Uncertainty as a Modality over T-norm Based Logics

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Abstract

In this work we propose a general approach for representing uncertainty measures in the framework of t-norm based logics. This approach is extended also to classes of measures like probability, possibility, necessity, lower and upper probability. We show that, under certain conditions, the logical consistency of a theory of uncertainty is tantamount to the coherence of a related assessment of rational values. Finally, we characterize the basic requirements that guarantee the compactness of coherent assessments.

Keywords: T-norm based logics, Fuzzy measures, Coherence, Compactness.

1 Introduction

Measures of uncertainty aim at formalizing the strength of our beliefs in the occurrence of some events by assigning to those events a degree of uncertainty. From the mathematical point of view a measure of uncertainty is a real-valued function that gives an event a value from the real unit interval $[0, 1]$. A well-known example is given by probability measures which try to capture our degree of confidence in the occurrence of events by real-valued assessments. Esteva, Hájek, and Godo proposed in [16, 12] a new interpretation of measures of uncertainty in the framework of t-norm based logics. Given a sentence as "The proposition φ is plausible (probable, believable)", its degrees of truth can be interpreted as the degree of uncertainty of the proposition φ. Indeed, the higher is our degree of confidence in φ, the higher the degree of truth of the above sentence will result. In some sense, the predicate "is plausible (believable, probable)" can be regarded as a modal operator over the proposition φ. Then, given a measure of uncertainty μ, we can define modal many-valued formulas $\kappa(\varphi)$, whose interpretation is given by a real number corresponding to the degree of uncertainty assigned to φ under μ. Furthermore, we can translate the peculiar axioms governing the behavior of an uncertainty measure into formulas of a certain t-norm based logic, depending on the operations we need to represent.

Previous particular results concerning the representation of measures of uncertainty were presented in several works. We can mention the treatment of probability measures, necessity measures and belief functions proposed by Esteva, Hájek, and Godo in [16, 15, 12], [16], and [13], respectively; the treatment of conditional probability proposed by the present author and Godo in [14]; the treatment of (generalized) conditional possibility and necessity given by the present author in [22]; and finally the treatment of simple and conditional non-standard probability given by Flaminio and Montagna in [10].

Here, our aim will consist in giving a general and comprehensive treatment of the representation of measures of uncertainty. In particular, we will show how it is possible to represent classes of measures such as probabilities, lower and upper probabilities, possibilities and necessities. Important properties of the functions of t-norm based logics will then be useful in order to prove relevant features of the classes of measures represented.

Recall that, given a set W of possible situations, a *fuzzy measure* [23] is a mapping μ from the Boolean algebra of subsets of W into the real unit interval $[0, 1]$ satisfying the following properties: (i) $\mu(\bot) = 0$, (ii) $\mu(\top) = 1$, (iii) if $\vdash \varphi \to \psi$ then $\mu(\varphi) \leq \mu(\psi)$.

Besides such common properties, each kind of fuzzy measure differs from the others in how the measure associated to compound propositions is computed from the marginal ones. In other words, what specifies the behavior of a fuzzy measure is the way how from assessments of uncertainty concerning separated events we can determine the degree associated to their combination. In a certain sense we can say that classes of fuzzy measures are characterized by the satisfaction of some compositional properties. However, it is well known that fuzzy measures cannot be fully compositional. This means that the degree of confidence in any compound proposition φ cannot be always computed from the degree assigned to its subformulas (see [6]).

Now, we briefly recall the definitions of some classes of measures we will deal with in the following sections.

Probability measures [21] are fuzzy measures defined over a *σ-algebra*[1]. Probabilities measures over σ-algebras are fuzzy measures which satisfy the law of countable additivity, i.e.: if $\bigwedge_{i=1}^{n} \varphi_i \leftrightarrow \bot$ then $\mu\left(\bigvee_{i=1}^{n} \varphi_i\right) = \sum_{i=1}^{n} \varphi_i$, for all $n \in \mathbb{N}$. Those measures are also called *countably additive probabilities.* If we do not require the algebra to be closed under countable unions, we define the class of probability measures, called *finitely additive probabilities*, as the class of all those fuzzy measures (over Boolean algebras) which satisfy the law of finite additivity: if $\vdash \varphi \wedge \psi \leftrightarrow \bot$ then $\mu(\varphi \vee \psi) = \mu(\varphi) + \mu(\psi)$. Here we deal with finitely additive probabilities only. We denote the class of probability measures by $\mathcal{P}$, and each measure in $\mathcal{P}$ by P.

