M-probability theory on IF-events

Beloslav Riečan
Faculty of Natural Sciences, Matej Bel University
Department of Mathematics
Tajovského 40
974 01 Banská Bystrica, Slovakia
riecan@fpv.umb.sk
and
Mathematical Institute of Slovak Acad. of Sciences
Štefánikova 49
SK-81473 Bratislava

Abstract

Following M. Krachounov ([5]), max and min operations with fuzzy sets are considered instead of Lukasiewicz ones ([6], [7], [8], [9]). Her the domain $\mathcal{F}$ of a probability $m : \mathcal{F} \to [0,1]$ consists on IF-events $A = (\mu_A, \nu_A)$ i.e. mappings from a measurable space $(\Omega, \mathcal{S})$ to the unit square ([1]). Local representation of sequences of M-observables by random variables is constructed and three kinds of convergences are characterized: convergence in distribution, convergence in measure, and almost everywhere convergence.

Keywords: IF-events, M-states, M-observables.

1 Introduction

Although the notion of an IF set is given uniquelly, operations on them present a large variety. In this paper we shall consider max and min operations. First let us repeat the basic definitions.

Let $(\Omega, \mathcal{S})$ be a measurable space. By an IF-event we mean any pair

$$A = (\mu_A, \nu_A)$$

of $\mathcal{S}$-measurable functions, such that $\mu_A \geq 0, \nu_A \geq 0$, and

$$\mu_A + \nu_A \leq 1.$$

An important notion is the ordering

$$A \leq B \Longleftrightarrow \mu_A \leq \mu_B, \nu_A \geq \nu_B.$$

We shall use the following connectives for $a, b \in R$:

$$a \vee b = \max(a, b),$$

$$a \wedge b = \min(a, b).$$

Hence consider a measurable space $(\Omega, \mathcal{S})$, where $\mathcal{S}$ is a σ-algebra of subsets of Ω, $\mathcal{F} = \{A = (\mu_A, \nu_A); \mu_A, \nu_A$ are non-negative, $\mathcal{S}$-measurable functions, $\mu_A + \nu_A \leq 1$ $\}$. According to [4] we shall define the probability on $\mathcal{F}$ using max-min connectives instead of the Lukasiewicz connectives,

$$A \vee B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B),$$

$$A \wedge B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B).$$

For distinguishing the two theories (max and min operations instead of Lukasiewicz operations) we shall speak about M-probability.

For the first time the probability of an IF-event $A = (\mu_A, \nu_A)$ was defined in [4] as the interval

$$\mathcal{P}(A) = [\int_\Omega \mu_A dP, 1 - \int_\Omega \nu_A dP].$$

As we have shown in [8] it is a special case of a general definition given axiomatically. Of course, also in the axiomatic approach, the probability $A \longmapsto \mathcal{P}(A) = [\mathcal{P}^\flat(A), \mathcal{P}^\sharp(A)]$ can be reduced to two states $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{F} \to [0,1]$. On the other hand the study of states on IF-events cannot be reduced to two coordinate mappings on fuzzy sets (see Proposition 2.3 and Example 2.5).

Of course, from the classical point of view the probability theory has two basic notions: probability, and random variable, in the non-commutative case state and observable. This approach in the IF -probability case is very fruitful. The space of IF-events can be embedded in a suitable MV-algebra, hence all results of the MV-algebra probability theory ([11], [12]) can be applied to the IF-case ([6], [7], [8], [9], [10]).

On the other hand the Krachounov proposal [5] need probably another approach. Generally probability can be defined for any t-norm and t-conorm ([2]). Of course, in this moment we are able to prove the existence of the joint observable only in the case of Lukasiewicz connectives and the Zadeh max - min connectives ([10]). The M-probability theory on IF-events cannot be reduced to the corresponding theory on fuzzy sets. On the other hand the obtained results working for IF-events are new also for M-probabilities on fuzzy events.

The existence of the joint observable is the key to the basic assertions of the probability theory. Our paper anobles to translate some known convergence theorems of the Kolmogorov theory to the M-probability case. We present as the main result the translation formulas (Theorem 3.3). ¿From the convergence of suitable sequence of random variables the corresponding convergence of a sequence of M-observables follows. In Section 2 some notions and preliminary useful results are presented.

2 States and observables

Definition 2.1 *A mapping $p : \mathcal{F} \to [0.1]$ is called M-state if the following properties are satisfied:*

(i) $m((1_\Omega, 0_\Omega)) = 1, m((0_\Omega, 1_\Omega)) = 0;$

(ii) $m(A) + m(B) = m(A \vee B) + m(A \wedge B)$ *for any* $A, B \in \mathcal{F}$*;*

(iii) $A_n \nearrow A, B_n \searrow B \Longrightarrow m(A_n) \nearrow m(A), m(B_n) \searrow m(B)$.

