MP and MT-implications on a finite scale

M. Mas
Dpt. de Matemàtiques i Inf.
Universitat de les Illes Balears.
07122 Palma de Mallorca. Spain
dmimmg0@uib.es

M. Monserrat
Dpt. de Matemàtiques i Inf.
Universitat de les Illes Balears.
07122 Palma de Mallorca. Spain
dmimma0@uib.es

J. Torrens
Dpt. de Matemàtiques i Inf.
Universitat de les Illes Balears.
07122 Palma de Mallorca. Spain
dmijts0@uib.es

Abstract

This paper is devoted to the study of discrete implications that satisfy modus ponens (MP), modus tollens (MT) or both (MPT). The main goal is to characterize all R, S, QL and D-implications on a finite chain L satisfying these properties for a given smooth t-norm T_1. The non-smooth case is also discussed for a special family of t-norms.

Keywords: Discrete implications, modus ponens, modus tollens, smooth t-norms, finite scale.

1 Introduction

It is well known that fuzzy implication functions (see the survey [12]) are used in approximate reasoning, not only to represent fuzzy conditional statements of the form "If p then q" (with p, q fuzzy statements), but also to perform inferences in any fuzzy rule based system. In this inference process, the two main classical rules are *modus ponens* (MP) and *modus tollens* (MT) that allow to perform, respectively, forward and backward inferences. In terms of fuzzy logic, these implications are operators $I : [0,1]^2 \to [0,1]$ extending the classical material implication, that is, satisfying $I(0,0) = I(0,1) = I(1,1) = 1$ and $I(1,0) = 0$.

Since conjunctions, disjunctions and negations are usually performed by t-norms (T), t-conorms (S) and strong negations (N), in fuzzy set theory as much as in fuzzy logic and approximate reasoning, the majority of the known implication functions are directly derived from these operators[1]. The four most usual ways to define these implication functions are:

i) *R-implications* defined by

$$I(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\} \qquad (1)$$

for all $x, y \in [0,1]$.

ii) *S-implications* defined by

$$I(x,y) = S(N(x), y), \qquad x, y \in [0,1]. \qquad (2)$$

iii) *QL-implications* defined by

$$I(x,y) = S(N(x), T(x,y)), \qquad x, y \in [0,1]. \qquad (3)$$

iv) *D-implications*, that are the contraposition with respect to N of QL-implications and are given by

$$I(x,y) = S(T(N(x), N(y)), y), \qquad x, y \in [0,1]. \qquad (4)$$

Moreover, in this context, given any t-norm T_1 and any strong negation N_1, modus ponens and modus tollens for an implication I can be written as

$$T_1(x, I(x,y)) \leq y, \qquad x, y \in [0,1] \qquad (5)$$

and

$$T_1(N_1(y), I(x,y)) \leq N_1(x), \qquad x, y \in [0,1] \qquad (6)$$

respectively. These two equations have been recently solved in [15] for the first three mentioned classes of implications.

On the other hand, the study of operators defined on finite scales is an area of increasing interest (see [2], [4], [5], [8], [9], [10], and [13]). Mainly, because it allows to deal with finite families of linguistic labels avoiding numerical interpretations (necessaries in the fuzzy logic approach). In this context, the book chapter [13] brings a survey on (smooth) discrete t-norms and t-conorms on a finite chain L, and the four classes of implications, R, S, QL and D-implications derived from discrete t-norms, are studied in [9] and

[1]Although some authors have derived also implications from other aggregation functions, specially uninorms (see [1], [11],[14]).

[10]. From these discrete implications, new possibilities for approximate reasoning with finite families of linguistic labels appear with their consequent applications in computing with words.

However, in this line, the study of MP and MT rules for discrete implications, equivalent to the one given in [15] for fuzzy implications, is essential. This is precisely the main goal of this paper. After some preliminaries given in section 2, we devote section 3 to characterize those R, S, QL and D-implications on a finite chain L, derived from smooth t-norms, that satisfy equation (5), equation (6) or both. Finally, section 4 is devoted to the non-smooth case for a special family of t-norms.

