integral reduces to a weighted average (see [6]). A great deal of attention has been paid to the evaluation of negative scores, in which case the Möbius inversion of the measure yields a useful representation. In fact, if the integrand is real-valued, then the Choquet integral allows for both a symmetric and an asymmetric form (see [7]).

The aim of this paper is to present a novel aggregation technique which constitutes, in fact, an extension of the Choquet integral. More precisely, given some n-set N, this technique allows to aggregate the 2^n values taken by a (monotone or antitone) function defined on subsets $A \subseteq N$ w.r.t. (with respect to) a fuzzy measure, taking values on such subsets as well. Yet, if the former function (i.e., the integrand) is additive, then this technique yields the same result as the Choquet integral.

Such an extension is shown to inherit two main features from the Choquet integral. Firstly, for given integrand, the extended Choquet integral (as defined below) w.r.t. a convex measure equals the minimum of this integral over all probabilities in the core of the measure. Secondly, this novel aggregation operator is an extension of the measure: it coincides with this latter on the vertices of the n-dimensional unit hypercube. In addition, it may be represented in terms of the Möbius inversions of both the integrand and the measure, and when the former is real-valued it allows for both a symmetric and an asymmetric form.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a finite set (of states or criteria). Any function $f : N \to \mathbb{R}$ is, in fact, a collection $f(1), \ldots, f(n)$ of n numbers. A weighted average or expectation (if f is a random variable) of these numbers takes form $E_p(f) = \sum_{i \in N} p(i) \cdot f(i)$, where $p(i) \geq 0$ for all $i \in N$ and $\sum_{i \in N} p(i) = 1$, that is, p is a probability. Perhaps, this is the most natural way of aggregating such n numbers $f(1), \ldots, f(n)$. Also, for $g, f : N \to \mathbb{R}$ and given p, if $E_p(g) \geq E_p(f)$, then g is ranked no lower than f.

The Choquet integral is a more sophisticated aggregation technique. It uses fuzzy measures (or capacities, see [3], or non-additive probabilities, see [8]). These are *monotone* $\gamma : 2^N \to [0, 1]$ (i.e., $A \subseteq B \in 2^N$ implies $\gamma(A) \leq \gamma(B)$, where $2^N = \{A : A \subseteq N\}$) such that $\gamma(\emptyset) = 1 - \gamma(N) = 0$. In DUU subsets $A \in 2^N$ are *events*, and fuzzy measures are interpreted as follows: $\gamma(A)$ quantifies the belief that the 'true' state i (that will occur) satisfies $i \in A$. On the other hand, in MCDM $\gamma(A)$ quantifies the importance (or worth) of all criteria $i \in A$ considered together.

Given that for functions taking both positive and negative values the Choquet integral may be computed either symmetrically or else asymmetrically (see [7] and below), firstly consider the case of integrands $f : N \to \mathbb{R}_+$ taking only positive values. Formally, the Choquet integral of f w.r.t. γ is

$$\int_N^C f d\gamma = \sum_{1 \leq i \leq n} \left[f((i)) - f((i-1))\right] \cdot \gamma(\{(i), \ldots, (n)\})$$

$$= \sum_{1 \leq i \leq n} f((i)) \cdot \left[\gamma(\{(i), \ldots, (n)\}) - \gamma(\{(i+1), \ldots, (n)\})\right]$$

where $(\cdot) : N \to N$ is any permutation such that $f((n)) \geq \cdots \geq f((i)) \geq \cdots \geq f((1)) \geq f((0)) := 0$. Also, $\gamma(\{(i+1), \ldots, (n)\})$ clearly vanishes for $i = n$. Measure γ is additive when for all $A, B \in 2^N$

$$\gamma(A \cap B) + \gamma(A \cup B) = \gamma(A) + \gamma(B),$$

i.e., when $\gamma(A) = \sum_{i \in A} \gamma(\{i\})$ for all $A \in 2^N$. If γ is additive, then the Choquet integral reduces to

$$\int_N^C f d\gamma = \sum_{1 \leq i \leq n} \gamma(\{i\}) \cdot f(i).$$

Out of the above two expressions for the Choquet integral, the latter links this aggregation operator and the *core* of *convex* (or *supermodular*) measures (see [5], theorem 2.2). The core $\mathcal{C}(\gamma)$ of a measure γ is the set of probabilities p such that $\sum_{i \in A} p(i) \geq \gamma(A)$ for all $A \in 2^N$ (and thus $\sum_{i \in N} p(i) = \gamma(N) = 1$). In general, the core may well be empty. Yet, if the measure is convex, that is, if

$$\gamma(A \cup B) + \gamma(A \cap B) \geq \gamma(A) + \gamma(B)$$

for all $A, B \in 2^N$, then $\mathcal{C}(\gamma) \neq \emptyset$ and

$$\int_N^C f d\gamma = \bigwedge_{p \in \mathcal{C}(\gamma)} E_p(f)$$

for any integrand f, where $\wedge$ (and $\vee$) denote the min or inf (and max or sup) operator.

