On the k-additive core of capacities

Michel GRABISCH
University of Paris I – Panthéon-Sorbonne, CERMSEM
106-112 Bd de l'Hôpital, 75013 Paris, France
michel.grabisch@lip6.fr

Pedro MIRANDA
Universidad Complutense de Madrid
Plaza de Ciencias, 3, 28040 Madrid, Spain
pmiranda@mat.ucm.es

Abstract

We investigate in this paper the set of k-additive capacities dominating a given capacity, which we call the k-additive core. We study its structure through achievable families, which play the role of maximal chains in the classical case ($k = 1$), and show that associated capacities are elements (possibly a vertex) of the k-additive core when the capacity is $(k+1)$-monotone. As a particular case, we study the set of k-additive belief functions dominating a belief function. The problem of finding all vertices of the k-additive core is still an open question.

Keywords: k-additive capacity, core, belief function

1 Introduction

The core of a capacity or a game is a fundamental concept, both in decision making theory and in cooperative game theory. In decision making, it is the set of probability measures which are coherent with the information given by the capacity in the representation of uncertainty [15]. In game theory, it is the set of imputations (additive games) that can be given to players so that no subcoalition of the grand coalition has interest to form.

The properties of the core are well known, most of them have been shown by Shapley [13]. In many cases, it happens that the core is empty. A sufficient and necessary condition for nonemptiness is known for capacities and games, which is called balancedness. In particular, convex capacities have a nonempty core.

Since having an empty core is not a favorable situation, either in decision making or in game theory, it may be an alternative solution to look for more general concepts. For example, since the core contains additive capacities or games, we may relax additivity to a weaker notion: k-additivity, proposed by Grabisch [5]. We may call this new notion the *k-additive core*.

Some studies on the k-additive core have already been done by the authors, see, e.g., [6, 11]. It happens that the structure of the k-additive core is much more complex than the one of the classical core. In particular, the set of its vertices is not known. The aim of this paper is to provide new insights in this direction, and to complete results shown in [9].

2 Background

Throughout the paper, we consider a finite universal set X, with $|X| = n$. We use indifferently 2^X or $\mathcal{P}(X)$ to denote the set of subsets of X, and the set of subsets of X containing at most k elements is denoted by $\mathcal{P}^k(X)$, while $\mathcal{P}^k_*(X) := \mathcal{P}^k(X) \setminus \{\emptyset\}$. A *set function* on X is a function $\mu : 2^X \to \mathbb{R}$.

Definition 1 *[3, 14] A* fuzzy measure *or* capacity *μ on X is a nonnegative set function on X such that $\mu(\emptyset) = 0$, and $\mu(A) \leq \mu(B)$ whenever $A \subseteq B$ (*monotonicity*). A capacity is* normalized *if $\mu(X) = 1$.*

We assume in this paper that capacities are normalized. The set of capacities (or fuzzy measures) on X is denoted by $\mathcal{FM}(X)$. For any $A \in 2^X \setminus \{\emptyset\}$, the *unanimity game centered on A* is defined by $u_A(B) = 1$ iff $B \supseteq A$, and 0 otherwise.

Definition 2 *A capacity μ on X is said to be:*

(i) additive *if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$;*

(ii) convex *if $\mu(A \cup B) + \mu(A \cap B) \geq \mu(A) + \mu(B)$, for all $A, B \subseteq X$;*

(iii) k-monotone *for $k \geq 2$ if for any family of k sub-*

sets $A_1, \ldots A_k$, it holds

$$\mu(\bigcup_{i=1}^{k} A_i) \geq \sum_{\substack{K\subseteq\{1,\ldots,k\}\\ K\neq\emptyset}} (-1)^{|K|+1}\mu(\bigcap_{j\in K} A_j)$$

(iv) totally monotone *if it is k-monotone for all $k \geq 2$.*

Totally monotone capacities are also called *belief functions* [12]. By extension, we define 1-monotone capacities as monotone capacities. We will denote the set of belief functions on X by $\mathcal{BEL}(X)$. Remark that k-monotonicity implies k'-monotonicity for all $2 \leq k' \leq k$. Also, for $n > 3$, $(n-2)$-monotone capacities are totally monotone [1].

Definition 3 *Let μ be a set function on X. The* Möbius transform *of μ is a set function $m : 2^X \to \mathbb{R}$ defined by:*

$$m(A) := \sum_{B\subseteq A} (-1)^{|A\setminus B|}\mu(B).$$

The Möbius transform is invertible since one can recover μ from m by:

$$\mu(A) = \sum_{B\subseteq A} m(B).$$

If μ is an additive capacity, then m is non null only for singletons, and $m(\{i\}) = \mu(\{i\})$. The Möbius transform of u_A is given by $m(A) = 1$ and m is 0 otherwise.

Definition 4 *A set function μ vanishing at the empty set is said to be* k-additive *for some integer $k \in \{1,\ldots,n\}$ if $m(A) = 0$ whenever $|A| > k$, and there exists some A such that $|A| = k$, and $m(A) \neq 0$.*

Clearly, 1-additive capacities are additive measures, and a k-additive set function needs only $\sum_{i=1}^{k}\binom{n}{i} - 2$ values to be defined. The set of capacities on X being at most k-additive is denoted by $\mathcal{FM}^k(X)$. Similarly, we denote by $\mathcal{BEL}^k(X)$ the set of belief functions being at most k-additive.