2 Preliminaries

We recall here the smooth t-norms and the smooth t-conorms on a finite chain L and their characterization, that will be used along the paper. It is well known that for our purposes (see [13]) all finite chains with the same number of elements are equivalent and then, from now on, we will deal with the simplest finite chain of $n+1$ elements:

$$L = \{0, 1, 2 \ldots, n-1, n\}$$

where $n \geq 1$. Such an L can be understood as a set of linguistic terms or "labels".

The following two definitions are adapted from [5] (see also [13]).

Definition 1 *A function $f : L \to L$ is said to be smooth if it satisfies one of the following conditions:*

- *f is nondecreasing and $f(x) - f(x-1) \leq 1$ for all $x \in L$ with $x \geq 1$.*
- *f is nonincreasing and $f(x-1) - f(x) \leq 1$ for all $x \in L$ with $x \geq 1$.*

Definition 2 *A binary operator F on L is said to be smooth if it is smooth in each variable.*

The importance of the smoothness condition lies in the fact that it is generally used as a discrete counterpart of continuity on [0,1].

Although t-norms, t-conorms and strong negations are usually binary operators on [0,1], they can be defined as in [2] on any bounded partially ordered set and, in particular, on L. In this last case, they are usually known as discrete t-norms and discrete t-conorms. In this way, recall that smoothness for discrete t-norms (and also for t-conorms) is equivalent to the *divisibility* condition, that is, $x \leq y$ if and only if there exists $z \in L$ such that $T(y, z) = x$, (see [13]).

Proposition 1 *There is one and only one strong negation on L that is given by*

$$N(x) = n - x \quad \textit{for all } x \in L. \tag{7}$$

From now on, N will always denote the negation on L given by (7). Smooth t-norms have been characterized as ordinal sums of Archimedean ones as follows.

Proposition 2 (See [13]). *There is one and only one Archimedean smooth t-norm on L, denoted by $T_{\mathbf{L}}$, given by*

$$T_{\mathbf{L}}(x, y) = \max\{0, x + y - n\} \quad \textit{for all } x, y \in L \tag{8}$$

which is known as the Łukasiewicz t-norm. Moreover, given any subset J of L containing $0, n$, there is one and only one smooth t-norm on L that has J as the set of idempotent elements, that will be denoted by T_J. In fact, if J is the set

$$J = \{0 = i_0 < i_1 < \ldots < i_{m-1} < i_m = n\}$$

then T_J is given by:

$$T_J(x, y) = \begin{cases} \max\{i_k, x + y - i_{k+1}\} & \textit{if } x, y \in [i_k, i_{k+1}] \\ & \textit{for some } i_k \in J \\ \min\{x, y\} & \textit{otherwise} \end{cases} \tag{9}$$

Smooth t-conorms have a classification theorem like the above one for t-norms which can be easily deduced by N-duality. The expression of the only Archimedean smooth t-conorm on L is given by

$$S_{\mathbf{L}}(x, y) = \min\{n, x + y\} \quad \text{for all } x, y \in L \tag{10}$$

which is also known as the Łukasiewicz t-conorm. In general, we have

Proposition 3 (See [13]). *Given any subset J of L containing $0, n$, there is one and only one smooth t-conorm on L, S_J, that has J as the set of idempotent elements. In fact, if J is the set*

$$J = \{0 = i_0 < i_1 < \ldots < i_{m-1} < i_m = n\}$$

then S_J is given by:

$$S_J(x, y) = \begin{cases} \min\{i_{k+1}, x + y - i_k\} & \textit{if } x, y \in [i_k, i_{k+1}] \\ & \textit{for some } i_k \in J \\ \max\{x, y\} & \textit{otherwise} \end{cases} \tag{11}$$

Note that with these notations, the Łukasiewicz t-norm and t-conorm can be written, respectively as $T_{\mathbf{L}} = T_{\{0,n\}}$ and $S_{\mathbf{L}} = S_{\{0,n\}}$. The following result follows from the previous propositions.