Possibility measures (see [24, 5]) are a class of fuzzy measures satisfying the following law of composition w.r.t. the maximum t-conorm: $\mu(\varphi \vee \psi) = \max(\mu(\varphi), \mu(\psi))$. We denote the class of possibility measures by $\mathcal{P}i$, and each measure in $\mathcal{P}i$ by Π. Similarly, *Necessity measures* [5] are fuzzy measures satisfying the following law of composition w.r.t. the minimum t-norm: $\mu(\varphi \wedge \psi) = \min(\mu(\varphi), \mu(\psi))$. We denote the class of necessity measures by $\mathcal{N}$, and each measure in $\mathcal{N}$ by N. Possibility and Necessity measures are dual in the sense that, given a possibility measure Π (a necessity measure N), one can derive its dual necessity measure (possibility measure) from it by means of the standard involutive negation $n_s(x) = 1 - x$. Indeed, $N(\varphi) = 1 - \Pi(\neg\varphi)$ and $\Pi(\varphi) = 1 - N(\neg\varphi)$. Probability measures are, on the contrary, self-dual, since the dual measure of a probability measure still is a probability measure: $P(\varphi) = 1 - P(\neg\varphi)$.

Given a set of probability measures P_i over the same Boolean algebra, the *upper probability* $P^{\triangle}(\varphi)$ is defined as $\sup\{P_i(\varphi)\}$ and the *lower probability* $P_{\triangledown}(\varphi)$ is defined as $\inf\{P_i(\varphi)\}$ (see [17]). Upper and lower probabilities are dual, since from an upper probability we can define a lower probability as $P_{\triangledown}(\varphi) = 1 - P^{\triangle}(\neg\varphi)$, and viceversa.

Upper and lower probabilities can be also seen as classes of fuzzy measures. Indeed, as shown by Anger and Lembcke in [1], any upper probability is a fuzzy measure μ such that for all natural numbers m, n, k, and all $\varphi_1, \ldots, \varphi_m$, if $\{\{\varphi_1, \ldots, \varphi_m\}\}$ is an (n, k)-cover[2] of $(\varphi, \top)$, then

$$(\sharp) \qquad k + n\mu(\varphi) \leq \sum_{i=1}^{m} \mu(\varphi_i).$$

Notice that Halpern and Pucella proved in [18] that when the sample space is finite there are only finitely many instances of the the above property. Indeed, there exists constants $k_0, k_1, \ldots$ such that if W is a finite set, for all natural numbers $m, n, k \leq k_{|W|}$, and all $\varphi_1, \ldots, \varphi_m$, if $\{\{\varphi_1, \ldots, \varphi_m\}\}$ is an (n, k)-cover of $(\varphi, \top)$, then $(\sharp)$ holds.

Similarly we can see any lower probability as a fuzzy measure μ such that for all natural numbers m, n, k, and all $\varphi_1, \ldots, \varphi_m$, if $\{\{\varphi_1, \ldots, \varphi_m\}\}$ is an (n, k)-cover of $(\varphi, \top)$, then

$$(\sharp\sharp) \qquad k + n\mu(\varphi) \geq \sum_{i=1}^{m} \mu(\varphi_i).$$

This paper is organized as follows. In the next section we introduce the uncertainty logic $\mathcal{M}(\mathcal{L})$ based on a t-norm based logic $\mathcal{L}$ in order to reason about the class $\mathcal{M}$ of fuzzy measures. We give a completeness result, and show how this logic can be easily extended to represent classes of measures like probabilities, possibilities, necessities, lower and upper probabilities. Furthermore, we briefly mention expressive power issues related to the possibility of defining the notion of conditioning and having rational truth-constants in the language. In Section 3 we deal with the notions of coherence and compactness of rational assessments and show their connections with suitable theories defined in the logical framework. We end with some final remarks.

Notice that, due to space reasons, we do not provide any background notion concerning t-norm based logics. The interested reader can find all the required concepts in the papers cited throughout this work.

[1] Recall that, given a set W, a σ-algebra is a collection of subsets of W closed under complementation and countable unions.

[2] A proposition φ is said to be *covered* n times by a multiset $\{\{\varphi_1, \ldots, \varphi_m\}\}$ of propositions if every situation in which φ is true makes true at least n propositions from $\varphi_1, \ldots, \varphi_m$ as well. An (n, k)-*cover* of $(\varphi, \top)$ is a multiset $\{\{\varphi_1, \ldots, \varphi_m\}\}$ that covers $\top$ k times and covers φ $n + k$ times.

2 Logics of Uncertainty

2.1 The base logic

Let $\mathcal{L}$ be any t-norm based logic [15, 7], or any of its expansions, and let $\mathcal{M}$ be any class of fuzzy measures. $\mathcal{M}(\mathcal{L})$ is built up over $\mathcal{L}$ extending its language by including modal formulas which represent the uncertainty given by a fuzzy measure $\mu \in \mathcal{M}$. We define the language in two steps. First, we have classical Boolean formulas φ, ψ, etc., defined in the usual way from the classical connectives $(\wedge, \neg)$ and from a countable set V of propositional variables $p, q, \ldots$, etc. The set of Boolean formulas is denoted by L. Moreover given any set $D \subseteq L$, we denote by $Con(D)$ the set of sentences which logically follow from D in classical logic. Moreover, $Taut(L)$ will denote the set of classical tautologies.