Definition 2.2 *A mapping $m : \mathcal{F} \to [0, 1]$ is an IF-state, if the following properties are satisfied:*

(i) $m((1_\Omega, 0_\Omega)) = 1, m((0_\Omega, 1_\Omega)) = 0;$

(ii) $m(A) + m(B) = m(A \oplus B) + m(A \odot B))$ *for any* $A, B \in \mathcal{F}$*;*

(iii) $A_n \nearrow A, B_n \searrow B \Longrightarrow m(A_n) \nearrow m(A), m(B_n) \searrow m(B)$.

Here

$$A \oplus B = (\mu_A \oplus \mu_B, \nu_A \odot \nu_B)$$

$$A \odot B = (\mu_A \odot \mu_B, \nu_A \oplus \nu_B)$$

$$f \oplus g = \min(f + g, 1),$$

$$f \odot g = \max(f + g - 1, 0)$$

Proposition 2.3 *A mapping $m : \mathcal{F} \to [0, 1]$ is an IF-state if and only if there exist a probability $P : \mathcal{S} \to [0, 1]$ and $\alpha \in [0, 1]$ such that*

$$m(A) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP) \qquad (1)$$

for any $A = (\mu_A, \nu_A) \in \mathcal{F}$.

Proof. [8]

Proposition 2.4 *Any IF - state is an M-state.*

Proof. It follows by Prop.2.3.

Example 2.5 *Let $\mathcal{T}$ be the tribe of all $\mathcal{S}$-measurable functions $f : \Omega \to [0, 1]$. Let m be an IF-state on $\mathcal{F}, m : \mathcal{F} \to [0, 1]$. Evidently $(f, 1 - f) \in \mathcal{F}$, hence we can define the mapping $\overline{m} : \mathcal{T} \to [0, 1]$ by the formula*

$$\overline{m}(f) = m((f, 1 - f)).$$

Evidently $\overline{m}$ is a state on $\mathcal{T}$, hence by the Butnariu - Klement theorem ([2]) there exists a probability $P : \mathcal{S} \to [0, 1]$ such that

$$\overline{m}(f) = \int_\Omega f dP, \qquad (2)$$

for any $f \in \mathcal{T}$. Of course, the formula (2) does not imply (1), we see that the IF-approach cannot be coordinatwisely reduced to the fuzzy approach.

Example 2.6 *Fix $x_0 \in \Omega$ and put*

$$m(A) = \frac{1}{2}(\mu_A^2(x_0) + 1 - \nu_A^2(x_0)).$$

Since $(\mu_A \vee \mu_B)^2 + (\mu_A \wedge \mu_B)^2 = \mu_A^2 + \mu_B^2$, it is not difficult to see that m is an M-state. Put

$$\mu_A(x) = \mu_B(x) = \frac{1}{4}, \nu_A(x) = \nu_B(x) = \frac{1}{2}$$

for any $x \in \Omega$. Then

$$m(A) = m(B) = \frac{13}{32}.$$

On the other hand

$$A \oplus B = ((\frac{1}{2})_\Omega, 0_\Omega), A \odot B = (0_\Omega, 1_\Omega),$$

hence

$$m(A \oplus B) + m(A \odot B) = \frac{5}{8} + 0 \neq \frac{13}{32} + \frac{13}{32} = m(A) + m(B).$$

Although the probability theory on IF-events studied in [6, 7, 8, 9, 10] seems to be satisfactory, the previous facts lead us to an experience to create basic instruments for an alternative M-probability theory. Of course, the crucial notion is the notion of an M-observable.

Definition 2.7 *An M-observable is a mapping $x : \mathcal{B}(R) \to \mathcal{F}$ satisfying the following conditions:*

(i) $x(R) = (1_\Omega, 0_\Omega), x(\emptyset) = (0_\Omega, 1_\Omega);$

(ii) $x(A \cup B) = x(A) \vee x(B), x(A \cap B) = x(A) \wedge x(B)$ *for any* $A, B \in \mathcal{B}(R)$*;*

(iii) $A_n \nearrow A, B_n \searrow B \Longrightarrow x(A_n) \nearrow x(A), x(B_n) \searrow x(B)$.