Proposition 4 (See [13]). *There are exactly 2^{n-1} different smooth t-norms (t-conorms) on L.*

Definition 3 *A binary operator $I : L \times L \to L$ is said to be a (discrete) implication if it satisfies:*

(I1) I is nonincreasing in the first variable and nondecreasing in the second one.

(I2) $I(0,0) = I(n,n) = n$ and $I(n,0) = 0$.

Note that, from the definition, it follows that $I(0,x) = n$ and $I(x,n) = n$ for all $x \in L$.

The four ways to define fuzzy implications apply here to define discrete implications. However, since the only strong negation on L is the one given by (7) and we deal with a finite scale, in our case they can be rewritten as follows:

$$I(x,y) = \max\{z \in L \mid T(x,z) \leq y\}, \quad x,y \in L. \quad (12)$$

$$I(x,y) = S(n-x,y), \qquad x,y \in L. \quad (13)$$

$$I(x,y) = S(n-x,T(x,y)), \qquad x,y \in L. \quad (14)$$

$$I(x,y) = S(T(n-x,n-y),y), \qquad x,y \in L. \quad (15)$$

All these classes of discrete implications have been already studied: R and S-implications in [9], and QL and D-implications in [10]. Thus we refer to these cited papers for details on these kinds of discrete implications that we will use in the paper. Although the non-smooth case is considered in these references, in the present work we will deal only with R, S, QL and D-operators derived from smooth t-norms and smooth t-conorms.

3 Main results

In this section we deal with (implication) operators on the finite chain L that satisfy the modus ponens, the modus tollens or both, with respect to a smooth t-norm T_1. Again, since we have only one strong negation on L, in our case equation (6) can be rewritten depending only on the t-norm T_1 and thus, we can adopt the following definitions:

Definition 4 *Let T_1 be a t-norm on L. A function $I : L^2 \to L$ will be called:*

- *an MP-operator for T_1 whenever it satisfies*

$$T_1(x, I(x,y)) \leq y \quad \text{for all } x,y \in L \qquad (16)$$

- *an MT-operator for T_1 whenever it satisfies*

$$T_1(n-y, I(x,y)) \leq n-x \quad \text{for all } x,y \in L \quad (17)$$

- *an MPT-operator for T_1 whenever it is both, an MP and an MT-operator.*

Moreover, we will say that I is an MP-implication (or MT, or MPT-implication) if it is an implication and also an MP-operator (or MT or MPT-operator, respectively).

Note that whereas all R and S-operators (given by equations (12) and (13), respectively) are always implications, this is not the case for QL and D-operators (given by equations (14) and (15), respectively), see for instance [10]. Thus, we will divide our study of properties MP, MT and MPT in three subsections, the first one devoted to R-implications, the second one devoted to S-implications and, finally, the third one devoted to QL and D-operators in general, including of course QL and D-implications.

Before this, let us begin with two easy but important propositions in the discussion of the mentioned properties. The first one deals with MP-operators.

Proposition 5 *Let T_1 be a t-norm on L and $I : L^2 \to L$ an MP-operator for T_1. Then,*

$$T_1(x, I(x,0)) = 0 \quad \text{for all } x \in L.$$

The second one deals with MT-operators.

Proposition 6 *Let T_1 be a t-norm on L and $I : L^2 \to L$ an MT-operator for T_1. Then,*

$$T_1(n-y, I(n,y)) = 0 \quad \text{for all } y \in L.$$

3.1 R-implications

Given any t-norm T we will denote by I_T its residual implication, that is, the operator given by equation (12):

$$I_T(x,y) = \max\{z \in L \mid T(x,z) \leq y\}.$$

We deal in this subsection with implications $I_T : L^2 \to L$, where T is a smooth t-norm. Let us recall here the structure of these implications. For an easier understanding we give their graphical structure in Figure 1 instead of their formulas, that can be found in [9]. Thus, R and S-implications can be viewed in Figures 1 and 2, respectively.

