Decomposable Signed Fuzzy Measures

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Abstract

In this paper some types of decomposable signed fuzzy measures have been presented. We discuss properties of $\otimes$-and $\oplus$-decomposable signed fuzzy measures. We consider their relationship with bi-capacities.

Keywords: signed fuzzy measure, bi-capacity, pseudo-addition

1 Introduction

There are many generalizations of the concept of a classical (σ-additive) measure studied by various authors, [7, 15, 14, 2, 8]. A generalization of measures by allowing that m can take negative values, leads to the notion of signed measures, see [7]. In [8], authors presented two concepts of generalized measures and integrals, one of them is the concept of S-measures, i.e. S-decomposable fuzzy measures, and (S,T)-integrals, based on an appropriate t-conorm S and a t-norm T. On the other hand, in [9, 4] generalizations of a fuzzy measure, i.e. of a non-negative, monotone set function, vanishing at the empty set, have been introduced. A generalized fuzzy measure, *a signed fuzzy measure*, has been introduced by Liu in [9]. Murofushi et al. in [10] used the term *a non-monotonic fuzzy measure* to denote a general real-valued set function, vanishing at the empty set. As an another generalization of fuzzy measures, the concepts of bi-capacities and bi-cooperative games have been introduced in [4].

This papers discusses decomposable signed fuzzy measures. In the next section the short overview of basic notions and definitions is given. Section 3 introduces $\oplus$- and $\otimes$-decomposable set functions. Several properties of such set functions are investigated and concrete examples are given. Section 4 discusses the relationship of decomposable signed fuzzy measures with special classes of bi-capacities.

2 Preliminaries

Let X be a universal set. Let $\mathcal{P}(X)$ be the class of subsets of the universal set X.

Definition 1 *[9, 11] A real-valued set function $m : \mathcal{P}(X) \to \mathbb{R}$, is a signed fuzzy measure if it satisfies*
(i) $m(\emptyset) = 0$
(ii) (RM) *If $E, F \in \mathcal{P}(X)$, $E \cap F = \emptyset$, then*
a) $m(E) \geq 0$, $m(F) \geq 0$, $m(E) \vee m(F) > 0 \Rightarrow$

$$m(E \cup F) \geq m(E) \vee m(F);$$

b) $m(E) \leq 0$, $m(F) \leq 0$, $m(E) \wedge m(F) < 0 \Rightarrow$

$$m(E \cup F) \leq m(E) \wedge m(F);$$

c) $m(E) > 0$, $m(F) < 0 \Rightarrow$

$$m(F) \leq m(E \cup F) \leq m(E).$$

The property (RM) of m is called the revised monotonicity, see [11, 13].

Definition 2 *The dual set function of a real-valued set function m, $m : \mathcal{P}(X) \to \mathbb{R}$ is defined by $\bar{m}(E) = m(X) - m(\bar{E})$, where $\bar{E}$ denotes the complement set of E, $\bar{E} = X \setminus E$.*

Obviously, if m is a fuzzy measure, $\bar{m}$ is a fuzzy measure, too. However, if m is a signed fuzzy measure, its dual set function $\bar{m}$ need not be a signed fuzzy measure and this fact will be discussed in the next section.

The *symmetric maximum* $\otimes : [-1,1]^2 \to [-1,1]$, originally introduced in [3], can be represented by:

$$a \otimes b = (|a| \vee |b|) \operatorname{sign}(a+b),$$

The *pseudo-addition* $\oplus : [-1,1]^2 \to [-1,1]$, associated to a continuous t-conorm S, introduced in [6], is defined by:

$$a \oplus b = \begin{cases} S(a,b) & (a,b) \in [0,1]^2 \\ -S(-a,-b) & (a,b) \in [-1,0]^2 \\ d & (a,b) \in [0,1] \times (-1,0], a \geq -b \\ e & (a,b) \in [0,1] \times (-1,0], a \leq -b \\ f & a = 1, b = -1 \end{cases}$$

where $f = 1$ or $f = -1$, $d = inf\{c \,|\, S(-b, c) \geq a\}$ and $e = -inf\{c \,|\, S(a, c) \geq -b\}$, and the remaining cases being determined by the commutativity of $\oplus$.
The binary operations, $\oslash$ and $\oplus$ are commutative, isotonic, with neutral element 0. The second one is associative and the first one is not. For more details we recommend [4, 6].