Elementary modal sentences are formulas of the form $\kappa(\varphi)$, where κ is a unary operator taking as arguments Boolean sentences. Compound modal formulas are built by means of the $\mathcal{L}$-connectives. Nested modalities are not allowed.

Definition 2.1 The axioms of the logic $\mathcal{M}(\mathcal{L})$ are the following:

(i) The set $Taut(L)$ of classical Boolean tautologies

(ii) Axioms of $\mathcal{L}$ for modal formulas

(iii) The following axiom:

$(\mathcal{M}1) \qquad \neg\kappa(\bot)$

Deduction rules of $\mathcal{M}(\mathcal{L})$ are those of $\mathcal{L}$, plus:

(iv) *modalization*: from $\vdash \varphi$ (i.e. φ is derivable in Classical Logic) derive $\kappa(\varphi)$

(v) *monotonicity*: from $\vdash \varphi \to \psi$ derive $\kappa(\varphi) \to \kappa(\psi)$.

The language we have defined clearly is a hybrid language. Indeed, any theory (set of formulas) we will deal with will be of the form $\Gamma = D \cup T$, where D contains only non-modal formulas and T contains only modal formulas. In the following D will always refer to a recursive non-modal theory. Notice that there is no direct interaction between non-modal and modal formulas, with the exception of the application of the above rules of inference. The role of modalization and monotonicity only consists in generating new modal formulas which can then be used in the deduction. Therefore, we are led to define in $\mathcal{M}(\mathcal{L})$ the notion of proof from a theory, written $\vdash_{\mathcal{M}(\mathcal{L})}$, in a non-standard way, at least when the theory contains non-modal formulas.

Definition 2.2 The proof relation $\vdash_{\mathcal{M}(\mathcal{L})}$ between sets of formulas and formulas is defined by:

1. $D \cup T \vdash_{\mathcal{M}(\mathcal{L})} \varphi$ if $\varphi \in Con(D)$;
2. $T \vdash_{\mathcal{M}(\mathcal{L})} \Phi$ if Φ follows from T in the usual way from the above axioms and rules;
3. $D \cup T \vdash_{\mathcal{M}(\mathcal{L})} \Phi$ if $T \cup D^{\mathcal{M}} \vdash_{\mathcal{M}(\mathcal{L})} \Phi$;

where $D^{\mathcal{M}} = \{\kappa(\varphi) : \varphi \in Con(D)\}$.

We now define the semantics for $\mathcal{M}(\mathcal{L})$ by introducing *$\mathcal{M}$-Kripke structures.*

Definition 2.3 A $\mathcal{M}$-Kripke model is a structure $K = \langle W, \mathcal{U}, e, \mu \rangle$, where:

- W is a non-empty set of possible worlds.
- $\mathcal{U}$ is a Boolean algebra of subsets of W.
- $e : V \times W \to \{0, 1\}$ is a *Boolean* evaluation of the propositional variables, that is, $e(p, w) \in \{0, 1\}$ for each propositional variable $p \in V$ and each world $w \in W$. Any given truth-evaluation $e(\cdot, w)$ is extended to Boolean propositions as usual. For a Boolean formula φ, we will denote by $[\varphi]_W$ the set of worlds in which φ is true, i.e. $[\varphi]_W = \{w \in W \mid e(\varphi, w) = 1\}$.
- $\mu : \mathcal{U} \to [0, 1]$ is a fuzzy measure over $\mathcal{U}$, such that $[\varphi]_W$ is μ-measurable for any non-modal φ.
- $e(\cdot, w)$ is extended to elementary modal formulas by defining

  $$e(\kappa(\varphi), w) = \mu([\varphi]_W),$$

  and to arbitrary modal formulas according to the $\mathcal{M}(\mathcal{L})$-semantics.

A structure K is a model for Φ, written $K \models \Phi$, if $e^K(\Phi) = 1$. If T is a set of formulas, we say that K is a model of T if $K \models \Phi$ for all $\Phi \in T$. The notion of logical entailment relative to a class of structures $\mathcal{K}$, written $\models_{\mathcal{K}}$, is then defined as follows:

$$\Gamma \models_{\mathcal{K}} \Phi \text{ iff } K \models \Phi \text{ for each } K \in \mathcal{K} \text{ model of } \Gamma.$$

If $\mathcal{K}$ denotes the whole class of $\mathcal{M}$-Kripke structures we shall write $\Gamma \models_{\mathcal{M}(\mathcal{L})} \Phi$. When $\models_{\mathcal{K}} \Phi$ holds we will say that Φ is *valid* in $\mathcal{K}$, i.e. when Φ gets value 1 in all structures $K \in \mathcal{K}$.