We recall the fundamental following result.

Proposition 1 *[2] (i) Let μ be a set function on X such that $\mu(\emptyset) = 0$. Then monotonicity is equivalent to*

$$\sum_{i\in L\subseteq B} m(L) \geq 0, \quad \forall B \subseteq X, \quad \forall i \in B.$$

(ii) Let μ be a capacity. Then for $2 \leq k \leq n$, k-monotonicity is equivalent to

$$\sum_{A\subseteq L\subseteq B} m(L) \geq 0, \quad \forall A \subseteq B \subseteq X, \quad |A| \leq k.$$

Clearly, a totally monotone capacity has a non nonnegative Möbius transform.

3 The core of capacities

Definition 5 *Let μ be a capacity on X. The* core *of μ is defined by:*

$$\mathcal{C}(\mu) := \{\nu \in \mathcal{FM}^1(X) \mid \nu \geq \mu\},$$

where $\nu \geq \mu$ means $\nu(A) \geq \mu(A)$ for all $A \subseteq X$.

A *maximal chain* in 2^X is a sequence of subsets $A_0 := \emptyset, A_1, \ldots, A_{n-1}, A_n := X$ such that $A_i \subset A_{i+1}$, $i = 0,\ldots,n-1$. The set of maximal chains of 2^X is denoted by $\mathcal{M}(2^X)$.

To each maximal chain $C := \{\emptyset, A_1, \ldots, A_n = X\}$ in $\mathcal{M}(2^X)$ corresponds a unique permutation σ on X such that $A_1 = \sigma(1)$, $A_2 \setminus A_1 = \sigma(2)$, …, $A_n \setminus A_{n-1} = \sigma(n)$. The set of all permutations over X is denoted by $\mathfrak{S}(X)$. Let μ be a capacity. To each permutation σ (or maximal chain C) we assign a *marginal worth vector* p^σ (or p^C) in $\mathbb{R}^n$ defined by:

$$p^\sigma_{\sigma(i)} := \mu(\{\sigma(1),\ldots,\sigma(i)\}) - \mu(\{\sigma(1),\ldots,\sigma(i-1)\})$$

or

$$p^C_{\sigma(i)} := \mu(A_i) - \mu(A_{i-1})$$

with the above notation. Any marginal worth vector forms a probability distribution over X, and hence defines an additive capacity. The following is immediate.

Proposition 2 *Let μ be a capacity on X, and C a maximal chain of 2^X. Then*

$$p^C(A) = \mu(A), \quad \forall A \in C.$$

Theorem 1 *The following are equivalent.*

(i) *μ is a convex capacity*

(ii) *all marginal worth vectors p^σ, $\sigma \in \mathfrak{S}(X)$ belong to the core of μ*

(iii) $\mathcal{C}(\mu) = \mathrm{co}(\{p^\sigma\}_{\sigma\in\mathfrak{S}(X)})$

(iv) $\mathrm{ext}(\mathcal{C}(\mu)) = \{p^\sigma\}_{\sigma\in\mathfrak{S}(X)}$,

where $\mathrm{co}(K)$ and $\mathrm{ext}(K)$ denote respectively the convex hull and the extreme points of some convex set K.

(i) ⇒ (ii) and (i) ⇒ (iv) are due to Shapley [13], while (ii) ⇒ (i) was proved by Ichiishi [10].

4 The k-additive core of capacities

Definition 6 *[6] Let μ be a capacity on X, and $1 \leq k \leq n-1$. The* k-additive core *of μ is defined by:*

$$\mathcal{C}^k(\mu) := \{\nu \in \mathcal{FM}^k(X) \mid \nu \geq \mu\}.$$

Note that $\mathcal{C}^1(\mu) = \mathcal{C}(\mu)$. Similarly, we introduce $\mathcal{BC}^k(\mu)$ the set of k-additive belief functions dominating μ.

Let us start introducing some notations. For a given k, $1 \leq k \leq n$, we define

$$\Lambda^k_\cap := \{\lambda : \mathcal{P}(X) \times \mathcal{P}^k_*(X) \to \mathbb{R} \,|\, \forall B \in \mathcal{P}(X), \sum_{A \cap B \neq \emptyset} \lambda(B,A) = 1, \lambda(B,A) = 0 \text{ if } A \cap B = \emptyset\}.$$

From this set and for a given capacity μ, we define a set of k-additive set functions $\mathcal{M}_{\Lambda^k_\cap}(\mu) := \{\mu_\lambda \mid \lambda \in \Lambda^k_\cap\}$, with μ_λ defined by its Möbius transform m_λ:

$$m_\lambda(A) = \sum_{B \in \mathcal{P}(X)} \lambda(B,A) m(B).$$

Similarly, we can define $\Lambda^k_\subseteq$ and $\Lambda^k_\supseteq$, from which we can derive the corresponding sets $\mathcal{M}_{\Lambda^k_\subseteq}(\mu)$ and $\mathcal{M}_{\Lambda^k_\supseteq}(\mu)$, by replacing $\cap$ by $\subseteq, \supseteq$. When λ is restricted to non-negative values (then it becomes a weight function), we will use the notations $\Lambda^k_{\cap,+}, \Lambda^k_{\subseteq,+}, \Lambda^k_{\supseteq,+}$, and the corresponding $\mathcal{M}_{\Lambda^k_{\cap,+}}(\mu), \mathcal{M}_{\Lambda^k_{\subseteq,+}}(\mu)$ and $\mathcal{M}_{\Lambda^k_{\supseteq,+}}(\mu)$.