The characteristic function $\chi_A : N \to \{0, 1\}$ associated with each subset $A \in 2^N$ is defined by $\chi_A(i) = 1$ if $i \in A$ and $\chi_A(i) = 0$ if $i \notin A$. Hence, there is a bijection between 2^N and the set $\{0, 1\}^n$ of vertices of the n-dimensional unit hypercube $[0, 1]^n$. This yields that the Choquet integral is an extension of the measure w.r.t. which integration is performed, as the former coincides with the latter on vertices $\chi_A \in \{0,1\}^n, A \in 2^N$. That is, for $A = \{i_1, \ldots, i_{|A|}\}$, $\int_N^C \chi_A d\gamma =$

$$\sum_{1 \le h \le |A|} \left[\gamma(\{i_h, \ldots, i_{|A|}\}) - \gamma(\{i_{h+1}, \ldots, i_{|A|}\})\right] = \gamma(A)$$

(with $\gamma(\{i_{h+1}, \ldots, i_{|A|}\})$ vanishing for $h = |A|$). In words, as the integrand varies over all $\chi_A, A \in 2^N$, the Choquet integral of such functions w.r.t. to any measure γ yields precisely $\gamma(A), A \in 2^N$. Also, there are $|A|!(n-|A|)!$ distinct permutations $(\cdot)$ that may be interchangeably used for computing the Choquet integral of χ_A. This is the number of distinct ways of putting all $j \in A^c = N \backslash A$ first and all $i \in A$ last.

Permutations bijectively correspond to *maximal chains* $K = \{A_0, A_1, \ldots, A_n\} \subset 2^N$, where

$$N = A_n \supset^* \cdots \supset^* A_k \supset^* \cdots \supset^* A_0 = \emptyset$$

and $\supset^*$ is the *covering relation* (see [1]), that is to say, $A \supset^* B \Leftrightarrow A \supset B, |A| = |B| + 1$ and reads "A covers B", with $A \supset B \Leftrightarrow A \supseteq B, A \neq B$.

Let $\mathcal{V}_m \subset \mathbb{R}_+^{2^n}$ denote the vector space of monotone $v : 2^N \to \mathbb{R}_+$, with $\gamma \in \mathcal{V}_m$ for all measures γ. The *Möbius inversion* $\mu^v : 2^N \to \mathbb{R}$ of $v \in \mathcal{V}_m$ is given by $\mu^v(A) = \sum_{B \subseteq A} (-1)^{|A \backslash B|} \cdot v(B)$ for all $A \in 2^N$. A basis of $\mathcal{V}_m$ is $\{u_A : A \in 2^N\}$, where $u_A(B) = 1$ if $A \subseteq B$ and 0 otherwise. In fact, $v = \sum_{A \in 2^N} \mu^v(A) \cdot u_A$. That is, $v(A) = \sum_{B \in 2^N} \mu^v(B) \cdot u_B(A) = \sum_{B \subseteq A} \mu^v(B)$ for all $A, B \in 2^N$ (see [1]). The Choquet integral of f w.r.t. γ may be expressed in terms of μ^γ as follows

$$\int_N^C f d\gamma = \sum_{A \in 2^N} \mu^\gamma(A) \bigwedge_{i \in A} f(i),$$

where $\mu^\gamma(\emptyset) = \gamma(\emptyset) = 0$ (see [2] and [7]).

3 The extension

The aggregation technique proposed below enables to integrate monotone set functions $v \in \mathcal{V}_m$ w.r.t. fuzzy measures γ. Firstly note that, given monotonicity, $0 \le v(\emptyset) \le v(A)$ for all $A \in 2^N$. In fact, setting $v(\emptyset) = 0$ yields no loss of generality.

For any $v \in \mathcal{V}_m$, let $\mathcal{K}^v$ denote the set of *admissible maximal chains* $K^v = \{A_0^v, A_1^v, \ldots, A_n^v\}$, defined to be those satisfying

$$v(A_i^v) = \bigwedge_{A \supset^* A_{i-1}^v} v(A) \text{ for } 1 \le i \le n,$$

where $\supset^*$ is the covering relation defined above.

Remark 1 *By construction, $\mathcal{K}^v \neq \emptyset$. In particular, $1 \le |\mathcal{K}^v| \le n!$. To see this, let $N = \{1,2,3\}$ and consider the two extreme examples $v, w : 2^N \to \mathbb{R}_+$ where $v(A) = \left(\sum_{i \in A} i\right)^2$ and $w(A) = \left(|A|\right)^2$ for all $A \in 2^N$. In the former case, $|\mathcal{K}^v| = 1$, and the unique maximal chain $K^v \in \mathcal{K}^v$ is $K^v = \{\emptyset, \{1\}, \{1,2\}, N\}$. Conversely, $\mathcal{K}^w$ contains all the $3!$ available maximal chains of lattice 2^N.*

For the i-th increment $\Delta(v(A_i^v)) = v(A_i^v) - v(A_{i-1}^v)$ of v along any $\{A_0^v, A_1^v, \ldots, A_n^v\} = K^v \in \mathcal{K}^v$, define

$$\Delta^2(v(A_i^v)) := \Delta(v(A_i^v)) - \Delta(v(A_{i-1}^v)) =$$

$= v(A_i^v) - 2v(A_{i-1}^v) + v(A_{i-2}^v)$. This is the i-th difference, $1 \le i \le n$, between consecutive increments.