In the next example, we consider a set function which satisfies the conditions given in Definition 1, i.e. a signed fuzzy measure.

Example 1 *Let X be a set of k elements. Let $A, B \subset X$ such that $X = A \cup B$, $A \cap B = \emptyset$, A, $B \neq \emptyset$ and $card(A) = n$, $card(B) = k - n$. We define a set function $m : \mathcal{P}(X) \to [-1, 1]$ by:*

$$m(E) = \begin{cases} 0, & E = X \\ \left(\frac{card(A \cap E)}{n}\right) \oplus_{S_P} \left(-\frac{card(B \cap E)}{k-n}\right), & else. \end{cases}$$

$S_P : [0, 1]^2 \to [0, 1]$ is the probabilistic sum, defined by $S_P(x, y) = x + y - xy$. The set function m is a signed fuzzy measure.

As an example, we consider $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4\}$. We have, e.g. $m(A) = 1$, $m(B) = -1$, $m(X) = 0$, $m(\{a_1, b_1\}) = \frac{1}{3} \oplus_{S_P} -\frac{1}{4} = \frac{1}{9}$, $m(\{a_1, a_2, b_1\}) = \frac{2}{3} \oplus_{S_P} -\frac{1}{4} = \frac{5}{9}$, $m(\{a_1, a_2\}) = \frac{2}{3} \oplus_{S_P} 0 = \frac{2}{3}$, etc. Let us indicate the sets A and B, the sets of two types of drugs influential in duration of some medical procedures(e.g. the duration of lungs harmful fluidity throwing out). It is known that the first one, A, decreases, and the second one, B, increases the duration of the considered procedure. It is also known that simultaneously taking drugs A and B has no effects in the duration of the procedure, i.e. the duration is unchanged. For the reasons of patients health, sometimes the procedure must be slower and sometimes, faster. The set function m in the above example is a mathematical model for the described situation.

3 Decomposable signed fuzzy measures

Definition 3 *A set function $m : \mathcal{P}(X) \to [-1, 1]$, is i) a $\oslash$-decomposable set function if it satisfies*

$$m(E \cup F) = m(E) \oslash m(F)$$

for all E, $F \in \mathcal{P}(X)$, $E \cap F = \emptyset$.
ii) a $\oplus$-decomposable set function if it satisfies

$$m(E \cup F) = m(E) \oplus m(F)$$

for all E, $F \in \mathcal{P}(X)$, $E \cap F = \emptyset$.

Example 2 *Let $X = \{x_1, x_2, \ldots, x_n\}$. Let m be a set function $m : \mathcal{P}(X) \longrightarrow [-1, 1]$ with $m(\emptyset) = 0$, defined by:*

$$m(E) = \begin{cases} \frac{1}{\min\limits_{x_i \in E} i} & if \ \min\limits_{x_i \in E} i = 2k \\ -\frac{1}{\min\limits_{x_i \in E} i} & if \ \min\limits_{x_i \in E} i = 2k+1 \end{cases}$$

m is a $\oslash$-decomposable set function.

Example 3 *Let m be a set function defined on $\mathcal{B}([-1, 1])$, the class of Borel subsets of $[-1, 1]$, and λ the usual Lebesgue measure. We denote $\lambda_1(E) = \lambda(E \cap [0, 1])$ and $\lambda_2(E) = \lambda(E \cap [-1, 0))$. We define $m : \mathcal{B}([-1, 1]) \longrightarrow [-1, 1]$ by*

$$m(E) = sign(\lambda_1(E) - \lambda_2(E)) \left(1 - e^{-|\lambda_1(E) - \lambda_2(E)|}\right).$$

m is a $\oplus_{S_P}$-decomposable signed fuzzy measure.

In the sequel of this section we will consider a signed fuzzy measure m with $m(X) = 0$. We will examine when its dual set function $\bar{m}$ is a signed fuzzy measure, too. Note that for a non-negative (non-positive) signed fuzzy measure m, the condition $m(X) = 0$ implies $m(E) = 0$ for all $E \in \mathcal{P}(X)$. We suppose that $m : \mathcal{P}(X) \to \mathbb{R}$ is a signed fuzzy measure of a non-constant sign. We easily obtain the next lemma by the definition of a signed fuzzy measure and the condition $m(X) = 0$.