Proposition 2.4 (Soundness) *The logic $\mathcal{M}(\mathcal{L})$ is sound with respect to the class of $\mathcal{M}$-Kripke structures.*

Let $D \subset L$ be any given non-modal (propositional) theory (possibly empty). For any $\varphi, \psi \in L$, define $\varphi \sim_D \psi$ iff $\varphi \leftrightarrow \psi$ follows from D in classical propositional logic, i.e. if $\varphi \leftrightarrow \psi \in Con(D)$. The relation $\sim_D$ is an equivalence relation in L and $[\varphi]_D$ will denote the equivalence class of φ. Obviously, the quotient set $L/_{\sim_D}$ forms a Boolean algebra which is isomorphic to a subalgebra $\mathcal{B}(\Omega_D)$ of the power set of the set Ω_D of Boolean interpretations of the crisp language L which are model of D. For each $\varphi \in L$, we shall identify the equivalence class $[\varphi]_D$ with the set $\{\omega \in \Omega_D \mid \omega(\varphi) = 1\} \in \mathcal{B}(\Omega_D)$ of models of D that make φ true. We shall denote by $\mathcal{M}(D)$ the set of fuzzy measures defined over $L/_{\sim_D}$ or, equivalently, on $\mathcal{B}(\Omega_D)$.

Notice that each fuzzy measure $\mu \in \mathcal{M}(D)$ induces a $\mathcal{M}$-Kripke structure $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\mu, \mu \rangle$ where $e_\mu(p, \omega) = \omega(p) \in \{0, 1\}$ for each $\omega \in \Omega_D$ and each propositional variable p. We shall denote by $\mathcal{K}_D$ the class of $\mathcal{M}$-Kripke structures which are models of D, and by $\mathcal{MS}(D)$ the class of $\mathcal{M}$-Kripke models $\{(\Omega_D, \mathcal{B}(\Omega_D), e_\mu, \mu) \mid \mu \in \mathcal{M}(D)\}$. Obviously, $\mathcal{MS}(D) \subset \mathcal{K}_D$.

Abusing the language, we will say that a fuzzy measure $\mu \in \mathcal{M}(D)$ is a *model* of a modal theory T whenever the induced Kripke structure $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\mu, \mu \rangle$ is a model of T (obviously $\langle \Omega_D, \mathcal{B}(\Omega_D), e_\mu, \mu \rangle$ is a model of D as well).

Given the above notions, we now prove a completeness result for $\mathcal{M}(\mathcal{L})$.

Theorem 2.5 ((Finite) Strong completeness) *Let $\mathcal{L}$ be any t-norm based logic (or any of its expansions). If $\mathcal{L}$ is finitely strongly standard complete, then let T be a finite modal theory over $\mathcal{M}(\mathcal{L})$, D a finite non-modal theory and Φ a modal formula. Then*

$$T \cup D \vdash_{\mathcal{M}(\mathcal{L})} \Phi \textit{ iff } e_\mu(\Phi) = 1$$

for each fuzzy measure $\mu \in \mathcal{M}(D)$ model of T.

Moreover, if $\mathcal{L}$ is strongly standard complete the same holds for infinite theories.

2.2 Classes of measures

We now see how we can easily extend the above logic in order to treat particular classes of measures of uncertainty. Let $\mathcal{L}$ be any t-norm based logic (or any of its expansions), and let $\mathcal{M}'$ be a class of fuzzy measures. We say that $\mathcal{L}$ is *compatible* with $\mathcal{M}'$ if the real valued operations needed to compute values of compound propositions are definable in $\mathcal{L}$. Clearly, a careful analysis of which functions are definable in certain t-norm based logics can help us know which among those logics are suitable for representing certain classes of measures.

Probability. As for probabilities, we need the sum and the standard involutive negation, which are only available in expansions of the Łukasiewicz logic [15], that are then the only t-norm based logics compatible with probability measures.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{P}$. Then, the logic $\mathcal{P}(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the following axioms:

$$(\mathcal{M}2)\ \kappa(\varphi \to \psi) \leftrightarrow (\kappa(\varphi) \to \kappa(\varphi)),$$

$$(\mathcal{M}3)\ \kappa(\varphi \vee \psi) \leftrightarrow (\kappa(\varphi) \to \kappa(\varphi \wedge \psi)) \to \kappa(\psi),$$

$$(\mathcal{M}4)\ \kappa(\neg\varphi) \leftrightarrow \neg\kappa(\varphi).$$

Notice that in presence of axiom $(\mathcal{M}2)$ the monotonicity rule is derivable.

$\mathcal{P}$-Kripke models are $\mathcal{M}$-Kripke models where μ is a finitely additive probability measure.