Proposition 2.8 *If $x : \mathcal{B}(R) \to \mathcal{F}$ is an M-observable, and $m : \mathcal{F} \to [0,1]$ is an M-state, then $m \circ x : \mathcal{B}(R) \to [0,1]$ is a probability measure.*

Proof. Evidently $m(x(R)) = m(1_\Omega) = 1$. Also continuity of $m \circ x$ is clear. Let $A \cap B = \emptyset$. Then $x(A) \wedge x(B) = x(\emptyset) = (0_\Omega, 1_\Omega)$. Therefore

$$\begin{aligned} m(x(A \cup B)) &= m(x(A) \vee x(B)) + m(x(A) \wedge x(B)) = \\ &= m(x(A)) + m(x(B)). \end{aligned}$$

Definition 2.9 *Let $x, y : \mathcal{B}(R) \to \mathcal{F}$ be M-observables. The joint M-observable of x and y is a mapping $h : \mathcal{B}(R^2) \to \mathcal{F}$ satisfying the following conditions:*

(i) $h(R^2) = (1_\Omega, 0_\Omega), h(\emptyset) = (0_\Omega, 1_\Omega)$;

(ii) $h(A \cup B) = h(A) \vee h(B), h(A \cap B) = h(A) \wedge h(B)$ *for any* $A, B \in \mathcal{B}(R^2)$;

(iii) $A_n \nearrow A, B_n \searrow B \Longrightarrow h(A_n \nearrow h(A), h(B_n) \searrow h(B)$;

(iv) $h(C \times D) = \min(x(C), y(D))$ *for any* $C, D \in \mathcal{B}(R)$.

Theorem 2.10 *For any M-observables there exists their joint M observable.*

Proof. [10] Theorem 2.2.2.

3 Convergence of M-observables

The aim of the paper is a characterization of sequences on M-observables by the convergence of random variables.

Definition 3.1 *Let $y_1, y_2, ...$ be a sequence of M-observables, $y_n : \mathcal{B}(R) \to \mathcal{F}, p : \mathcal{F} \to [0,1]$ be an M-state.*

(i) *The sequence is said to converge in distribution to a function $F : R \to [0,1]$ if for each $t \in R$*

$$\lim_{n\to\infty} p(y_n((-\infty, t))) = F(t).$$

(ii) *The sequence is said to converge in measure to 0 if for each $\varepsilon > 0$*

$$\lim_{n\to\infty} p(y_n((-\varepsilon, \varepsilon))) = 1.$$

(iii) *The sequence converges to 0 almost everywhere, if*

$$\lim_{p\to\infty} \lim_{k\to\infty} \lim_{i\to\infty} p(\bigvee_{n=k}^{k+i} y_n((-\frac{1}{p}, \frac{1}{p}))) = 1.$$

Definition 3.2 *Let $x_1, ..., x_n : \mathcal{B}(R) \to \mathcal{F}$ be M-observables, $h_n : \mathcal{B}(R^n) \to \mathcal{F}$ be their joint M-observable, $g_n : R^n \to R$ be a Borel function. Then we define $g_n(x_1, ..., x_n)$ by the formula*

$$g_n(x_1, ..., x_n) = h_n \circ g_n^{-1}.$$

It is easy to see that the mapping $y_n : \mathcal{B}(R) \to \mathcal{F}$ is an M-observable.

Theorem 3.3 *Let (x_n) be a sequence of M-observables, $x_n : \mathcal{B}(R) \to \mathcal{F}, h_n : \mathcal{B}(R^n) \to \mathcal{F}$ their joint M-observables $(n = 1, 2, ...), g_n : R^n \to R$ Borel functions, $y_n = g_n(x_1, ..., x_n), n = 1, 2, ...$ Then there exists a probability space $(X, \mathcal{S}, P)$ and a sequence (ξ_n) of random variables, $\xi_n : X \to R$ such that if $\eta_n = g_n(\xi_1, ..., \xi_n), n = 1, 2, ...)$, then*

(i) *the sequence $y_1, y_2, ...$ converges in distribution to a function F if and only if so does the sequence $\eta_1, \eta_2, ...$;*

(ii) *$y_1, y_2, ...$ converges to 0 in measure p if and only if $\eta_1, \eta_2, ...$ converges to 0 in measure P;*

(iii) *if $\eta_1, \eta_2, ...$ converges P-almost everywhere to 0, then $y_1, y_2, ...$ converges p-almost evrywhere to 0.*

Proof. Put $X = R^N.\mathcal{S} = \sigma(\mathcal{C})$, where $\mathcal{C}$ is the family of all cylinders in R^N. Put $p_n = p \circ h_n$. Then $\{p_n; n \in N\}$ form a consistent family of probability measures $p_n : \mathcal{B}(R^n) \to [0,1]$, i.e.