Now, the values taken by set function v may be aggregated w.r.t. fuzzy measure γ by means of the *extended Choquet integral* defined as follows:

$$\int_{2^N}^{EC} v d\gamma := \bigwedge_{K^v \in \mathcal{K}^v} \sum_{1 \le i \le n} \Delta^2(v(A_i^v)) \cdot \gamma(N \backslash A_{i-1}^v),$$

with $K^v = \{A_0^v, A_1^v, \ldots, A_n^v\}$ and $v(A_{-1}^v) := 0$.

Notice immediately that $\wedge$ is over the set $\mathcal{K}^v$ of *admissible* maximal chains only. In particular, $\int_{2^N}^{EC} v d\gamma$ works as follows. Given monotonicity, the integrand v takes non-decreasing values along any maximal chain. Firstly, the focus is placed on the set $\mathcal{K}^v$ of maximal chains along which increments $\Delta(v(A_i^v)), 1 \le i \le n$ are minimal. Secondly, precisely in order to minimize over such chains $\{A_0^v, A_1^v, \ldots, A_n^v\} = K^v \in \mathcal{K}^v$, the focus turns on those satisfying $\gamma(N \backslash A_i^v) \ge \gamma(N \backslash B_i^v)$ for every $\{B_0^v, B_1^v, \ldots, B_n^v\} \in \mathcal{K}^v$ and $1 \le i \le n$. That is to say, increments

$$\Delta(\gamma(N \backslash A_i^v)) = \gamma(N \backslash A_i^v) - \gamma(N \backslash A_{i+1}^v)$$

must be maximal for (increasing) $0 \le i \le n-1$. Once a maximal chain $\{A_0^v, A_1^v, \ldots, A_n^v\}$ satisfying this sort of $\min - \max$ criterion is found (there surely exists at least one), $\int_{2^N}^{EC}$ is simply the sum of all differences $\Delta^2(v(A_i^v))$ between consecutive increments of the integrand, multiplied each by the value $\gamma(N \backslash A_{i-1}^v)$ taken by the measure on A_{i-1}^v's complement.

Also note that, although both the integrand and the measure take 2^n values, only $n+1$ values (for each) are used for aggregation. This is because both the integrand and the measure are not defined on a generic set. Conversely, they are defined on a subset lattice. In fact, this technique may be adapted to any lattice (or even any poset) satisfying the Jordan-Dedekind condition (that is, any two maximal chains must have equal length, see [1]).

Remark 2 *Note that there may be two or more maximal chains satisfying the above* min − max *criterion. In fact, if both the integrand and the measure are symmetric (that is, if* $v(A) = \eta_v(|A|), \gamma(A) = \eta_\gamma(|A|)$ *for some* $\eta_v, \eta_\gamma : \{0, 1, \ldots, n\} \to \mathbb{R}_+$ *and all* $A \in 2^N$*), then all* $n!$ *maximal chains satisfy such a criterion (see remark 1 above). Still, given* v *and* γ*, for any such a maximal chain* $\int_{2^N}^{EC} v d\gamma$ *yields the same result.*

Any $f : N \to \mathbb{R}_+$ extends to the whole power set 2^N through additivity, i.e., as $v_f : 2^N \to \mathbb{R}_+$ defined by $v_f(A) = \sum_{i \in A} f(i), A \in 2^N$. Also, such an extension v_f is monotone (for all f with positive range). Let $\mathcal{V}_m^a \subset \mathcal{V}_m$ denote the space of additive set functions, and for every $v \in \mathcal{V}_m^a$ let f_v denote its restriction to singletons, that is to say, $f_v(i) = v(\{i\}), 1 \leq i \leq n$.

Claim 3 *If* $v \in \mathcal{V}_m^a$*, then* $\int_{2^N}^{EC} v d\gamma = \int_N^C f_v d\gamma$.

Proof: If $v \in \mathcal{V}_m^a$, then any permutation $(\cdot) : N \to N$ satisfying $v(\{(1)\}) \leq \cdots \leq v(\{(n)\})$ yields that maximal chain $K^v = \{A_0^v, \ldots, A_n^v\}$ obtained by setting $A_i^v = \{(1), \ldots, (i)\}$ for $1 \leq i \leq n$ satisfies $K^v \in \mathcal{K}^v$. Also, $\Delta(v(A_i^v)) = v(\{(i)\})$ for $1 \leq i \leq n$. Substitution completes the proof. ■

Hence, $\int^{EC}$ extends $\int^C$, regarded as an aggregation operator, from N to 2^N. For this reason, in the sequel the former shall be termed extended Choquet (EC) integral.

Claim 4 *If* γ *is convex, then for all* $v \in \mathcal{V}_m$

$$\int_{2^N}^{EC} v d\gamma = \bigwedge_{p \in \mathcal{C}(\gamma)} \int_{2^N}^{EC} v dp,$$

where $p(A) = \sum_{i \in A} p(i), A \in 2^N$.