We begin with the property of MP-implications. Given any t-norm T, let us denote also by $Idemp_T$ the set of all idempotent elements of the t-norm T, that is:

$$Idemp_T = \{x \in L \mid T(x,x) = x\}.$$

Then we have the following proposition.

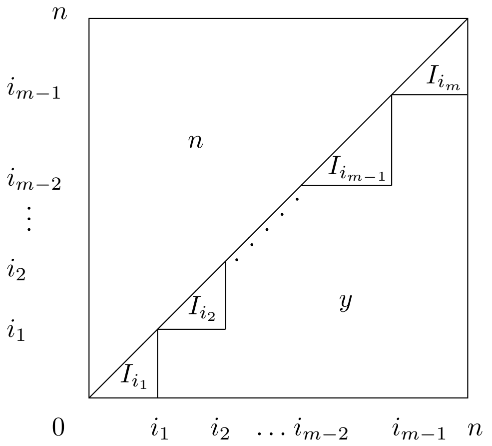

Figure 1. The structure of the R-implication derived from T_J where $J = \{0 = i_0 < i_1 < \ldots < i_{m-1} < i_m = n\}$ and $I_{i_{k+1}}(x, y) = i_{k+1} + y - x$ for $k = 0, ..., m-1$.

Proposition 7 *Let T, T_1 be smooth t-norms on L and $I_T : L^2 \to L$ the R-implication associated to T. The following statements are equivalent:*

i) *I_T is an MP-implication for T_1.*

ii) *$I_T(x, y) \leq I_{T_1}(x, y)$ for all $x, y \in L$.*

iii) *$Idemp_{T_1} \subseteq Idemp_T$.*

Thus, it is clear that we can deduce the following particular cases.

Corollary 1 *Let T, T_1 be smooth t-norms on L and $I_T : L^2 \to L$ the R-implication associated to T. Then,*

i) *If $T = \min$, $I_{\min}$ is an MP-implication for any smooth t-norm T_1.*

ii) *If $T = T_{\mathbf{L}}$, $I_{T_{\mathbf{L}}}$ is an MP-implication for T_1 if and only if $T_1 = T_{\mathbf{L}}$.*

iii) *If $T_1 = T_{\mathbf{L}}$, I_T is an MP-implication for any smooth t-norm T.*

iv) *If $T_1 = \min$, I_T is an MP-implication for $\min$ if and only if $T = \min$.*

With respect to MT-implications we obtain solutions only when T_1 is the Łukasiewicz t-norm and then I_T works for any smooth t-norm T. This can be proved through the following two propositions.

Proposition 8 *Let T, T_1 be t-norms on L with T smooth and let $I_T : L^2 \to L$ be the R-implication associated to T. If I_T is an MT-implication for T_1 then necessarily $T_1(x, n - x) = 0$ for all $x \in L$.*

Remark 1 *It is proved in Lemma 1 of [9] that, for smooth t-norms, the previous condition is equivalent to be T_1 the Lukasiewicz t-norm. That is, for smooth t-norms,*

$$T_1(x, n - x) = 0 \text{ for all } x \in L \iff T_1 = T_{\mathbf{L}}.$$

Proposition 9 *Let T, T_1 be smooth t-norms on L and $I_T : L^2 \to L$ the R-implication associated to T. Then I_T is an MT-implication for T_1 if and only if $T_1 = T_{\mathbf{L}}$.*

Now, jointly the obtained results for MP and for MT-implications we obtain the following corollary.

Corollary 2 *Let T, T_1 be smooth t-norms on L and $I_T : L^2 \to L$ the R-implication associated to T. Then,I_T is an MPT-implication for T_1 if and only if $T_1 = T_{\mathbf{L}}$.*

3.2 S-implications

Let us now deal with S-implications. Given any t-conorm S we will denote by I_S the corresponding S-implication given by equation (13). That is:

$$I_S(x, y) = S(n - x, y) \quad \text{for all } x, y \in L.$$

The structure of S-implications can be viewed in Figure 2 (their expression can be found in [9]).