Lemma 1 *Let m be a signed fuzzy measure, $m(X) = 0$. $m(E)$ and $m(\bar{E})$ are of opposite sign values, i.e. $(\forall E \in \mathcal{P}(X))$ $(m(E) > 0 \Leftrightarrow m(\bar{E}) < 0)$.*

Definition 4 *We say that a real-valued set function m, $m(\emptyset) = 0$ satisfies the intersection property if for all $E, F \in \mathcal{P}(X)$, $E \cap F \neq \emptyset$ and $E \cup F = X$ we have*

$$\begin{aligned} a)\ & m(E) \geq 0, m(F) \geq 0, m(E) \vee m(F) > 0 \Rightarrow \\ & \qquad m(E \cap F) \geq m(E) \vee m(F); \\ b)\ & m(E) \leq 0, m(F) \leq 0, m(E) \wedge m(F) < 0 \Rightarrow \\ & \qquad m(E \cap F) \leq m(E) \wedge m(F); \\ c)\ & m(E) > 0, m(F) < 0 \Rightarrow \\ & \qquad m(F) \leq m(E \cap F) \leq m(E). \end{aligned}$$

We have the next theorem.

Proposition 1 *Let m be a signed fuzzy measure, $m(X) = 0$. m satisfies the intersection property if and only if the dual set function $\bar{m}$ of m is a signed fuzzy measure.*

`Proof.` Let m be a signed fuzzy measure such that $m(X) = 0$.

($\Longrightarrow$) First, we suppose that m satisfies the intersection property. We will prove that $\bar{m}$ is a signed fuzzy measure.
(i) Directly by the definition of $\bar{m}$ we have $\bar{m}(\emptyset) = 0$.
(ii) In order to prove condition (RM) *a)* let $E, F \in \mathcal{P}(X)$ such that $E \cap F = \emptyset$ and $\bar{m}(E) \geq 0$, $\bar{m}(F) \geq 0$, $\bar{m}(E) \vee \bar{m}(F) > 0$. We have $\bar{E} \cup \bar{F} = X$ and $m(\bar{E}) \leq 0$, $m(\bar{F}) \leq 0$ and $m(\bar{E}) \wedge m(\bar{F}) < 0$. $(*)$
If we suppose that $\bar{E} \cap \bar{F} = \emptyset$ then we have $F = \bar{E}$. By Lemma 1. we obtain that the values $m(F)$ and $m(\bar{F})$ are of opposite sign values and it is a contradiction with $(*)$. Therefore, $\bar{E} \cap \bar{F} \neq \emptyset$. By the intersection property of m we have:

$$m(\bar{E} \cap \bar{F}) \leq m(\bar{E}) \wedge m(\bar{F}) \Longleftrightarrow$$
$$\Longleftrightarrow \quad m(\overline{E \cup F}) \leq m(\bar{E}) \wedge m(\bar{F})$$
$$\Longleftrightarrow \quad -\bar{m}(E \cup F) \leq (-\bar{m}(E)) \wedge (-\bar{m}(F))$$
$$\Longleftrightarrow \quad \bar{m}(E \cup F) \geq \bar{m}(E) \vee \bar{m}(F).$$

Hence, we have that $\bar{m}$ satisfies condition (RM) a). Similarly we obtain that $\bar{m}$ satisfies conditions (RM) b) and c), hence, $\bar{m}$ is a signed fuzzy measure.
($\Longleftarrow$) Let $\bar{m}$ be a signed fuzzy measure, i.e. $\bar{m}$ is a revised monotone set function and $\bar{m}(\emptyset) = 0$. We obtain the claim directly by the definition of the intersection property and the above consideration. □

Directly by the definitions we have that $\oplus$- and $\circledvee$-decomposable set functions are signed fuzzy measures. The condition of the intersection property we will replace by the condition $m(E \cap F) = m(E) \oplus m(F)$ for all $E, F \in \mathcal{P}(X)$, $E \cup F = X$ for a $\oplus$-decomposable set function and respectively with $m(E \cap F) = m(E) \circledvee m(F)$ for all $E, F \in \mathcal{P}(X)$, $E \cup F = X$ for a $\circledvee$-decomposable set function. Hence, we have the next corollaries of Proposition 1.