Possibility and Necessity. As for possibility measures we only need the minimum t-norm, hence every t-norm based logic is compatible with $\mathcal{P}i$.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{P}i$. Then, the logic $\mathcal{P}i(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the following axiom:

$$(\mathcal{M}5)\ \kappa(\varphi \vee \psi) \leftrightarrow \kappa(\varphi) \vee \kappa(\psi).$$

$\mathcal{P}i$-Kripke models are $\mathcal{M}$-Kripke models where μ is a possibility measure.

As for necessity measures we only need the maximum t-conorm [20], hence every t-norm based logic is compatible with $\mathcal{N}$.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{N}$. Then, the logic $\mathcal{N}(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the following axiom:

$$(\mathcal{M}6)\ \kappa(\varphi \wedge \psi) \leftrightarrow \kappa(\varphi) \wedge \kappa(\psi).$$

$\mathcal{N}$-Kripke models are $\mathcal{M}$-Kripke models where μ is a necessity measure.

Lower and Upper Probability. As for upper probabilities, notice that the condition $(\sharp)$ is equivalent to $\frac{k}{m} + \frac{n}{m}\mu(\varphi) \leq \sum_{i=1}^{m} \frac{\mu(\varphi_i)}{m}$, given that $n, k \leq m$. It is then clear that $\sum_{i=1}^{m} \frac{\mu(\varphi_i)}{m} \leq 1$, and so it makes sense to rely on t-norm based logics. It is evident that a logic is compatible with the class $\mathcal{P}^\Delta$ only if it allows the representation rational numbers, the product of rationals and formulas, and addition. Thus, any extension (or expansion) of RŁ [11], or RPPŁ$'_\Delta$ [19] represents a suitable choice. Furthermore, the presence of the standard involutive negation makes possible to define also lower probabilities.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{P}^\Delta$. The logic $\mathcal{P}^\Delta(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the rule (UP): if

$$\varphi \to \bigvee_{\left(\substack{J\subseteq\{1,\ldots,m\}\\|J|=k+n}\right)} \bigwedge_{j\in J}\varphi_j, \quad \text{and} \quad \bigvee_{\left(\substack{J\subseteq\{1,\ldots,m\}\\|J|=k}\right)} \bigwedge_{j\in J}\varphi_j$$

are propositional tautologies, then derive

$$k.\delta_m \oplus n.\delta_m\kappa(\varphi) \to \bigoplus_{j=1}^{m}\delta_m\kappa(\varphi_j),$$

if $\mathcal{L}$ is an extension (expansion of RŁ), or derive

$$\overline{\tfrac{k}{m}} \oplus \left(\overline{\tfrac{n}{m}} *_\pi \kappa(\varphi)\right) \to \bigoplus_{j=1}^{m}\left(\overline{\tfrac{1}{m}} *_\pi \kappa(\varphi_j)\right),$$

if $\mathcal{L}$ is an extension (expansion) of RPPŁ$'_\Delta$.

The semantics for $\mathcal{P}^\Delta$ is given by $\mathcal{P}^\Delta$-Kripke models, i.e. $\mathcal{M}$-Kripke models where μ is an upper probability measure.

As for the class of lower probability measures, given that they are fuzzy measures characterized by (♯♯), it is obvious that the logics compatible with $\mathcal{P}^\Delta$ are the same that are compatible with $\mathcal{P}_\nabla$. Furthermore, notice that (♯♯) is equivalent to $\sum_{i=1}^{m}\frac{\mu(\varphi_i)}{m} \ominus \frac{k}{m} \leq \frac{n}{m}\mu(\varphi)$. It is then clear that $\frac{n}{m}\mu(\varphi) \leq 1$, and so, again, it makes sense to rely on t-norm based logics.

Let $\mathcal{L}$ be a t-norm based logic compatible with $\mathcal{P}_\nabla$. The logic $\mathcal{P}_\nabla(\mathcal{L})$ is obtained from $\mathcal{M}(\mathcal{L})$ by adding the rule (LP): if

$$\varphi \to \bigvee_{\left(\substack{J\subseteq\{1,\ldots,m\}\\|J|=k+n}\right)} \bigwedge_{j\in J}\varphi_j, \quad \text{and} \quad \bigvee_{\left(\substack{J\subseteq\{1,\ldots,m\}\\|J|=k}\right)} \bigwedge_{j\in J}\varphi_j$$

are propositional tautologies, then derive

$$\bigoplus_{j=1}^{m}\delta_m\kappa(\varphi_j) \ominus k.\delta_m \to n.\delta_m\kappa(\varphi),$$

if $\mathcal{L}$ is an extension (expansion of RŁ), or derive

$$\bigoplus_{j=1}^{m}\left(\overline{\tfrac{1}{m}} *_\pi \kappa(\varphi_j)\right) \ominus \overline{\tfrac{k}{m}} \to \left(\overline{\tfrac{n}{m}} *_\pi \kappa(\varphi)\right),$$

if $\mathcal{L}$ is an extension (expansion) of RPPŁ$'_\Delta$.