Given a capacity μ, the problem of obtaining the set of all probability measures dominating μ has been addressed by several authors [2, 4]. Chateauneuf and Jaffray proved in [2] the following result:

Theorem 2 *[2] Let μ be a capacity on X, m its Möbius transform, and suppose $P \in \mathcal{C}^1(\mu)$. Then, $P \in \mathcal{M}_{\Lambda^1_{\subseteq,+}}(\mu)$.*

Dempster has shown the same result in [4] and also Shapley in [13], but both of them only for belief functions. In general, $\mathcal{C}^1(\mu) \subseteq \mathcal{M}_{\Lambda^1_{\subseteq,+}}(\mu)$, and equality holds when $\mu \in \mathcal{BEL}(X)$.

Grabisch has extended Theorem 2 for the k-additive case in [8].

Theorem 3 *[8] Let μ be a capacity, and suppose that $\mu^* \in \mathcal{C}^k(\mu)$, for some $1 \leq k \leq n-1$. Then, necessarily, μ^* belongs to $\mathcal{M}_{\Lambda^k_\cap}(\mu)$.*

As shown in [11], $\mathcal{C}^k(\mu) \subseteq \mathcal{M}_{\Lambda^k_\cap}(\mu)$, and equality holds only for the unanimity game centered on X. The following can be proved.

Theorem 4 *[11] Let μ be a belief function, and suppose $\mu^* \in \mathcal{BC}^k(\mu)$. Then, necessarily μ^* belongs to $\mathcal{M}_{\Lambda^k_{\cap,+}}(\mu)$.*

Remark that $\mathcal{M}_{\Lambda^k_{\cap,+}}(\mu)$ generalizes $\mathcal{M}_{\Lambda^1_{\subseteq,+}}(\mu)$ for the general k-additive case. However, similarly as Theorem 3, Theorem 4 provides a very large class of functions, as shown by the following result.

Proposition 3 *Let μ be a capacity. Then, $\mathcal{M}_{\Lambda^k_{\cap,+}}(\mu) \subseteq \mathcal{BC}^k(\mu)$ if and only if $\mu(A) = 0, \forall A \neq X$.*

Thus, as pointed out in [11], it is not possible to generalize the good properties obtained for probabilities in [2]. For $\Lambda^k_{\subseteq,+}$, the following can be proved.

Proposition 4 *[7] If $\mu \in \mathcal{BEL}(X)$ and $\mu^* \in \mathcal{M}_{\Lambda^k_{\subseteq,+}}(\mu)$, then $\mu^* \in \mathcal{BC}^k(\mu)$.*

On the other hand, it can be seen that this set does not cope in general with the set of all dominating k-additive measures [7] ($k > 1$), i.e., the conditions on λ given in Theorem 4 cannot be strengthened. Then, if μ is a belief function,

$$\mathcal{M}_{\Lambda^k_{\subseteq,+}}(\mu) \subseteq \mathcal{BC}^k(\mu) \subseteq \mathcal{M}_{\Lambda^k_{\cap,+}}(\mu),$$

where inclusion is strict in general. Next result shows that all k-additive belief measures dominating another belief function μ can be generated from $\mathcal{M}_{\Lambda^k_{\subseteq,+}}(\mu)$.

Theorem 5 *[11] Let μ be a belief function, and suppose that $\mu^* \in \mathcal{BC}^k(\mu)$. Then, there exists $\mu' \in \mathcal{BC}^k(\mu)$ such that μ' belongs to $\mathcal{M}_{\Lambda^k_{\subseteq,+}}(\mu)$ and μ^* belongs to $\mathcal{M}_{\Lambda^k_{\supseteq,+}}(\mu')$.*

This result is explained in Figure 1 for $|X| = 3$ and $k = 2$. m, m', and m^* are respectively the Möbius transforms of μ, μ', and μ^*.

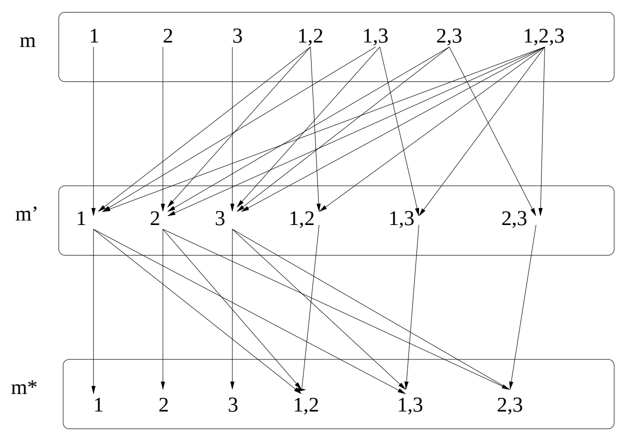

Figure 1: Illustration of Theorem 5.

5 Vertices of the k-additive core

We put $N(k) := \binom{n}{1} + \cdots + \binom{n}{k}$. A first fact is the following.