Corollary 1 *The dual set function $\bar{m}$ of a $\circledvee$-decomposable set function m, $m(\emptyset) = m(X) = 0$ is a $\circledvee$-decomposable iff $m(E \cap F) = m(E) \circledvee m(F)$ for all $E, F \in \mathcal{P}(X)$, $E \cup F = X$.*

Corollary 2 *Let m be a $\circledvee$-decomposable set function, $m(\emptyset) = m(X) = 0$, such that $m(E \cap F) = m(E) \circledvee m(F)$ for all $E, F \in \mathcal{P}(X)$, $E \cup F = X$. Then m is a self-dual set function, i.e. $m = \bar{m}$.*

Analogously, we have the next two corollaries related to $\oplus$- decomposable set functions.

Corollary 3 *The dual set function $\bar{m}$ of a $\oplus$-decomposable set function m, $m(\emptyset) = m(X) = 0$, $1 \notin Ran(m)$, is a $\oplus$-decomposable iff $m(E \cap F) = m(E) \oplus m(F)$ for all $E, F \in \mathcal{P}(X)$, $E \cup F = X$.*

Corollary 4 *Let m be a $\oplus$-decomposable set function, $m(\emptyset) = m(X) = 0$, and $1 \notin Ran(m)$, such that $m(E \cap F) = m(E) \oplus m(F)$ for all $E, F \in \mathcal{P}(X)$, $E \cup F = X$. Then m is a self-dual set function, i.e. $m = \bar{m}$.*

As an illustration, in the next example a ternary signed fuzzy measure m which is $\oplus$-decomposable, where $\oplus$ is the pseudo-addition related to an arbitrary continuous t-conorm S, is given. The values of a decomposable set function m are determined by its values on singletons. The considered signed fuzzy measure m is a self-dual set function.

Example 4 *Let $X = \{x_1, x_2, x_3, x_4\}$ and let m be a $\oplus$-decomposable set function with $m(\emptyset) = 0$, $m : \mathcal{P}(X) \to [-1, 1]$, defined by:*

$$m(\{x_1\}) = a, \qquad m(\{x_2\}) = 0,$$
$$m(\{x_3\}) = -a, \qquad m(\{x_4\}) = 0.$$

where $0 < a < 1$. Obviously, m is a self-dual signed fuzzy measure.

4 Decomposable signed fuzzy measures and bi-capacities

In this section we consider the relationship of a decomposable signed fuzzy measure m and a bi-capacity $\mathbf{m}$. Let X be a finite set. Bi-capacities are a real-valued functions defined on the set $\mathcal{Q}(X) = \{(E, F) \in \mathcal{P}(X) \times \mathcal{P}(X) | \, E \cap F = \emptyset\}$, which are non-decreasing in the first variable and non-increasing in the second one.

Definition 5 *[4] A bi-capacity is a real-valued function $\mathbf{m} : \mathcal{Q}(X) \longrightarrow \mathbb{R}$ with the following properties:*
(BC1) $\mathbf{m}(\emptyset, \emptyset) = 0$,
(BC2) $E \subset F \Longrightarrow \mathbf{m}(E, \cdot) \leq \mathbf{m}(F, \cdot)$ *and* $\mathbf{m}(\cdot, E) \geq \mathbf{m}(\cdot, F)$ *for all* $(E, F) \in \mathcal{Q}(X)$

The structure $(\mathcal{Q}(X), \leq)$ is the lattice of the type 3^n, for more details see [4]. For $n = 2$ and $X = \{1, 2\}$ the Hasse diagram of $(\mathcal{Q}(X), \leq)$ is

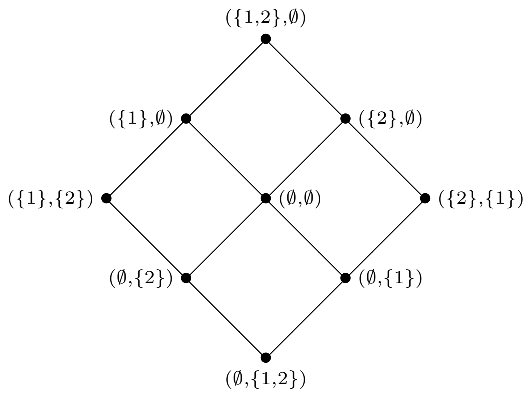

Figure 1. $(\mathcal{Q}(X), \leq)$ for $n = 2$.