The semantics for $\mathcal{P}_\nabla$ is given by $\mathcal{P}_\nabla$-Kripke models, i.e. $\mathcal{M}$-Kripke models where μ is a lower probability measure.

We can now prove the following completeness theorem.

Theorem 2.6 *Let $\mathcal{L}$ be any t-norm based logic, and let $\mathcal{M}'$ be any class of measures among $\mathcal{P}$, $\mathcal{P}i$, $\mathcal{N}$, $\mathcal{P}^\Delta$, and $\mathcal{P}_\nabla$. If the following conditions are satisfied:*

1. *$\mathcal{L}$ is compatible with $\mathcal{M}'$,*
2. *$\mathcal{L}$ is (finitely) strongly standard complete,*

then $\mathcal{M}'(\mathcal{L})$ is (finitely) strongly standard complete.

2.3 Expansions: truth-constants and definability of conditional measures

The expressive power of the above logics can be significantly increased if we aim at representing other features of certain fuzzy measures. First of all, notice that relying on a t-norm based logic including rational truth-constants (see [9]) would allow to represent several statements concerning assessment of rational values like

- "the degree of uncertainty of φ is 0.8" as $\kappa(\varphi) \leftrightarrow \overline{0.8}$,
- "the degree of uncertainty of φ is at least 0.8" as $\overline{0.8} \to \kappa(\varphi)$,
- "the degree of uncertainty of φ is at most 0.8" as $\kappa(\varphi) \to \overline{0.8}$.

Not having truth constants would yield a purely qualitative treatment in which only comparative statements can be expressed. The advantage of the presence of truth constants will be made even clearer in next section.

Another increase of expressive power would consist in allowing the definition of conditional measures from the simple marginal measures represented in the logic.

The degree of confidence in the occurrence of an event might have to be changed when new information comes at hand. This results in an update of the sample space which is commonly treated in theories of uncertainty by the concept of conditioning.

Conditional probability. The updated probability measure $P(\cdot|\chi)$ (i.e. the probability of an event given the occurrence of χ) called *conditional probability*, is defined as $P(\varphi|\chi) = \frac{P(\varphi\wedge\chi)}{P(\chi)}$, provided that $P(\chi) > 0$. If $P(\chi) = 0$ the conditional probability remains then undefined.

In order to define conditioning from marginal probabilities we need division. Clearly, the only possibility to express division in t-norm based logics is given by logics containing the product implication. Hence, any extension (or expansion) of Łukasiewicz logic having the product implication, like ŁΠ and ŁΠ$\frac{1}{2}$ (see [8]), represents a suitable choice.

Conditional possibility. In general, the conditional possibility $\Pi(\varphi|\chi)$ can be viewed as the solution to the equation $\Pi(\varphi \wedge \chi) = x * \Pi(\chi)$, where $*$ is a continuous t-norm (continuity guarantees the existence of a solution), and $\Pi(\varphi|\chi)$ is defined as the greatest solution. This would then be equivalent to the following equation: $\Pi(\varphi|\chi) = \Pi(\chi) \Rightarrow_* \Pi(\varphi \wedge \chi)$, where $\Rightarrow_*$ is the residuum of the t-norm $*$.

A classical treatment, proposed by Dubois and Prade [4], consists in taking the minimum t-norm, to obtain a qualitative definition. This, however, yields some technical problems when the sample space is infinite, given that Gödel implication is not continuous. This can be avoided by defining a probability-like conditioning by means of the product t-norm. Indeed, as shown in [3], not any t-norm can be used if we want the mapping $\Pi(\cdot|\chi)$ to be a possibility measure. If we rely on an arbitrary space, $*$ must be a strict t-norm, i.e. continuous, Archimedean and without zero-divisors [20]. If the universe is finite, $*$ needs not be Archimedean, and then we can recover the minimum t-norm.

Hence, two natural definitions for conditional possibility are obtained from the above equation by taking $\Rightarrow_*$ as the Gödel implication or the product implication. Hence, such types of derived conditioning can be framed in any extension (or expansion) of Gödel logic, the former, and in any extension (or expansion) of Product logic, the latter [15].

Notice that conditioning for necessity measure is not in general defined from marginal necessities, but it is derived from conditional possibilities (see [5], for the details). Furthermore, there is no clear notion of conditional lower or upper probability as derived from a single measure. Conditional lower and upper probabilities are rather defined from a set of probabilities (see [17]).

3 Rational Assessments: Compactness, Coherence and Consistency

In this section we lay out a link between consistency of modal theories and the coherence of rational assessments of fuzzy measures and conditional measures. In order to do so, we need some previous notions and results concerning satisfiability, compactness and consistency.