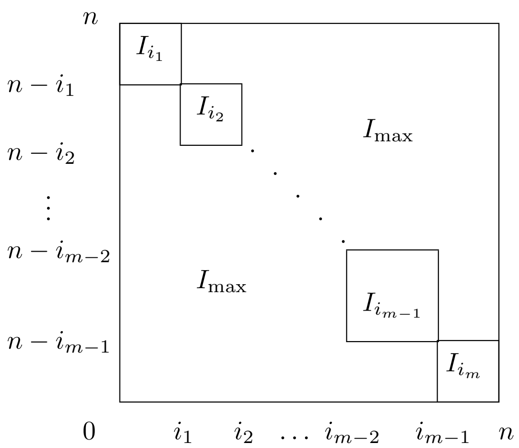

Figure 2. The structure of the S-implication derived from the dual t-conorm of T_J, where $J = \{0 = i_0 < i_1 < \ldots < i_{m-1} < i_m = n\}$, $I_{\max}(x, y) = \max\{n - x, y\}$ and $I_{i_{k+1}}(x, y) = \min\{n - i_k, i_{k+1} + y - x\}$ for $k = 0, ..., m-1$.

For S-implications, the study of the modus ponens is solved by the following two propositions.

Proposition 10 *Let T_1 be a t-norm, S a smooth t-conorm on L and $I_S : L^2 \to L$ the corresponding S-*

implication. If I_S is an MP-implication for T_1 then necessarily $T_1(x, n-x) = 0$ for all $x \in L$.

Proposition 11 *Let T_1 be a smooth t-norm and S a smooth t-conorm on L and $I_S : L^2 \to L$ the corresponding S-implication. Then I_S is an MP-implication for T_1 if and only if $T_1 = T_{\mathbf{L}}$.*

In this case, the study of modus tollens gives exactly the same solutions. Specifically,

Proposition 12 *Let T_1 be a t-norm and S a smooth t-conorm on L and $I_S : L^2 \to L$ the corresponding S-implication. If I_S is an MT-implication for T_1 then necessarily $T_1(x, n-x) = 0$ for all $x \in L$.*

Proposition 13 *Let T_1 be a smooth t-norm and S a smooth t-conorm on L and $I_S : L^2 \to L$ the corresponding S-implication. Then I_S is an MT-implication for T_1 if and only if $T_1 = T_{\mathbf{L}}$.*

And consequently we have:

Corollary 3 *Let T_1 be a smooth t-norm and S a smooth t-conorm on L and $I_S : L^2 \to L$ the corresponding S-implication. Then I_S is an MPT-implication for T_1 if and only if $T_1 = T_{\mathbf{L}}$.*

3.3 QL and D-operators

In this subsection we deal with QL and D-operators given by equations (14) and (15), respectively. As we have commented, not all of them are implications in the sense of Definition 3. In fact, it is proved in [10] that this occurs in the smooth case (for both QL and D) if and only if $S = S_{\mathbf{L}}$. However we will study the properties MP, MT and MPT in general for all QL and D-operators, regardless of they are or they are not implications.

Again we can begin with this first proposition.

Proposition 14 *Let T_1 be a t-norm, I_{QL} a QL-operator and I_D a D-operator. Then:*

i) *If I_{QL} (I_D) is an MP-operator for T_1 then necessarily $T_1(x, n-x) = 0$ for all $x \in L$.*

ii) *If I_{QL} (I_D) is an MT-operator for T_1 then necessarily $T_1(x, n-x) = 0$ for all $x \in L$.*

When we deal with QL and D-implications, since then the t-conorm S in equations (14) and (15) must be $S = S_{\mathbf{L}}$, they are derived simply from a smooth t-norm as:

$$I_{QL}(x, y) = S_{\mathbf{L}}(n-x, T(x, y)) = n - x + T(x, y) \quad (18)$$

for all $x, y \in L$, and

$$I_D(x, y) = S_{\mathbf{L}}(T(n-x, n-y), y) = y + T(n-x, n-y) \quad (19)$$

for all $x, y \in L$, respectively. Moreover, it is proved in [10] that the set of QL-implications and the set of D-implications coincide when we derive them from smooth t-norms and t-conorms, and consequently we can study both kind of implications at the same time. Let us recall the structure of QL and D-implications in Figure 3 (again their formulas can be found in [10]).