Proof: Any convex γ has a non-empty core whose extreme points $p_K \in ex(\mathcal{C}(\gamma))$ get defined each through a maximal chain $K = \{A_0, \ldots, A_n\}$ by

$$p_K(i) = \gamma(A_k) - \gamma(A_{k-1}) : A_k \backslash A_{k-1} = i, 1 \leq i \leq n$$

(see [11]). Given any $v \in \mathcal{V}_m$ with the associated set $\mathcal{K}^v \ni K^v = \{A_0^v, \ldots, A_n^v\}$ of maximal chains as above (i.e., $v(A_k^v) = \bigwedge_{A \supset^* A_{k-1}^v} v(A), 1 \leq k \leq n$), consider those corresponding $\overline{K^v} = \{N \backslash A_n^v, \ldots, N \backslash A_0^v\}$ obtained by substituting each element with its complement. Clearly, $\gamma(N \backslash A_k^v) = p_{\overline{K^v}}(N \backslash A_k^v), 1 \leq k \leq n$. It only remains to observe that the EC integral obtains by minimizing over $\mathcal{K}^v$. ■

The EC integral inherits another main property from the Choquet integral: it constitutes an extension of the measure, as it coincides with this latter on the vertices of the hypercube. Yet, as integrands must be set functions, each $\chi_A : N \to \{0, 1\}, A \in 2^N$ must be turned into some v_{χ_A} taking values on 2^N. This is most naturally achieved through additivity:

$$v_{\chi_A}(B) = \sum_{i \in B} \chi_A(i) \text{ for all } A, B \in 2^N.$$

In this way, $v_{\chi_A} : 2^N \to \{0, 1, \ldots, |A|\}, A \in 2^N$.

Claim 5 *For any measure* γ*,*

$$\int_{2^N}^{EC} v_{\chi_A} d\gamma = \gamma(A) \text{ for all } A \in 2^N.$$

Proof: For any integrand $v_{\chi_A}, A \in 2^N$, admissible maximal chains $K^{v_{\chi_A}} \in \mathcal{K}^{v_{\chi_A}}$ bijectively correspond to those $(n - |A|)!|A|!$ permutations $(\cdot) : N \to N$ where all $j \notin A$ come first and all $i \in A$ come last, that is to say, $\{(1), \ldots, (n - |A|)\} = N \backslash A$ as well as $\{(n - |A| + 1), \ldots, (n)\} = A$. Also, along any such a maximal chain $\{B_0^v, \ldots, B_n^v\} \in \mathcal{K}^{v_{\chi_A}}$,

$$\sum_{1 \leq k \leq n} \Delta^2(v_{\chi_A}(B_k^v)) \cdot \gamma(N \backslash B_{k-1}^v) =$$

$$\sum_{n-|A|+1 \leq h \leq n} \left[\gamma(\{(h), \ldots, (n)\}) - \gamma(\{(h+1), \ldots, (n)\})\right]$$

$= \gamma(A)$ (with $\gamma(\{(n+1), \ldots, (n)\})$ vanishing). ■

3.1 An extension through averaging

Let $N^{(k)} = \{A \in 2^N : |A| = k\}$ denote the k-th level of subset lattice $(2^N, \cap, \cup)$, containing all k-cardinal subsets $A \subseteq N$, with $|N^{(k)}| = \binom{n}{k}$ for $0 \leq k \leq n$ (see [1]). An integrand $v \in \mathcal{V}_m$ may well attain the same value on several elements of each level, for each level. That is, for each k, it may be $v(A_1) = \cdots = v(A_h)$ with $A_1, \ldots, A_h \in N^{(k)}$ and $1 << h \leq \binom{n}{k}$. If this is the case, then the set $\mathcal{K}^v$ of admissible maximal chains is rather likely to contain many elements (at most, $|\mathcal{K}^v| = n!$ for symmetric v; see remarks 1 and 2 above). Hence, for given γ, define $g_\gamma : \mathcal{K}^v \to \mathbb{R}_+$ by

$$g_\gamma(K^v) = \sum_{1 \leq k \leq n} \Delta^2(v(B_k^v)) \cdot \gamma(N \backslash B_{k-1}^v)$$

for every $\{B_0^v, \ldots, B_n^v\} = K^v \in \mathcal{K}^v$. In general, $g_\gamma(K^v)$ yields different values for different admissible chains. Clearly, in order to define an integration technique, these values must be aggregated. Above, this is achieved through the min operator $\wedge$, as the EC integral is conceived for dealing with DUU, where $\wedge$ seems the most natural way of formalizing risk-aversion. An alternative aggregation technique is the following: if g_γ takes $q \leq |\mathcal{K}^v|$ different values $g_\gamma^1, \ldots, g_\gamma^q$, then set

$$\widetilde{\int_{2^N}^{EC}} v d\gamma = \sum_{1 \leq h \leq q} \frac{|\mathcal{K}_h^v|}{|\mathcal{K}^v|} \cdot g_\gamma^h,$$

where $\mathcal{K}_h^v = \{K^v \in \mathcal{K}^v : g_\gamma(K^v) = g_\gamma^h\}$. This is the weighted average where the weight of each value $g_\gamma^h, 1 \leq h \leq q$ is the ratio of the number of admissible chains yielding that value to the total number of admissible chains. A somehow extreme case is that where v is symmetric but γ is not. Then, $|\mathcal{K}^v| = n!$, and it might well be $q = n!$. Accordingly, the EC integral as defined above through $\wedge$ considers only the minimum over all these $n!$ values, while this (modified) $\widetilde{\text{EC}}$ integral considers the average of all these values, associating weight $(n!)^{-1}$ to each of them.