A detailed investigation of compactness of many logics based on continuous t-norms was presented in [2]. The notion of *satisfiability* proposed there generalizes the classical one by admitting various degrees of simultaneous satisfiability.

Definition 3.1 [2] For a set Γ of formulas in a t-norm based logic and $K \subseteq [0,1]$, we say that Γ is *K-satisfiable* if there exists an evaluation e such that $e(\varphi) \in K$ for all $\varphi \in \Gamma$. The set Γ is said to be *finitely K-satisfiable* if each finite subset of Γ is K-satisfiable. We say that a logic is *K-compact* if K-satisfiability is equivalent to finite K-satisfiability. A logic satisfies the *compactness property* if it is K-compact for each closed subset of $[0,1]$.

In particular we should mention that t-norm based logics only having continuous truth-functions, like Łukasiewicz Logic, do enjoy the compactness property.

Theorem 3.2 ([2]) *Let $\mathcal{L}$ be a given t-norm based logic whose connectives only have continuous truth-functions. Then $\mathcal{L}$ has the compactness property.*

The above result clearly still holds when we deal with theories in which the interpretations of all connectives correspond to continuous truth functions.

In many real-life situations assessments of uncertainty are not precisely made over a set of events with a specific algebraic structure. Still, such assessments must be required to be coherent, that is: they must satisfy the axioms of a fuzzy measure whenever they are extended over the whole Boolean algebra generated by those events.

Definition 3.3 Let $\mathcal{M}$ be a class of fuzzy measures, $\mathcal{C}$ be a countable set of events, and μ be a real-valued assessment defined on $\mathcal{C}$. We call μ a *$\mathcal{M}$-coherent fuzzy measure* if there is a fuzzy measure $\mu' \in \mathcal{M}$ over the Boolean algebra generated by $\mathcal{C}$ such that $\mu(\varphi) = \mu'(\varphi)$ for all $\varphi \in \mathcal{C}$.

It is clear that by relying on a t-norm based logic $\mathcal{L}$ in which rational truth constants are definable we can represent rational assessments w.r.t. to a fuzzy measure. This will allow us to show that checking the coherence of a rational assessment over a countable set of events is tantamount to checking consistency of a suitably defined theory in $\mathcal{M}(\mathcal{L})$.

First of all, a clarification has to be made. Here, given a class of fuzzy measures $\mathcal{M}'$, $\mathcal{M}'(\mathcal{L})$ will denote an extension of $\mathcal{M}(\mathcal{L})$ over a t-norm based logic with rational truth constants compatible with $\mathcal{M}'$ being complete w.r.t. to $\mathcal{M}'$-Kripke models. For instance, $\mathcal{M}'(\mathcal{L})$ might correspond to either $\mathcal{P}(\mathcal{L})$, $\mathcal{P}i(\mathcal{L})$, or $\mathcal{N}(\mathcal{L})$. Now, we need theories of the form $\Gamma = \{\kappa(\varphi_i) \leftrightarrow \overline{\alpha_i}\}$ in order to have models in which assessments of fuzzy measures are not only 1-valued. Of course, we cannot take into account real-valued assessments, since we only have rational truth-constants in our language. Then we obtain that for any rational assessment its coherence is equivalent to the consistency of the respective theory in $\mathcal{M}'(\mathcal{L})$, given that its extension induces a $\mathcal{M}'$-Kripke structure which is a model of such a theory. However, there is an important restriction. Indeed, to obtain the mentioned result we need the logic $\mathcal{L}$ to have the compactness property, or the connective $\leftrightarrow$ to be continuous, since we need to exploit the above compactness results (see [22]). Since $\varphi \leftrightarrow \psi$ is defined as $(\varphi \leftrightarrow \psi)\&(\psi \leftrightarrow \varphi)$, it is obvious that both & and $\rightarrow$ must have continuous truth functions. Up to isomorphism, the only continuous t-norm having a continuous residuum is the Łukasiewicz t-norm. This implies, in this case, that $\mathcal{L}$ must be an extension (or expansion) of RPL [15].

Theorem 3.4 *Let $\theta = \{\mu^*(\varphi_i) = \alpha_i\}$ be a rational assessment, and let $\mathcal{M}'$ be a class of fuzzy measures. Suppose that the following conditions are satisfied:*

i. $\mathcal{L}$ is a t-norm based logic with rational truth constants

ii. $\mathcal{L}$ either has the compactness property or is an extension (or expansion) of RPL

iii. $\mathcal{L}$ has a finitary notion of derivability

iv. $\mathcal{L}$ is compatible with $\mathcal{M}'$

v. $\mathcal{M}'(\mathcal{L})$ is (finitely) strongly complete w.r.t. $\mathcal{M}'$-Kripke structures.