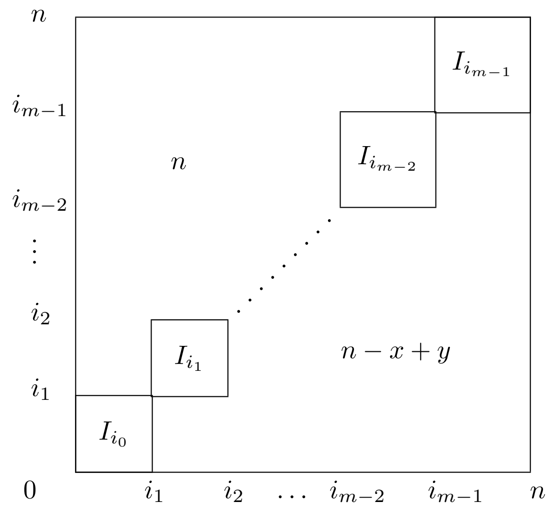

Figure 3. The structure of the QL-implication derived from T_J where $J = \{0 = i_0 < i_1 < \ldots < i_{m-1} < i_m = n\}$. For $j = 0, ..., m-1$, each I_{i_j} is given by $I_{i_j}(x, y) = \max\{n - x + i_j, n + y - i_{j+1}\}$ for all $x, y \in [i_j, i_{j+1}]$. The same structure corresponds to the D-implication derived from $T_{N(J)}$.

In the study of QL and D-implications that are MP, MT and MPT for T_1 we obtain always the same result. Any of these conditions is satisfied if and only if $T_1 = T_{\mathbf{L}}$. For QL-operators and D-operators in general, the result is the same, but now we need to study both kinds of operators separately. In any case, we have:

Proposition 15 *Let T_1 be a smooth t-norm, I_{QL} a QL-operator and I_D a D-operator. Then, the following statements are equivalent:*

i) *I_{QL} (I_D) is an MP-operator for T_1.*

ii) *I_{QL} (I_D) is an MT-operator for T_1.*

iii) *I_{QL} (I_D) is an MPT-operator for T_1.*

iv) *$T_1 = T_{\mathbf{L}}$.*

A table summarizing all results in this section can be viewed in Table 1.

	MP for T_1	MT for T_1	MPT for T_1
I_T	$Idemp_{T_1} \subseteq Idemp_T$	$T_1 = T_{\mathbf{L}}$	$T_1 = T_{\mathbf{L}}$
I_S	$T_1 = T_{\mathbf{L}}$	$T_1 = T_{\mathbf{L}}$	$T_1 = T_{\mathbf{L}}$
I_{QL}	$T_1 = T_{\mathbf{L}}$	$T_1 = T_{\mathbf{L}}$	$T_1 = T_{\mathbf{L}}$
I_D	$T_1 = T_{\mathbf{L}}$	$T_1 = T_{\mathbf{L}}$	$T_1 = T_{\mathbf{L}}$

Table 1. Characterization of R, S, QL and D-operators that are MP, MT and MPT-operators for a smooth t-norm T_1.

4 The non-smooth case

In our previous study we have seen that, given any t-norm T_1 and any binary operator $I : L^2 \to L$, to be I an MP, MT, or MPT-operator, the condition

$$T_1(x, n-x) = 0 \quad \text{for all} \quad x \in L \tag{20}$$

is necessary in almost all cases. From Remark 1 we know that in the smooth case this is equivalent to be T_1 the Łukasiewicz t-norm. However, in the non-smooth case we have many others t-norms on L satisfying such condition. Namely, for any $k \in L$ such that $n-k \leq k$ we have the following indexed family of t-norms T^k given by

$$T^k(x,y) = \begin{cases} 0 & \text{if } x+y \leq n \\ x+y-k & \text{if } x+y > n \text{ and} \\ & \ n-k \leq x,y \leq k \\ \min\{x,y\} & \text{otherwise.} \end{cases} \tag{21}$$

Each t-norm of this family satisfies equation (20), see for instance [9]. In fact, these t-norms T^k are the discrete counterpart of the family of t-norms introduced by J. C. Fodor in [3] (see also [6]) when he studies contrapositive symmetry (genuine property of S-implications) for R-implications. For each t-norm of this family the corresponding R and S-implications coincide (see also [7], Jenei family of t-norms in pages 97–98). This fact is also true for our family T^k in the discrete case (see [9]).