Inspection reveals that while claims 2 and 4 remain valid for this $\widetilde{\text{EC}}$ integral, claim 3 does not. Formally, for given γ, if v is additive (through f_v), then

$$\widetilde{\int_{2^N}^{EC}} v d\gamma = \int_N^C f_v d\gamma.$$

$$\text{Also, } \widetilde{\int_{2^N}^{EC}} v_{\chi_A} d\gamma = \gamma(A) \text{ for all } A \in 2^N.$$

$$\text{Yet, } \widetilde{\int_{2^N}^{EC}} v d\gamma \neq \bigwedge_{p \in \mathcal{C}(\gamma)} \widetilde{\int_{2^N}^{EC}} v dp$$

for generic convex γ.

3.2 Antitone integrands

The EC technique may also be applied to antitone integrands $v : 2^N \backslash \{\emptyset\} \to \mathbb{R}_+$, where $A \subseteq B \in 2^N$ implies $v(A) \geq v(B)$ for all $A, B \in 2^N \backslash \{\emptyset\}$, with $v(\emptyset) = 0$. (Note that $v(\emptyset) = 0$ implies that v is antitone on $2^N \backslash \{\emptyset\}$ but not on the whole power set 2^N).

In order to achieve this, the general procedure above is somehow reversed. Firstly, for taking into account both the maximum (i.e., $\bigvee_{i \in N} v(\{i\})$) and the minimum (i.e., $v(N)$) values taken by v on $2^N \backslash \{\emptyset\}$, admissible chains $\{A_0^v, \ldots, A_n^v\} = K^v \in \mathcal{K}^v$ satisfy

$$v(A_i^v) = \bigvee_{A \supset^* A_{i-1}^v} v(A) \text{ for } \textit{increasing } 1 \leq i \leq n.$$

Next, for $0 < i \leq n$, define

$$\Delta^2(v(A_i^v)) = v(A_i^v) - 2v(A_{i+1}^v) + v(A_{i+2}^v),$$

with $v(A_{n+1}^v) = v(A_{n+2}^v) = 0$. Then, given any capacity γ, the EC integral of v w.r.t. γ is

$$\int_{2^N}^{EC} v d\gamma = \bigwedge_{K^v \in \mathcal{K}^v} \sum_{0 \leq i < n} \Delta^2(v(A_{n-i}^v)) \cdot \gamma(A_{n-i}^v),$$

with $K^v = \{A_0^v, \ldots, A_n^v\}$.

This EC integral, with integrand v antitone on $2^N \backslash \{\emptyset\}$, may be used for quantifying the certainty equivalent of a seemingly sophisticated DUU problem (see subsection 4.2 below).

3.3 Real-valued integrands

There are two alternative techniques for integrating real-valued functions $f : N \to \mathbb{R}$ w.r.t. fuzzy measures γ, namely, the Šipoš or symmetric Choquet integral, and the asymmetric Choquet integral. In order to formalize these operators, define $f^+(i) = 0 \vee f(i)$ and $f^-(i) = 0 \vee -f(i)$ for $1 \leq i \leq n$. Also, for any measure γ, consider its conjugate $\bar{\gamma}$ defined, for every $A \in 2^N$, by $\bar{\gamma}(A) = \gamma(N) - \gamma(N \backslash A)$. Then, the symmetric Choquet integral is

$$\int_N^{C_{sy}} f d\gamma = \int_N^C f^+ d\gamma - \int_N^C f^- d\gamma,$$

while the asymmetric Choquet integral is

$$\int_N^{C_{asy}} f d\gamma = \int_N^C f^+ d\gamma - \int_N^C f^- d\bar{\gamma}$$

(see [7]).

Paralleling this in terms of the EC integral seems rather straightforward: given a real-valued, monotone $v : 2^N \to \mathbb{R}$, define $v^+(A) = 0 \vee v(A)$ as well as $v^-(A) = 0 \vee -v(A)$ for every $A \in 2^N$. Then, the symmetric EC integral is

$$\int_{2^N}^{EC_{sy}} v d\gamma = \int_{2^N}^{EC} v^+ d\gamma - \int_{2^N}^{EC} v^- d\gamma,$$

while the asymmetric EC integral is

$$\int_{2^N}^{EC_{asy}} v d\gamma = \int_{2^N}^{EC} v^+ d\gamma - \int_{2^N}^{EC} v^- d\bar{\gamma}.$$

The same argument may be applied, *mutatis mutandis*, to antitone integrands.

3.4 Möbius inversion

Like the Choquet integral, the EC integral may be expressed in terms of Möbius inversion. Yet, while in the former case only the measure has a non-trivial such an inversion (as the integrand is additive), in the latter case the integrand displays a non-trivial such an inversion as well.