Then θ is $\mathcal{M}'$-coherent iff the theory $\Gamma_\theta = \{\kappa(\varphi_i) \leftrightarrow \overline{\alpha}_i\}$ is consistent in $\mathcal{M}'(\mathcal{L})$, i.e. $\Gamma_\theta \nvdash_{\mathcal{M}'(\mathcal{L})} \overline{0}$.

Given the above theorems, it is now easy to prove a compactness result for coherent assessments. This means that when we have a rational assessment to a countable set of events, such an assessment is coherent if and only if its restriction to each finite subset of that set also is coherent. Indeed, since any of such coherent restrictions can be translated into a theory which is consistent by Theorem 3.4, the whole corresponding theory is consistent, and consequently, again by Theorem 3.4, the corresponding assessment is coherent. Notice that this result concerns rational assessments of fuzzy measures only, and it is proved by purely logical means.

Theorem 3.5 *Let $\mathcal{C} = \{\varphi_i\}$ be a countable family of events, let $\theta = \{\mu^*(\varphi_i) = \alpha_i\}$ be a rational assessment over $\mathcal{C}$, and let $\mathcal{M}'$ be a class of fuzzy measures. Suppose that the following conditions are satisfied:*

i. $\mathcal{L}$ is a t-norm based logic with rational truth constants

ii. $\mathcal{L}$ either has the compactness property or is an extension (or expansion) of RPL

iii. $\mathcal{L}$ has a finitary notion of derivability

iv. $\mathcal{L}$ is compatible with $\mathcal{M}'$

v. $\mathcal{M}'(\mathcal{L})$ is (finitely) strongly complete w.r.t. $\mathcal{M}'$-Kripke structures.

Let $\theta_{\downarrow\mathcal{I}}$ be the restriction of θ to each finite $\mathcal{I}$, such that $\mathcal{I} \subset \mathcal{C}$. Then:

θ is $\mathcal{M}'$-coherent iff $\theta_{\downarrow\mathcal{I}}$ is $\mathcal{M}'$-coherent for every $\mathcal{I}$.

4 Final Remarks

As far as we know, the only comprehensive logical treatment of uncertainty measures is the one proposed by Halpern [17]. In such a work, a modal operator ℓ, standing for likelihood, is applied over Boolean formulas, so that $\ell(\varphi)$ is a likelihood term interpreted as "the uncertainty of φ". A basic likelihood formula is an expression of the form $a_1\ell(\varphi_1) + \cdots + a_k\ell(\varphi_k) > b$, where $a_1, \ldots, a_k, b$ are real numbers and $k \geq 1$. Likelihood formulas are Boolean combinations of basic likelihood formulas. The language resulting from the foregoing description is called $\mathcal{L}^{QU}$, where QU stands for *quantitative uncertainty*. From $\mathcal{L}^{QU}$ we can then build up a logic for a class of measures, by introducing the adequate axioms. Given that likelihood formulas are linear inequalities, we also have to introduce all substitution instances of valid linear inequality formulas as axioms. The semantics for $\mathcal{L}^{QU}$ is given by Kripke models $\langle W, \mathcal{U}, e, \mu\rangle$ equipped with a measure belonging to a certain class. Halpern showed how to treat probabilities, possibilities, belief functions and upper probabilities obtaining sound and complete axiomatizations.

This approach is strongly based on the presence of axioms of linear inequalities which allow to represent basic operations between formulas. Our approach exploits the advantage given by the fact that in t-norm based logics the operations associated to the evaluation of the connectives are functions defined over $[0, 1]$, which correspond, directly or up to some combinations, to operations used to compute degrees of uncertainty. Then such algebraic operations can be embedded in the connectives of the many-valued logical framework, resulting in clear and elegant formalizations. Given that there is a whole family of t-norm based logics, the choice of the logic to exploit to represent a specific class of measures will clearly depend on the operations we need to represent. This permits to avoid the introduction of instances of linear inequalities, since they are directly given by the functions associated to the connectives of some logics. For instance, Łukasiewicz logic and its expansions allow the representation of piecewise linear functions, and hence are the most suitable choice for the representation of linear equalities and inequalities. Moreover, in the case of possibility and necessity measures, for instance, we might not even need to use linear inequalities. What we need are just the minimum and the maximum operators plus the possibility of expressing comparative statements which is immediately given by the implication connective.

Therefore, in our treatment we do not need to add axioms for having peculiar operations, since the possible presence of those operations just relies on an adequate choice of the base logic. Having functions embedded in our logics also implies that some properties of the chosen logic might be inherited by the kind of measures we define in it. Indeed, once proven the connection between the consistency of a suitably defined theory in our logic and the coherence of the related assessment, properties like compactness for those assessments can be easily studied by purely logical means.

To conclude, we would like to point out that we do not deem that the t-norm based approach is better than the others. The study carried out in this work might be just an overt example of the advantages t-norm based logics can provide.

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