Proposition 5 *For any capacity μ, $\mathcal{C}^k(\mu)$ and $\mathcal{BC}^k(\mu)$ are closed convex $(N(k)-1)$-dimensional polytopes.*

We denote by $\prec$ a total (strict) order on $\mathcal{P}^k_*(X)$, $\preceq$ denoting the corresponding large order.

Definition 7 *For any $B \in \mathcal{P}^k_*(X)$, we define*

$$\mathcal{A}(B) := \{A \subseteq X \mid A \supseteq B, \forall K \subseteq A, K \in \mathcal{P}^k_*(X), K \preceq B\}$$

the achievable family of B.

For example, taking $n = 3$, $k = 2$ and the order $1 \prec 2 \prec 12 \prec 13 \prec 23 \prec 3$, we get:

$$\mathcal{A}(1) = \{1\}, \quad \mathcal{A}(2) = \{2\}, \quad ,\mathcal{A}(12) = \{12\},$$
$$\mathcal{A}(13) = \emptyset, \quad \mathcal{A}(23) = \emptyset, \quad \mathcal{A}(3) = \{3, 13, 23, 123\}.$$

It is easy to see that $\{\mathcal{A}(B)\}_{B \in \mathcal{P}^k_*(X)}$ is a partition of $\mathcal{P}(X) \setminus \{\emptyset\}$.

A total order $\prec$ on $\mathcal{P}^k_*(X)$ is said to be *compatible* if for all $i, j \in X$, $i \prec j$ implies $S \cup i \prec S \cup j$, for any $S \in \mathcal{P}^{k-1}(X)$, $i, j \notin S$. It is said to be $\subseteq$*-compatible* if $A \subseteq B$ implies $A \prec B$. Lastly, $\prec$ is said to be *strongly compatible* if it is compatible and $\subseteq$-compatible, and *weakly compatible* if only compatible.

Proposition 6 *Assume $\prec$ is compatible. For any $B \in \mathcal{P}^k_*(X)$ such that $\mathcal{A}(B) \neq \emptyset$, $\mathcal{A}(B)$ endowed with inclusion is a Boolean lattice with bottom element B. The top element is denoted by $\breve{B}$.*

$\subseteq$-compatibility is a sufficient and necessary condition for the nonemptiness of all achievable families.

Let μ be a capacity on X, m its Möbius transform, and $\prec$ some total order on $\mathcal{P}^k_*(X)$. We define $\mu_\prec$ by its Möbius transform as follows:

$$m_\prec(B) := \begin{cases} \sum_{A \in \mathcal{A}(B)} m(A), & \text{if } \mathcal{A}(B) \neq \emptyset \\ 0, & \text{else} \end{cases} \tag{1}$$

for all $B \in \mathcal{P}^k_*(X)$, $m_\prec(\emptyset) := 0$. Since achievable families form a partition of 2^X, $m_\prec$ satisfies $\sum_{B \subseteq X} m_\prec(X) = 1$, hence $\mu_\prec(X) = 1$. The following can be shown.

Proposition 7 *If $\prec$ is compatible, then for any nonempty achievable family $\mathcal{A}(B)$, $\mu_\prec(\breve{B}) = \mu(\breve{B})$.*

(see the analogy with Prop. 2).

Proposition 8 *Let μ be a capacity on X. Then $\mu_\prec$ is a belief function for any compatible order $\prec$ if and only if μ is k-monotone.*

The following propositions are analogous to the Shapley-Ichiishi theorem above.

Proposition 9 *Let μ be a capacity on X. Then $\mu_\prec \in \mathcal{C}^k(\mu)$ for all compatible orders $\prec$ if and only if μ is $(k+1)$-monotone.*

Proposition 10 *Let μ be a $(k+1)$-monotone capacity. Then*

(i) If $\prec$ is strongly compatible, then $\mu_\prec$ is a vertex of $\mathcal{C}^k(\mu)$.

(ii) If $\prec$ is compatible, then $\mu_\prec$ is a vertex of $\mathcal{BC}^k(\mu)$.

However, there are many vertices that are not belief functions. Experiments conducted with the PORTA software finding vertices and facets of polyhedra show that, for example, the set of vertices of $\mathcal{C}^2(\mu)$ of the following 3-monotone capacity μ with $n = 3$

A	1	2	3	12	13	23	123
$m(A)$	0	0.1	0.2	0.1	0	0.2	0.4
$\mu(A)$	0	0.1	0.2	0.2	0.2	0.5	1