A bi-capacity $\mathbf{m}$ is $\vee$-CPT type if there exist two fuzzy measures m_1 and m_2 such that $\mathbf{m}(E,F) = m_1(E) \circledvee (-m_2(F))$ for all $(E,F) \in \mathcal{Q}(X)$ (see [5]).

Proposition 2 *Let $m : \mathcal{P}(X) \to [-1,1]$ be a $\circledvee$-decomposable signed fuzzy measure. Then the set function $\mathbf{m} : \mathcal{Q}(X) \to [-1,1]$ defined by*

$$\mathbf{m}(E,F) = \max_{A\subset E, m(A)\geq 0} m(A) \circledvee \min_{B\subset F, m(B)\leq 0} m(B) \quad (1)$$

is a bi-capacity.

`Proof.` We will show that $\mathbf{m}$ defined as above satisfies conditions (BC1) and (BC2).
(BC1) $\mathbf{m}(\emptyset,\emptyset) = 0$,
(BC2) Let $E \subset F$. For all $G \subset X$ such that (E,G), $(F,G) \in \mathcal{Q}(X)$ we have:

$$\begin{aligned} \mathbf{m}(E,G) &= \max_{A\subset E, m(A)\geq 0} m(A) \circledvee \min_{B\subset G, m(B)\leq 0} m(B) \\ &\leq \max_{A\subset F, m(A)\geq 0} m(A) \circledvee \min_{B\subset G, m(B)\leq 0} m(B) \\ &= \mathbf{m}(F,G) \end{aligned}$$

For all $H \subset X$, such that (H, E),$(H, F) \in \mathcal{Q}(X)$

$$\begin{aligned} \mathbf{m}(H,E) &= \max_{A\subset H, m(A)\geq 0} m(A) \circledvee \min_{B\subset E, m(B)\leq 0} m(B) \\ &\geq \max_{A\subset H, m(A)\geq 0} m(A) \circledvee \min_{B\subset F, m(B)\leq 0} m(B) \\ &= \mathbf{m}(H,F) \end{aligned}$$

□

We shall give the next simple example.

Example 5 *Let $X = \{1,2\}$ and let $m : \mathcal{P}(X) \to [-1,1]$ be a $\circledvee$-decomposable set function, defined by $m(\emptyset) = 0$, $m(\{1\}) = 0.2$, $m(\{2\}) = -0.3$ and $m(\{1,2\}) = -0.3$. By Proposition 2, the set function $\mathbf{m}$ determined by Eq.(1) is a bi-capacity. The bi-capacity $\mathbf{m}$ associated to m, is defined on $\mathcal{Q}(X)$, $card(\mathcal{Q}(X)) = 3^2 = 9$, illustrated by Figure 1.*
$\mathbf{m}(\{1,2\},\emptyset) = \mathbf{m}(\{1\},\emptyset) = 0.2$,
$\mathbf{m}(\emptyset,\{2\}) = \mathbf{m}(\emptyset,\{1,2\}) = \mathbf{m}(\{1\},\{2\}) = -0.3$, and $\mathbf{m}(E,F) = 0$ in the remaining cases.

Proposition 3 *Let $\mathbf{m} : \mathcal{Q}(X) \to [-1,1]$ be a bi-capacity such that for any nonempty index set $\mathcal{I} \subset \{1,2,\ldots,n\}$, we have*

$$\max_{i\in\mathcal{I}} \mathbf{m}(\{x_i\},\emptyset) \neq -\min_{i\in\mathcal{I}} \mathbf{m}(\emptyset,\{x_i\}). \quad (2)$$

Then the set function $m : \mathcal{P}(X) \to [-1,1]$ defined by

$$m(E) = \max_{x_i\in E} \mathbf{m}(\{x_i\},\emptyset) \circledvee \min_{x_i\in E} \mathbf{m}(\emptyset,\{x_i\})$$

is a $\circledvee$-decomposable signed fuzzy measure.