Note that T^k is non-smooth except for the case $k = n$ and in this extreme case T^n agrees with the Łukasiewicz t-norm $T_{\mathbf{L}}$. Note also that the nilpotent minimum (see again [9]) is obtained in the other extreme case given by $n - k = \lfloor n/2 \rfloor$ where $\lfloor n/2 \rfloor$ means the floor of $n/2$, that is, the greatest integer which is smaller than or equal to $n/2$.

Remark 2 *Note that, in the particular case when $n - k = k$, n must be an even number and $k = n/2$. Moreover, in this case both T^k and T^{k+1} coincide with the nilpotent minimum. Thus, from now on, we will consider only the cases when $n - k < k$ without any loose of generality.*

We will consider also $n \geq 3$ since for the cases $n = 1, 2$ it is always $T^k = T_{\mathbf{L}}$.

The indexed family of t-norms T^k can be viewed in Figure 4.

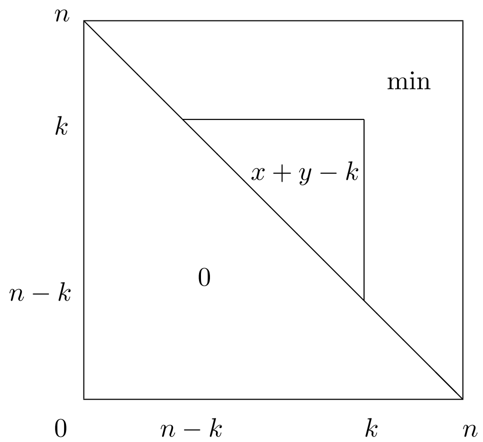

Figure 4. The structure of the t-norm T^k.

Thus, since each T^k satisfies the necessary condition (20), we can study which R, S, QL and D-implications derived from smooth t-norms are MP, MT or MPT-implications for each T^k.

In this line we have the following results with respect to the modus ponens. We begin with R-implications.

Proposition 16 *Let T be a smooth t-norm and I_T its corresponding R-implication. Then, I_T is an MP-implication for T^k if and only if $Idemp_T$ contains the set $[k, n] = \{x \in L \mid k \leq x\}$.*

The case of S-implications is solved in the following proposition.

Proposition 17 *Let S be a smooth t-conorm and I_S its corresponding S-implication. Then, I_S is an MP-implication for T^k if and only if $Idemp_S$ contains the set $[0, n-k]$.*

The case of QL and D-implications is also easy. Since they are derived from $S_{\mathbf{L}}$ and a smooth t-norm T, they are given by equations (18) and (19) respectively.

We will only deal with T^k when $n - k < k < n$ since the case $k = n$ corresponds to the Łukasiewicz t-norm that has been studied in the previous section.

Proposition 18 *Suppose $n - k < k < n$. Let T be a smooth t-norm and I_{QL} and I_D the corresponding QL* *and D-implications derived from T. Then, the following statements are equivalent:*

i) *I_{QL} is an MP-implication for T^k*

ii) *I_D is an MP-implication for T^k*

iii) *T is the Lukasiewicz t-norm $T_{\mathbf{L}}$.*

iv) *$I_{QL} = I_D = I$ is the Kleene-Dienes implication given by*

$$I(x, y) = \max(n - x, y).$$

In table 2 we can view summarized all results in this section concerning MP-implications for the t-norm T^k.