has 48 elements, whose only 3 are belief functions.

```
(  1)    0 1/10 1/ 5  1/10    0  3/ 5
(  2)    0 1/10 1/ 5  1/10  2/ 5  1/ 5
(  3)    0 1/10 1/ 5  1/ 2    0  1/ 5
(  4)    0 1/10 1/ 2  1/10  2/ 5 -1/10
(  5)    0 1/10 1/ 2  1/ 2    0 -1/10
(  6)    0 1/10 9/10  1/10    0 -1/10
(  7)    0 1/ 5 1/ 5     0    0  3/ 5
(  8)    0 1/ 5 1/ 5     0  1/ 2  1/10
(  9)    0 1/ 5 1/ 2     0  1/ 2 -1/ 5
( 10)    0 1/ 2 1/ 5     0  1/ 2 -1/ 5
( 11)    0 1/ 2 1/ 5  1/ 2    0 -1/ 5
( 12)    0 1/ 2 1/ 2     0  1/ 2 -1/ 2
( 13)    0 1/ 2 1/ 2  1/ 2    0 -1/ 2
( 14) 1/10 1/10 1/ 5     0 -1/10  7/10
( 15) 1/10 1/10 1/ 2     0  2/ 5 -1/10
( 16) 1/ 5 1/10 1/ 5 -1/10 -1/10  7/10
( 17) 1/ 5 1/10 1/ 5 -1/10  2/ 5  1/ 5
( 18) 1/ 5 1/10 1/ 5     0 -1/ 5  7/10
( 19) 1/ 5 1/10 1/ 5  1/ 2 -1/ 5  1/ 5
( 20) 1/ 5 1/10 2/ 5 -1/10  2/ 5     0
( 21) 1/ 5 1/10 9/10 -1/10 -1/10     0
( 22) 1/ 5 1/ 5 1/ 5 -1/ 5    0  3/ 5
( 23) 1/ 5 1/ 5 1/ 5 -1/ 5  1/ 2  1/10
( 24) 1/ 5 1/ 5 3/10 -1/ 5  1/ 2     0
( 25) 1/ 5 1/ 5 4/ 5 -1/ 5    0     0
( 26) 1/ 5 3/10 1/ 5  1/ 2 -1/ 5     0
( 27) 1/ 5 4/ 5 1/ 5     0 -1/ 5     0
( 28) 3/10 1/10 1/ 5 -1/10 -1/ 5  7/10
( 29) 3/10 3/10 1/ 5 -3/10  1/ 2     0
( 30) 3/10 1/ 2 1/ 5 -3/10  1/ 2 -1/ 5
( 31) 2/ 5 1/10 2/ 5  1/ 2 -2/ 5     0
( 32) 2/ 5 1/10 1/ 2  1/ 2 -2/ 5 -1/10
( 33) 1/ 2 1/ 2 1/ 2     0    0 -1/ 2
( 34) 4/ 5 1/10 1/ 5 -1/10 -1/ 5  1/ 5
( 35) 4/ 5 1/ 2 1/ 5 -3/10    0 -1/ 5
( 36) 4/ 5 4/ 5 1/ 5 -4/ 5    0     0
( 37) 9/10 1/10 1/ 2     0 -2/ 5 -1/10
( 38) 9/10 1/10 9/10     0 -9/10     0
( 39)    0 1/ 5    1     0    0 -1/ 5
```

```
( 40)    0    1 1/ 5    0    0 -1/ 5
( 41) 1/10 1/10    1    0 -1/10 -1/10
( 42) 4/ 5    1 1/ 5 -4/ 5    0 -1/ 5
( 43) 9/10 1/10    1    0 -9/10 -1/10
( 44)    1 1/10 2/ 5 -1/10 -2/ 5    0
( 45)    1 1/10 9/10 -1/10 -9/10    0
( 46)    1 3/10 1/ 5 -3/10 -1/ 5    0
( 47)    1 4/ 5 1/ 5 -4/ 5 -1/ 5    0
( 48)    0    1    1    0    0   -1
```

Let us examine more precisely the vertices induced by strongly compatible orders. In fact, there are much fewer than expected, since many strongly compatible orders lead to the same $\mu_{\prec}$ (hence the experimental result above). We can show the following.

Proposition 11 *The number of vertices of $\mathcal{C}^k(\mu)$ given by strongly compatible orders is at most $\frac{n!}{k!}$.*

We examine the case of weakly compatible orders.

Proposition 12 *Suppose μ is a $(k+1)$-monotone capacity which satisfies $\mu(\{i\}) > 0$ for all $i \in X$. Then no weakly compatible order can produce a vertex of $\mathcal{C}^k(\mu)$.*

Weakly compatible orders can produce vertices if $\mu(\{i\}) = 0$ for some $i \in X$. It suffices that it exists $B \in \mathcal{P}^k_*(X)$ such that $\mathcal{A}(B) = \emptyset$, and $i \in B$ such that $\mu(\{i\}) = 0$, and all subsets C such that $i \in C \subset B$ satisfy $m_{\prec}(B) = 0$. The above example with 48 vertices illustrates this. By Prop. 11, we know that 3 vertices are produced by the strongly compatible orders, with corresponding sequences of $\check{B}$'s:

$$1, 2, 3, 12, 13, 123 \quad \text{(this is vertex 1)}$$
$$2, 1, 3, 12, 23, 123 \quad \text{(this is vertex 2)}$$
$$3, 1, 2, 13, 23, 123 \quad \text{(this is vertex 3).}$$

Take the weakly compatible order $1 \prec 12 \prec 2 \prec 3 \prec 13 \prec 23$. Then achievable families are:

$$\mathcal{A}(1) = \{1\}, \quad \mathcal{A}(12) = \emptyset, \quad \mathcal{A}(2) = \{2, 12\},$$
$$\mathcal{A}(3) = \{3\}, \quad \mathcal{A}(13) = \{13\}, \quad \mathcal{A}(23) = \{23, 123\}.$$

This gives

A	1	2	3	12	13	23
$m_{\prec}(A)$	0	0.2	0.2	0	0	0.6

which is vertex 7.