$$p_{n+1}(A \times R) = p_n(A), A \in \mathcal{B}(R^n), n = 1, 2, ...$$

By the Kolmogorov theorem there exists exactly one probability measure $P : \sigma(\mathcal{C}) \to [0,1]$ such that

$$P \circ \pi_n^{-1} = p_n, n = 1, 2, ...$$

where $\pi_n : R^N \to R^n$ is the projection. Put

$$\xi_n : R^N \to R, \xi_n((u_i)_{i=1}^\infty)) = u_n, n = 1, 2, ...$$

Then

$$P(\eta_n^{-1}(A)) = P((g_n(\xi_1, ..., \xi_n))^{-1}(A)) =$$

$$P(\pi_n^{-1}(g_n^{-1}(A))) = p(h_n(g_n^{-1}(A))) = p(y_n(A))$$

Therefore

$$p(y_n(-\infty, t)) = P(\eta_n^{-1}(-\infty, t)),$$

$$p(y_n((-\varepsilon,\varepsilon))) = P(\eta_n^{-1}((-\varepsilon,\varepsilon))),$$

what implies (i) and (ii). Let now η_n converges to 0 P-almost everywhere. We have

$$P(\bigcap_{n=k}^{k+i} \eta_n^{-1}((-\frac{1}{p},\frac{1}{p}))) =$$

$$p(h_{k+i}(\bigcap_{n=k}^{k+i}\{(t_1,.,,,t_{k+i}) : g_n(t_1,...,t_n) \in (-\frac{1}{p},\frac{1}{p})\})) \leq$$

$$\leq p(\bigwedge_{n=k}^{k+i} h_{k+i}(\{(t_1,...,t_{k+i}) : (t_1,...,t_n) \in g_n^{-1}((-\frac{1}{p},\frac{1}{p}))\})) =$$

$$= p(\bigwedge_{n=k}^{k+i} h_n \circ g_n^{-1}((-\frac{1}{p},\frac{1}{p}))) =$$

$$= p(\bigwedge_{n=1}^{k+i} y_n((-\frac{1}{p},\frac{1}{p}))).$$

Therefore

$$1 \leq \lim_{p\to\infty}\lim_{k\to\infty}\lim_{i\to\infty} p(\bigwedge_{n=k}^{k+i} y_n((-\frac{1}{p},\frac{1}{p}))) \leq 1,$$

hence $(y_n)_{n=1}^{\infty}$ converges to 0 p-almost everywhere.

Acknowledgement

The paper was supported by Grant VEGA 1/2002/05.

References

[1] K. Atanassov: Intuitionistic Fuzzy Sets: Theory and Applications, Physica Verlag, New York 1999.

[2] D. Butnariu, E. P. Klement: Triangular norm-based measures and their markov kernel representation, J. Math. Anal. Appl. 162, 1991, 11 - 143.

[3] R. L. O. Cignoli, I. M. L. D'Ottaviano, D. Mundici: Algebraic Foundations of Many-valued Reasoning, Kluwer, Dordrecht 2000.

[4] P. Grzegorzewski, E. Mrowka: Probability of intuitionistic fuzzy events, in: *Soft Methods in Probability, Statistics and Data Analysis (P. Grzegorzewski et al. eds.).* Physica Verlag, New York 2002, 105 - 115.

[5] M. Krachounov: Intuitionistic probability and intuitionistic fuzzy sets, in: *First Intern. Workshop on IFS, Generalized Nets and Knowledge Engineering (E. El-Darzi, K. Atanassov, P. Chountas eds.).* Univ. of Westminster, London 2006, 18 - 24. 2006, 714-717.

[6] K. Lendelová: Measure theory on multivalued logics and its applications, PhD thesis, M.Bel University Banská Bystrica 2005.

[7] M. Renčová, B. Riečan: Probability on IF-sets: an elementary approach, in: *First Intern. Workshop on IFS, Generalized Nets and Knowledge Engineering (E. El-Darzi, K. Atanassov, P. Chountas eds.).* Univ. of Westminster, London 2006, 8 - 17.

[8] B. Riečan: On a problem of Radko Mesiar: general from of IF probabilities, Fuzzy Sets and Systems 152, 2006, 1485 - 1490.

[9] B. Riečan: On two concepts of probability on IF-sets, Notes on IFS 12, 2006, 69 - 72.

[10] B. Riečan: Probability theory on intuitionistic fuzzy events, in: *A volume in honour of Daniele Mundici's 60th birthday Lecture Notes in Computer Science*, Springer, Berlin 2007.

[11] B. Riečan, D. Mundici: Probability on MV-algebras, in: *Handbook of Measure Theory (E. Pap ed.).* Elsevier, Amsterdam 2002, 869 -909.

[12] B. Riečan, T. Neubrunn: Integral, Measure, and Ordering, Kluwer, Dordrecht 1997.