	MP-implication for T^k
R-implication from T	$[k, n] \subseteq Idemp_T$
S-implications from S	$[0, n-k] \subseteq Idemp_S$
QL-implications from T	$T = T_{\mathbf{L}}$
D-implications from T	$T = T_{\mathbf{L}}$

Table 2. Characterization of MP-implications for the t-norm T^k with $n - k < k < n$.

On the other hand, the general case of QL and D-operators is not so easy. That is, when these operators are derived from a smooth t-conorm S different from $S_{\mathbf{L}}$. In this case, we can give only some partial results, as follows:

- When $S = \max$, for any smooth t-norm T, the QL-operator given by

$$I_{QL}(x, y) = \max(n - x, T(x, y))$$

and the D-operator given by

$$I_D(x, y) = \max(T(n - x, n - y), y)$$

are MP-operators for T^k $(n - k \leq k \leq n)$.

- When $S(x, x) = x$ for all $x \leq n-k$, both the QL and the D-operator derived from S and any smooth t-norm T are MP-operators for T^k.

Finally, we want to deal with the MT-property for T^k.

From the duality between MP and MT, we will be able to derive identical results for the case of modus tollens, to the ones obtained for the modus ponens, just by contraposition.

The only exception of this is for R-implications. In this case the results can not be derived from contraposition and we need to study MT independently of MP. However, we also obtain an identical result to the one obtained for modus ponens.

Proposition 19 *Let T be a smooth t-norm and I_T its corresponding R-implication. Then, I_T is an MT-implication for T^k if and only if $Idemp_T$ contains the set $[k, n]$.*

In all remaining cases all results can be derived from contraposition. First of all, note that S-implications always satisfy contraposition with respect to the unique negation $N(x) = n - x$. On the other hand, D-operators are the contraposition (with respect to $N(x) = n - x$) of QL-operators and vice versa. Using these facts and the duality between MP and MT we can easily prove the results concerning MT.

Proposition 20 *Let S be a smooth t-conorm and I_S its corresponding S-implication. Then, the following statements are equivalent:*

i) *I_S is an MP-implication for T^k*

ii) *I_S is an MT-implication for T^k*

iii) *I_S is an MPT-implication for T^k*

iv) *$Idemp_S$ contains the set $[0, n-k]$.*

Proposition 21 *Suppose $n - k < k < n$. Let T be a smooth t-norm and I_{QL} and I_D the corresponding QL and D-implications derived from T. Then, the following statements are equivalent:*

i) *I_{QL} (and I_D) is an MP-implication for T^k*

ii) *I_{QL} (and I_D) is an MT-implication for T^k*

iii) *I_{QL} (and I_D) is an MPT-implication for T^k*

iv) *T is the Lukasiewicz t-norm $T_{\mathbf{L}}$.*

v) *$I_{QL} = I_D = I$ is the Kleene-Dienes implication.*

The results for MT are summarized now in table 3.

	MT-implication for T^k
R-implication from T	$[k, n] \subseteq Idemp_T$
S-implications from S	$[0, n-k] \subseteq Idemp_S$
QL-implications from T	$T = T_{\mathbf{L}}$
D-implications from T	$T = T_{\mathbf{L}}$

Table 3. Characterization of MT-implications for the t-norm T^k with $n - k < k < n$.

To finish, note that the general case of modus ponens for QL and D-operators can be translated also for modus tollens via duality, obtaining again exactly the same results.

In view of the characterizations of MP and MT conditions, since both coincide in all four cases, it is clear that in each case the corresponding characterization also works in fact for the MPT-condition.

5 Conclusion

The two main inference rules, modus ponens (MP) and modus tollens (MT), are studied for the four most usual classes of discrete implications: R, S, QL and D-implications. A characterization of MP and a characterization of MT is given for all these kinds of implications, obtaining in the majority of cases the condition $T_1(x, n-x) = 0$, which directly derives (in the smooth case) into the Łukasiewicz t-norm. For this reason, the non-smooth case is also studied for a general class of discrete t-norms T_1 that satisfy the condition above. In this study, a lot of new solutions among R, S, QL and D-implications, derived from smooth t-norms, is obtained for both properties MP and MT.

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