6 The case of belief functions

Let us now assume that μ is a belief function. We try to derive results on the vertices $\mathcal{BC}^k(\mu)$. We stress the fact that even if $\mathcal{BC}^k(\mu) \subseteq \mathcal{C}^k(\mu)$, a vertex of $\mathcal{BC}^k(\mu)$ is not necessarily a vertex of $\mathcal{C}^k(\mu)$.

As shown in the previous section, for a compatible order $\prec$, the measure $\mu_{\prec}$ determines an extreme point of $\mathcal{BC}^k(\mu)$ (Proposition 10). However, not all the vertices can be obtained this way.

Example 1 *Take $|X| = 4$ and consider a belief function μ whose Möbius transform is given by:*

$$m(i) = 0.1,\ m(i,j) = 0.1,\ m(A) = 0 \text{ otherwise.}$$

Consider the capacity whose Möbius transform is given by

$$m^*(1) = 0.1, m^*(2) = 0.2, m^*(3) = 0.1, m^*(1,3) = 0.2,$$

$$m^*(4) = 0.4, m^*(A) = 0 \text{ otherwise.}$$

We will show below (Proposition 14) that this is the Möbius transform of an extreme point of $\mathcal{BC}^2(\mu)$. However, it can be checked that this capacity cannot be obtained through a compatible order $\prec$.

Remark that this example does not invalidate Conjecture 1, which is only concerned with vertices of $\mathcal{C}^k(\mu)$.

On the other hand, it can be shown that if $\prec$ is not compatible, the belief function obtained is not necessarily an extreme point:

Example 2 *Consider $|X| = 4$ and the capacity whose Möbius transform is defined by*

$$m(i) = 0.1, m(i,j) = 0.1,\ \forall i, j = 1, 2, 3, 4, i \neq j.$$

Let us consider the 2-additive case and the order $\prec$ given by

$$\{1\} \prec \{1,2\} \prec \{2\} \prec \{1,3\} \prec \{2,3\} \prec$$

$$\{3\} \prec \{3,4\} \prec \{4\} \prec \{1,4\} \prec \{2,4\}.$$

For this order, the corresponding Möbius transform obtained through Equation (1) is:

$$m^*(1) = 0.1, m^*(2) = 0.2, m^*(3) = 0.3, m^*(4) = 0.2,$$

$$m^*(1,4) = 0.1, m^*(2,4) = 0.1, m^*(A) = 0, \text{ otherwise.}$$

However, this measure is not an extreme point of $\mathcal{BC}^2(\mu)$. It suffices to consider μ_1, μ_2 given by the following Möbius transforms:

$$m_1(4) = 0.1, m_1(1,4) = 0.2, m_1(A) = m^*(A), \text{ otherwise.}$$

$$m_2(4) = 0.3, m_2(1,4) = 0, m_2(A) = m^*(A), \text{ otherwise.}$$

It is clear that $m^*(A) = \dfrac{m_1(A) + m_2(A)}{2}, \forall A \subseteq X$. Moreover, they both dominate μ, whence the result.

This leads us to look for an alternative assignment. For a given order $\prec$, let us define recursively the following set function:

$$\underline{m}_{\prec}(B) := \max_{A\in\mathcal{A}(B)}\{\mu(A) - \sum_{K\subseteq A, K\in\mathcal{P}^k_*(X), K\prec B}\underline{m}_{\prec}(K), 0\} \quad (2)$$

if $\mathcal{A}(B) \neq \emptyset$, being 0 otherwise.

Proposition 13 *If $\prec$ is compatible, then the set functions defined in Equations* (1) *and* (2) *coincide.*

Let us study the properties of $\underline{\mu}_{\prec}$, the inverse transform of $\underline{m}_{\prec}$. Remark that $\underline{\mu}_{\prec} \geq \mu$ by construction of $\underline{\mu}_{\prec}$. Moreover, as $\underline{m}_{\prec}$ is non-negative, $\underline{\mu}_{\prec}$ is monotone. However, $\underline{\mu}_{\prec}$ is not necessarily a capacity, as it could be the case that $\underline{\mu}_{\prec}(X) > 1$ (see next example).

Example 3 *Let $|X| = 4$ and consider $\mu \in \mathcal{FM}(X)$ whose Möbius transform is given by:*

$$m(i) = 0.1,\ m(i,j) = 0.1,\ m(A) = 0 \text{ otherwise.}$$

Consider the order $\prec$ defined by:

$$1 \prec 1,2 \prec 2 \prec 3 \prec 2,3 \prec 1,3 \prec 2,4 \prec 4 \prec 1,4 \prec 3,4.$$

Then,

$$\underline{m}_{\prec}(1) = 0.1, \underline{m}_{\prec}(1,2) = 0, \underline{m}_{\prec}(2) = 0.2, \underline{m}_{\prec}(3) = 0.1,$$

$$\underline{m}_{\prec}(2,3) = 0, \underline{m}_{\prec}(1,3) = 0.2, \underline{m}_{\prec}(2,4) = 0, \underline{m}_{\prec}(4) = 0.1,$$
$$\underline{m}_{\prec}(1,4) = 0.2, \underline{m}_{\prec}(3,4) = 0.2$$

Therefore, $\underline{\mu}_{\prec}(X) = \sum_{A\subseteq X}\underline{m}_{\prec}(A) = 1.1.$

However, this procedure always leads to a belief function for compatible orders (Proposition 8).