Given $v, \gamma \in \mathcal{V}_m$, the EC integral of v w.r.t. γ may be written as follows $\int_{2^N}^{EC} v d\gamma =$

$$\bigwedge_{K^v \in \mathcal{K}^v} \sum_{1 \leq i \leq n} \Delta(v(A_i^v)) \cdot \left[\gamma(N \backslash A_{i-1}^v) - \gamma(N \backslash A_i^v)\right],$$

with $K^v = \{A_0^v, \ldots, A_n^v\}$. Also, for all $\emptyset \neq A \in 2^N$,

$$v(A) - v(A \backslash i) = \sum_{B \subseteq A : B \ni i} \mu^v(B) = \sum_{B \in 2^A \backslash 2^{A \backslash i}} \mu^v(B)$$

for all $i \in A$. Accordingly, $\int_{2^N}^{EC} v d\gamma =$

$$\bigwedge_{K^v \in \mathcal{K}^v} \sum_{1 \leq i \leq n} \Big[\sum_{\substack{B \subseteq A_i^v \\ B \not\subseteq A_{i-1}^v}} \mu^v(B) \Big] \cdot \Big[\sum_{\substack{B' \subseteq N \backslash A_{i-1}^v \\ B' \not\subseteq N \backslash A_i^v}} \mu^\gamma(B') \Big],$$

with $K^v = \{A_0^v, \ldots, A_n^v\}$.

Möbius inversion yields two further extensions $\widehat{\int_{2^N}^{EC}}$ and $\overline{\int_{2^N}^{EC}}$ of the Choquet integral. The $\widehat{\text{EC}}$ integral of v w.r.t. γ is

$$\widehat{\int_{2^N}^{EC}} v d\gamma = \sum_{A \in 2^N} \mu^\gamma(A) \bigwedge_{i \in A} \big(v(A) - v(A \backslash i)\big)$$

$$= \sum_{A \in 2^N} \mu^\gamma(A) \bigwedge_{i \in A} \Big(\sum_{B \in 2^A \backslash 2^{A \backslash i}} \mu^v(B) \Big).$$

To see that this actually constitutes an extension (i.e., when the integrand is additive it reduces to the Choquet integral), note that if v is additive through f_v (see above), then $v(A) - v(A \backslash i) = f_v(i)$ for all $A \in 2^N$ and all $i \in A$, so that the former expression above reduces to a well known representation of the Choquet integral in terms of the Möbius inversion of the measure (see [2], [7] and section 2 above). Concerning the latter expression, the Möbius inversion of additive set functions v satisfying $v(\emptyset) = 0$ lives only on 1-cardinal subsets (or atoms), i.e., $|A| \neq 1 \Rightarrow \mu^v(A) = 0$ (see [1] on valuations of distributive lattices, pp. 189-191). In particular, $v_{\chi_A}, A \in 2^N$ is an additive set function (see above), and it may be easily checked that

$$\widehat{\int_{2^N}^{EC}} v_{\chi_A} d\gamma = \sum_{B \in 2^A} \mu^\gamma(B) = \gamma(A).$$

On the other hand, the $\overline{\text{EC}}$ integral of v w.r.t. γ is

$$\overline{\int_{2^N}^{EC}} v d\gamma = \sum_{A \in 2^N} \mu^\gamma(A) \bigwedge_{i \in A} \Big(v\big((N \backslash A) \cup i\big) - v(N \backslash A) \Big).$$

Here again, $v\big((N \backslash A) \cup i\big) - v(N \backslash A) = f_v(i)$ for all $A \in 2^N$ and all $i \in A$ whenever v is additive (through f_v), in which case this latter expression reduces to the same well known representation of the Choquet integral (in terms of the Möbius inversion of the capacity) as above. Also note that

$$\bigwedge_{i \in B} \Big(v_{\chi_A}\big((N \backslash B) \cup i\big) - v_{\chi_A}(N \backslash B) \Big) = 1$$

if $B \subseteq A$ and 0 otherwise, with $A, B \in 2^N$. Therefore, for any integrand $v_{\chi_A}, A \in 2^N$, only the values $\mu^\gamma(B)$ such that $B \subseteq A$ are summed. Hence,

$$\overline{\int_{2^N}^{EC}} v_{\chi_A} d\gamma = \sum_{B \subseteq A} \mu^\gamma(B) = \gamma(A).$$

3.5 An example

In order to appreciate the difference between these four extensions of the Choquet integral (i.e., each coincides with this latter whenever the integrand is additive), consider the simple case where $N = \{1, 2, 3\}$ and the integrand v is symmetric; in particular, $v(A) = \left(\frac{|A|}{3}\right)^2$ for all $A \subseteq N$. (Note that this is a capacity itself.) Let γ denote the (fuzzy) measure w.r.t. which integration is performed, with

$$\gamma(\{1\}) = 0.1 \; ; \; \gamma(\{2\}) = 0.4 \; ; \; \gamma(\{3\}) = 0.6$$

$$\gamma(\{1,2\}) = 0.6 \; ; \; \gamma(\{1,3\}) = 0.8 \; ; \; \gamma(\{2,3\}) = 0.9.$$