`Proof.` Let us suppose that $\mathbf{m} : \mathcal{Q}(X) \to [-1,1]$ is such that inequality (2) holds for any nonempty set $\mathcal{I} \subset \{1,2,\ldots,n\}$. Let E, $F \in \mathcal{P}(X)$, $E\cap F = \emptyset$. Then we have

$$\begin{aligned} & m(E\cup F) = \\ &= \max_{x_i\in E\cup F} \mathbf{m}(\{x_i\},\emptyset) \circledvee \min_{x_i\in E\cup F} \mathbf{m}(\emptyset,\{x_i\}) = \\ &= \left(\max_{x_i\in E} \mathbf{m}(\{x_i\},\emptyset) \circledvee \max_{x_i\in F} \mathbf{m}(\{x_i\},\emptyset)\right) \circledvee \\ &\qquad \circledvee \left(\min_{x_i\in E} \mathbf{m}(\emptyset,\{x_i\}) \circledvee \min_{x_i\in F} \mathbf{m}(\emptyset,\{x_i\})\right) = \\ &= \left(\max_{x_i\in E} \mathbf{m}(\{x_i\},\emptyset) \circledvee \min_{x_i\in E} \mathbf{m}(\emptyset,\{x_i\})\right) \circledvee \\ &\qquad \circledvee \left(\max_{x_i\in F} \mathbf{m}(\{x_i\},\emptyset) \circledvee \min_{x_i\in F} \mathbf{m}(\emptyset,\{x_i\})\right) = \\ &= m(E) \circledvee m(F) \end{aligned}$$

□

Corollary 5 *Let $\mathbf{m} : \mathcal{Q}(X) \to [-1,1]$ be a bi-capacity of $\vee$-CPT type, such that m_1 and m_2 are two $\vee$-decomposable fuzzy measures and $m_1(E) \neq m_2(E)$ for all $E \in \mathcal{P}(X)$. Then the set function $m : \mathcal{P}(X) \to [-1,1]$ defined by*

$$m(E) = \mathbf{m}(E,\emptyset) \circledvee \mathbf{m}(\emptyset,E)$$

is a $\circledvee$-decomposable signed fuzzy measure.

`Proof.` Obviously, two $\vee$-decomposable fuzzy measures m_k, $k = 1,2$ satisfy for all $E \in \mathcal{P}(X)$:

$$m_k(E) = \bigvee_{x_i\in E} m_k(\{x_i\}).$$

It is clear that $\forall\, i$, $m_1(\{x_i\}) = \mathbf{m}(\{x_i\},\emptyset)$ and $m_2(\{x_i\}) = -\mathbf{m}(\emptyset,\{x_i\})$. Hence, by Proposition 3, we have the claim. □

We present two Propositions for a $\oplus$-decomposable signed fuzzy measure, where S is an arbitrary continuous t-conorm. The proofs are similarly to the proofs of Proposition 2 and Proposition 3, we remark that $\oplus$ is associative on $[-1,1]^2$, so in this case, a condition similar to condition (2) is omitted here.

Proposition 4 *Let $m : \mathcal{P}(X) \to [-1,1]$ be a $\oplus$-decomposable signed fuzzy measure. Then the set function $\mathbf{m} : \mathcal{Q}(X) \to [-1,1]$ defined by*

$$\mathbf{m}(E,F) = \left(\bigoplus_{A\subset E, m(A)\geq 0} m(A)\right) \oplus \left(\bigoplus_{B\subset F, m(B)\leq 0} m(B)\right)$$

is a bi-capacity.

Proposition 5 *Let $\mathbf{m} : \mathcal{Q}(X) \to [-1,1]$ be a bi-capacity, then the set function $m : \mathcal{P}(X) \to [-1,1]$ defined by*

$$m(E) = \left(\bigoplus_{x_i\in E} \mathbf{m}(\{x_i\},\emptyset)\right) \oplus \left(\bigoplus_{x_i\in E} \mathbf{m}(\emptyset,\{x_i\})\right)$$

is a $\oplus$-decomposable signed fuzzy measure.

Corollary 6 *Let $\mathbf{m} : \mathcal{Q}(X) \to [-1, 1]$ be a bi-capacity of $\oplus$-CPT type, i.e. $\mathbf{m}(E, F) = m_1(E) \oplus (-m_2(F))$, for all $(E, F) \in \mathcal{Q}(X)$, such that m_1 and m_2 are two S-decomposable fuzzy measures, then the set function $m : \mathcal{P}(X) \to [-1, 1]$ defined by*

$$m(E) = \mathbf{m}(E, \emptyset) \oplus \mathbf{m}(\emptyset, E)$$

is a $\oplus$-decomposable signed fuzzy measure.

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