Proposition 14 *Suppose that $\underline{\mu}_{\prec}$, whose Möbius transform $\underline{m}_{\prec}$ is obtained through Eq.* (2)*, is a normalized measure. Then, $\underline{\mu}_{\prec}$ is an extreme point of $\mathcal{BC}^k(\mu)$.*

If the order is compatible, this proposition is just Proposition 10. However, there are vertices of $\mathcal{BC}^k(\mu)$ that cannot be obtained by this procedure, as shown in the following example.

Example 4 *Let $|X| = 4$ and consider $\mu \in \mathcal{BEL}(X)$ whose Möbius transform is given by*

$$m(1,2) = 0.3, m(1,3) = 0.2, m(1,4) = 0.2, m(1,2,3) = 0.1,$$

$$m(1,2,4) = 0.1, m(1,3,4) = 0.1, m(A) = 0 \text{ otherwise.}$$

Consider μ^* whose Möbius transform is given by:

$$m^*(1,2) = 0.35, m^*(1,3) = 0.25, m^*(1,4) = 0.25,$$

$$m^*(X) = 0.15, m(A) = 0 \text{ otherwise.}$$

Thus, $\mu^* \in \mathcal{BEL}(X)$. Moreover, $\mu^* \geq \mu$. Let us now prove that μ^* is a vertex of $\mathcal{BC}^4(\mu)$. For this, let us suppose that there are $\mu_1, \mu_2 \in \mathcal{BC}^4(\mu)$ satisfying

$$\mu^* = \alpha\mu_1 + (1-\alpha)\mu_2,\ \alpha \in (0,1).$$

As μ_1, μ_2 and μ^* are belief functions, it follows that $m_1(A) = m_2(A) = 0$ when $m^*(A) = 0$. Therefore, the only subsets that can attain non-null Möbius inverse for μ_1 and μ_2 are $\{1,2\}, \{1,3\}, \{1,4\}$ and X.

On the other hand, as $\mu^* = \mu$ for $\{1,2,3\}, \{1,3,4\}$ and $\{1,2,4\}$, so are μ_1 and μ_2. But this implies that μ_1 and μ_2 are in the set of capacities whose Möbius transform m' satisfies:

$$\begin{array}{rll} m'(1,2)+ & m'(1,3) = & 0.6 \\ m'(1,2)+ & m'(1,4) = & 0.6 \\ m'(1,3)+ & m'(1,4) = & 0.5 \end{array}.$$

Since the determinant of the system is non-null, there is only one solution for the values of $m'(1,2), m'(1,3)$ and $m'(1,4)$, and this solution is given by the values of m^*.

Finally, as $\sum_{A\subseteq X} m_1(A) = \sum_{A\subseteq X} m_2(A) = \sum_{A\subseteq X} m^*(A) = 1$, we conclude that $m_1(X) = m_2(X) = m^*(X) = 0.15$. Hence, μ^* is an extreme point of $\mathcal{BC}^4(\mu)$.

However, there is no order $\prec$ on $\mathcal{P}^4_*(X)$ such that $\mu^* = \underline{\mu}_{\prec}$. First, remark that if such an order exists, then X is the last subset in the order $\prec$, as $m^*(X) \neq 0$.

The other subsets which need nonempty achievable families are $\{1,2\}, \{1,3\}$ and $\{1,4\}$. On the other hand, if the achievable family is nonempty, then there is a subset A in it such that $\underline{\mu}_{\prec}(A) = \mu(A)$ by construction. For μ^*, the equality $\mu^*(A) = \mu(A)$ is attained at:

$$\begin{array}{ll} \text{Supersets of } \{1,2\} \rightarrow & \{1,2,3\}, \{1,2,4\}, X \\ \text{Supersets of } \{1,3\} \rightarrow & \{1,2,3\}, \{1,3,4\}, X \\ \text{Supersets of } \{1,4\} \rightarrow & \{1,2,4\}, \{1,3,4\}, X. \end{array}$$

Suppose for example that $\{1,2\} \prec \{1,3\} \prec \{1,4\}$. In this case, neither $\{1,2,3\}$ nor $\{1,2,4\}$ belong to $\mathcal{A}(1,2)$, and μ^* cannot be recovered. The same can be done for the other possibilities. Therefore, μ^* cannot be obtained by this procedure.

7 Some results on the derivation of vertices of $\mathcal{BC}^k(\mu)$ from the core

Let us now treat the problem from a different point of view. Based on the results of Section 4, the following can be proved:

Proposition 15 *Let $\mu' \in \bigcup_{\mu^*\in\mathcal{M}_{\Lambda^k_{\subseteq,+}}(\mu)} \mathcal{M}_{\Lambda^k_{\supseteq,+}}(\mu^*)$. Then, there exists $P \in \mathcal{M}_{\Lambda^1_{\subseteq,+}}(\mu)$ such that $m' \in \mathcal{M}_{\Lambda^k_{\supseteq,+}}(P)$.*

Corollary 1 *Let μ' be in the conditions of Proposition 15. Then,*

$$\mu' \in \bigcup_{P \in \mathcal{C}^1(\mu)} \mathcal{M}_{\Lambda^k_{\supseteq,+}}(P).$$

Corollary 2 *Suppose $\mu' \in \mathcal{BC}^k(\mu)$. Then, $\mu' \in \bigcup_{P \in \mathcal{C}^1(\mu)} \mathcal{M}_{\Lambda^k_{\supseteq,+}}(P)$.*

However, it is not true that any extreme point of $\mathcal{BC}^k(\mu)$ can be obtained through $\Lambda^k_{\supseteq,+}$ from a suitable extreme point of the core, as next example shows.