Accordingly, $\mu^\gamma(A) = \gamma(A)$ if $|A| = 1$, and

$$\mu^\gamma(\{1,2\}) = 0.6 - 0.1 - 0.4 = 0.1$$

$$\mu^\gamma(\{1,3\}) = 0.8 - 0.1 - 0.6 = 0.1$$

$$\mu^\gamma(\{2,3\}) = 0.9 - 0.4 - 0.6 = -0.1$$

$$\mu^\gamma(\{1,2,3\}) = v(\{1,2,3\}) - \sum_{A \subset \{1,2,3\}} \mu^\gamma(A) =$$

$$= 1 - (0.1 + 0.4 + 0.6 + 0.1 + 0.1 - 0.1) = -0.2.$$

Simple computations yield

$$\int_{2^N}^{EC} v d\gamma = \frac{24}{90},$$

where the (admissible) maximal chain along which integration is performed is $\{\emptyset, \{3\}, \{2,3\}, N\}$. Also,

$$\widetilde{\int_{2^N}^{EC}} v d\gamma = \frac{1}{3!} \cdot \frac{40 + 36 + 38 + 28 + 30 + 24}{90} = \frac{98}{3 \cdot 90},$$

$$\text{while } \widehat{\int_{2^N}^{EC}} v d\gamma = \frac{4}{90} \text{ and } \overline{\int_{2^N}^{EC}} v d\gamma = \frac{56}{90}.$$

4 Applications

This final section is devoted to possible applications of the EC integral (or any variation $\widetilde{\mathrm{EC}}$, $\widehat{\mathrm{EC}}$ or $\overline{\mathrm{EC}}$). More generally, the focus turns on conceivable situations where integrating (i.e., aggregating) set functions w.r.t. capacities might be useful. In particular, two such applications are provided below. In both cases, the aggregation operator yields an expectation. In the former case the integrand is monotone, while in the latter it is antitone.

4.1 An expectation of diversity

The idea of measuring diversity within any subset $A \subseteq N$ of a finite population $N = \{1, \ldots, n\}$ has been recently formalized through *diversity functions* $v : 2^N \rightarrow \mathbb{R}_+$ which, by construction, are monotone (see [9]). This approach relies upon the multi-attribute model of diversity: any attribute characterizing any member of the population is identified with the set of members characterized by that attribute and, conversely, any subset A of the population identifies a conceivable attribute, i.e., belonging to A. Accordingly, the set of conceivable attributes is 2^N. For each attribute $A \in 2^N$ there is a weight $\lambda_A \geq 0$. In fact, $v : 2^N \rightarrow \mathbb{R}_+$ is defined to be a diversity function if there is a $\lambda : 2^N \rightarrow \mathbb{R}_+$ such that

$$v(B) = \sum_{\substack{A \in 2^N \\ B \cap A \neq \emptyset}} \lambda_A \text{ for all } B \in 2^N,$$

with $v(\emptyset) = \lambda_\emptyset = 0$. Furthermore, if this is the case, then λ is the *conjugate Möbius inverse* of v, given by

$$\lambda_A = \sum_{B \subseteq A} (-1)^{|A \backslash B|+1} \cdot v(N \backslash B) \text{ for all } A \in 2^N.$$

In turn, v has a positive conjugate Möbius inversion if and only if it is monotone and *totally submodular*, that is, for all collections $\{A_1, \ldots, A_k\} \subseteq 2^N$

$$v\Big(\bigcap_{1 \leq k' \leq k} A_{k'}\Big) \leq \sum_{\emptyset \subset L \subseteq \{1,\ldots,k\}} (-1)^{|L|+1} \cdot v\Big(\bigcup_{l \in L} A_l\Big),$$

(see [9] and [2]).

Hence, a diversity function v is a perfect candidate as a monotone integrand of the EC integral above. In particular, consider the case where one is interested in the expectation of diversity within a random sample of population N. For the sake of concreteness, assume, for example, that the population of a national park has to be created (or increased) by moving some random sample of an existing, known population from another park. Similarly, assume one has a given amount of time and resources to devote to survey a known population of animals moving freely within a wide region, and is interested in forming an expectation of the diversity that will be observed. In this case, the capacity γ w.r.t. which integration is performed is to be interpreted as follows: $\gamma(A), A \in 2^N$ quantifies the belief that the random sample will be precisely A. Clearly enough, there is no reason why such a capacity should be additive. Then, $\int_{2^N}^{EC} v d\gamma$ (or $\widetilde{\int_{2^N}^{EC}} v d\gamma$ or $\widehat{\int_{2^N}^{EC}} v d\gamma$ or $\overline{\int_{2^N}^{EC}} v d\gamma$) furnishes an expectation of the diversity in the random sample.

4.2 Certainty equivalent in DUU

Consider again the DUU situation described above, where a DM has to take action in response to an unknown state of Nature $i \in N$. In particular, let $\mathcal{A} = \{a_1, \ldots, a_n\}$ denote the set of available actions, and formalize preferences by $u : N \times \mathcal{A} \rightarrow \mathbb{R}$, with $u(i, a) = u_a(i)$ for all $(i, a) \in N \times \mathcal{A}$. For the sake of concreteness, for $1 \leq i \leq n$, assume $u_a(i) < u_{a_i}(i)$ for all $a \in \mathcal{A} \backslash a_i$. In words, every state $i \in N$ has its own, distinct optimal action $a_i \in \mathcal{A}$. Finally, beliefs are quantified by capacity γ.