Example 5 *Let us consider $|X| = 4$ and the belief function μ whose Möbius transform is given by:*

$$m(i) = 0.1, m(i,j) = 0.1, \forall i,j \in X,$$

and $m(A) = 0$ otherwise. Consider now μ^* whose Möbius transform is given by

$$m^*(i) = 0.2, \forall i \in X, m^*(X) = 0.2,$$

and $m^*(A) = 0$ otherwise. It is easy to check that $\mu^* \geq \mu$. Note that μ^* cannot be generated by any extreme probability, as if P is a vertex of $\mathcal{BC}^1(\mu)$, we know that P is determined by a permutation σ on X. If i is the first element according to σ, it is $P(i) = m(i) = 0.1$, and then, $m^*(i) = 0.2$ cannot be generated from $m(i)$.

Finally, it can be shown that μ^* is a vertex of $\mathcal{BC}^4(\mu)$. To see this, it suffices to remark that if $\mu^* = \alpha\mu_1 + (1-\alpha)\mu_2$, for some $\mu_1, \mu_2 \in \mathcal{BC}^4(\mu)$, it follows that $m_1(A) = m_2(A)$ when $m^*(A) = 0$. Moreover, $\mu^*(A) = \mu(A)$ if $|A| \geq 3$, and thus, so are μ_1 and μ_2. However, the only measure satisfying these two conditions is μ^*.

Consider $\mu \in \mathcal{BEL}(X)$. We already know by Corollary 2 that any $\mu^* \in \mathcal{BC}^k(\mu)$ can be obtained from the set of dominant probabilities. On the other hand, as remarked in Section 4, $\mathcal{BC}^1(\mu) = \mathcal{M}_{\Lambda^1_{\subseteq,+}}(\mu)$. Moreover, this set is a convex polytope whose vertices are the marginal worth vectors. We can obtain some vertices of $\mathcal{BC}^k(\mu)$ through the following result:

Proposition 16 *Let μ^* be the Möbius transform of an extreme point m^* of $\mathcal{M}_{\Lambda^k_{\supseteq,+}}(P) \cap \mathcal{BC}^k(\mu)$, with P an extreme point of $\mathcal{BC}^1(\mu)$. Then, μ^* is an extreme point of $\mathcal{BC}^k(\mu)$.*

References

[1] J.-P. Barthélemy. Monotone functions on finite lattices: an ordinal approach to capacities, belief and necessity functions. In J. Fodor, B. De Baets, and P. Perny, editors, *Preferences and Decisions under Incomplete Knowledge*, pages 195–208. Physica Verlag, 2000.

[2] A. Chateauneuf and J.-Y. Jaffray. Some characterizations of lower probabilities and other monotone capacities through the use of Möbius inversion. *Mathematical Social Sciences*, 17:263–283, 1989.

[3] G. Choquet. Theory of capacities. *Annales de l'Institut Fourier*, 5:131–295, 1953.

[4] A. P. Dempster. Upper and lower probabilities induced by a multivalued mapping. *Ann. Math. Statist.*, 38:325–339, 1967.

[5] M. Grabisch. k-order additive discrete fuzzy measures and their representation. *Fuzzy Sets and Systems*, 92:167–189, 1997.

[6] M. Grabisch. On lower and upper approximation of fuzzy measures by k-order additive measures. In *7th Int. Conf. on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'98)*, pages 1577–1584, Paris, France, July 1998.

[7] M. Grabisch. Upper approximation of non-additive measures by k-additive measures — the case of belief functions. In *Proc. of the 1st Int. Symp. on Imprecise Probabilities and Their Applications*, Ghent, Belgium, June 1999.

[8] M. Grabisch. On upper approximation of fuzzy measures by k-order additive measures. In B. Bouchon-Meunier, R. R. Yager, and L. Zadeh, editors, *Information, Uncertainty, Fusion*, pages 105–118. Kluwer Scientific Publ., 2000.

[9] M. Grabisch and P. Miranda. A study of the k-additive core of capacities through achievable families. In *SCIS-ISIS 2006, 3nd Int. Conf. on Soft Computing and Intelligent Systems and 7th Int. Symp. on Advanced Intelligent Systems*, Yokohama, Japan, September 2006.

[10] T. Ichiishi. Super-modularity: applications to convex games and to the greedy algorithm for LP. *J. Econom. Theory*, 25:283–286, 1981.

[11] P. Miranda, M. Grabisch, and P. Gil. Dominance of fuzzy measures by k-additive belief functions. In *Int. Fuzzy Systems Association World Congress (IFSA 2003)*, pages 143–146, Istanbul, Turkey, June 2003.

[12] G. Shafer. *A Mathematical Theory of Evidence.* Princeton Univ. Press, 1976.

[13] L. S. Shapley. Core of convex games. *Int. J. Game Theory*, 1:11–26, 1971.

[14] M. Sugeno. *Theory of fuzzy integrals and its applications.* PhD thesis, Tokyo Institute of Technology, 1974.

[15] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, London, 1991.