The DM chooses some action a^* maximizing her CEU, that is,

$$\int_N^C u_{a^*} d\gamma = \bigvee_{a \in \mathcal{A}} \int_N^C u_a d\gamma,$$

where $\int_N^C u_a d\gamma$ is the Choquet integral of u_a w.r.t. γ (see section 2 above). In fact, this is the certainty equivalent of the decision making problem. More precisely, assume that u is in money terms. Accordingly, if $\int_N^C u_{a^*} d\gamma > 0$, then the DM is willing to pay, at most, such a quantity in order to face this problem. Conversely, if $\int_N^C u_{a^*} d\gamma < 0$, then the DM is willing to pay, at most, such a quantity in order to avoid this problem. Finally, if $\int_N^C u_{a^*} d\gamma = 0$, then the DM is indifferent between facing this problem or not. In particular, as u is real-valued, the Choquet integral may be computed either symmetrically, or else asymmetrically (see subsection 3.3 above and [7]).

This framework can be enriched by introducing the idea of information, which may be formalized in alternative ways and raises the issue of updating non-additive beliefs (see [4]). Information may be modeled as a *field* of events and, in particular, as a field of events generated by some *partition* of N (see [8]). Yet, the focus here is placed on the situation where the DM knows, before taking action, that the 'true' state of Nature belongs to some subset $\emptyset \neq B \in 2^N$. Then, what action will be taken? One possibility is to choose an action that maximizes the conditional expectation defined by the Choquet integral (over B) of the restricted utility function $u : B \times \mathcal{A} \to \mathbb{R}$ w.r.t. the normalized restricted capacity $\gamma : 2^B \to [0,1]$. That is, the DM may choose an action a_B^* satisfying

$$\int_B^C u_{a_B^*} d\gamma_B = \bigvee_{a\in\mathcal{A}} \int_B^C u_a d\gamma_B,$$

where $\int_B^C u_a d\gamma_B =$

$$\sum_{1\leq i\leq |B|} \left[u_a((i)) - u_a((i-1))\right] \cdot \frac{\gamma(\{(i),\ldots,(|B|)\})}{\gamma(B)}$$

and $(\cdot) : B \to B$ is any permutation such that $u_a((|B|)) \geq \cdots \geq u_a((i)) \geq \cdots \geq u_a((1)) \geq 0$, with $u_a((0)) := 0$ and where the integrand is assumed to take only positive values for the sake of simplicity (and reasons of space). With a real-valued integrand, this conditioned Choquet integral may be computed either symmetrically or else asymmetrically (again, see subsection 3.3 above and [7]). Now, for $\emptyset \neq B \in 2^N$, define $v : 2^N \backslash \{\emptyset\} \to \mathbb{R}_+$ by

$$v(B) := \bigvee_{a\in\mathcal{A}} \int_B^C u_a d\gamma_B.$$

It seems rather evident that if $\emptyset \subset A \subseteq B \in 2^N$, then $v(A) \geq v(B)$. That is, v is antitone on $2^N \backslash \{\emptyset\}$. In fact, roughly speaking, the smaller the (non-void) subset, the better the DM can choose a corresponding optimal action. In particular, $v(\{i\}) = u_{a_i}(i), i \in N$.

Finally, consider the issue of quantifying some belief about what information (in the form of an event) will be available before taking action. In other terms, focus on how to define some capacity $\eta : 2^N \to [0,1]$ such that $\eta(A)$ quantifies the belief that before taking action the DM *will know* that the 'true' state belongs to A, with $\emptyset \neq A \in 2^N$. In fact, $\gamma(A)$ already quantifies the belief that the 'true' state belongs to A. Nevertheless, $\eta(A)$ must quantify the belief that, *in addition*, this will be known before taking action. Accordingly, it must be $\eta(A) \leq \gamma(A)$. For example, one may set $\eta(A) = (\gamma(A))^2$.

Then, the EC integral of v w.r.t. η as defined in subsection 3.2, i.e., $\int_{2^N}^{EC} v d\eta$, furnishes an expectation of the utility that the DM may receive by facing this DUU problem with (non-additive) beliefs about what information (in the form of an event) will be available before taking action. Accordingly, $\bigvee_{a\in\mathcal{A}} \int_N^C u_a d\gamma$ may be interpreted as the certainty equivalent of this DUU problem when the DM believes that no information will be available before taking action. That is, the certainty equivalent with trivial $\eta_{\mathcal{T}}$ such that $\eta_{\mathcal{T}}(A) = 0$ for all $A \subset N$ and $\eta_{\mathcal{T}}(N) = \gamma(N) = 1$, as

$$\int_{2^N}^{EC} v d\eta_{\mathcal{T}} = v(N) = \bigvee_{a\in\mathcal{A}} \int_N^C u_a d\gamma.$$

Conversely, $\int_{2^N}^{EC} v d\eta$ seems a suitable candidate as the certainty equivalent of this DUU problem with non-trivial beliefs $\eta \neq \eta_{\mathcal{T}}$ about what information (i.e., event) will be available before taking action.

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