Almost everywhere convergence in family of IF-events with product

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Abstract

The aim of this paper is to define the lower and upper limits on the family of IF-events with product. We compare two concepts of almost everywhere convergence and we show that they are equivalent, too.

Keywords: IF-event, Product, Lower limit, Upper limit, Almost everywhere convergence.

1 Introduction

In recent years the theory of IF-sets introduced by Atanassov ([1]) has been studied by many authors.

An *IF-set* A on a space Ω is a couple (μ_A, ν_A), where $\mu_A : \Omega \to [0,1]$, $\nu_A : \Omega \to [0,1]$ are functions such that $\mu_A(\omega) + \nu_A(\omega) \leq 1$ for each $\omega \in \Omega$ (see [1]). The function μ_A is called the membership function, the function ν_A is called the non membership function.

In [4] Grzegorzewski and Mrówka defined the probability on the family of IF-events

$$\mathcal{N} = \{(\mu_A, \nu_A) \,;\, \mu_A + \nu_A \leq 1\},$$

where μ_A, ν_A are $\mathcal{S}$-measurable, as a mapping $\mathcal{P}$ from the family $\mathcal{N}$ to the set of all compact intervals in $\mathbf{R}$ by the formula

$$\mathcal{P}((\mu_A, \nu_A)) = \left[\int_\Omega \mu_A \; dP, 1 - \int_\Omega \nu_A \; dP \right],$$

where $(\Omega, \mathcal{S}, P)$ is probability space. This IF-probability was axiomatically characterized by B. Riečan (see[13]).

More general situation was studied in [16], where the author introduced the notion of IF-probability on the family

$$\mathcal{F} = \{(f,g) \;;\; f, g \in \mathcal{T},\; f + g \leq 1\},$$

where $\mathcal{T}$ is a Lukasiewicz tribe, as a mapping $\mathcal{P}$ from the family $\mathcal{F}$ to the family $\mathcal{J}$ of all closed intervals $\langle a, b\rangle$ such that $0 \leq a \leq b \leq 1$. Variant of Central limit theorem and Weak law of large numbers were proved as an illustration of method applied on these IF-events. It can see in the papers [10, 11].

More general situation was used in [9]. The authors defined the IF-probability on the family $\mathcal{M} = \{(a,b) \in M, a + b \leq u\}$, where M is σ-complete MV-algebra, which can be identified with the unit interval of a unique ℓ-group G with strong unit u, in symbols,

$$M = \Gamma(G, u) = (\langle 0, u\rangle, 0, u, \neg, \oplus, \odot)$$

where $\langle 0, u\rangle = \{a \in G \;;\; 0 \leq a \leq u\}$, $\neg\, a = u - a$, $a \oplus b = (a + b) \wedge u$, $a \odot b = (a + b - u) \vee 0$ (see [17]). We say that G is the ℓ-group (with strong unit u) corresponding to M.

By an ℓ-group we shall mean a lattice-ordered Abelian group. For any ℓ-group G, an element $u \in G$ is said to be a strong unit of G, if for all $a \in G$ there is an integer $n \geq 1$ such that $nu \geq a$.

The independence of IF-observables, the convergence of IF-observables and the Strong law of large numbers were studied on this family of IF-events, see [6, 7].

In paper [8] we defined the product operation on the family $\mathcal{F}$ of IF-events

$$\mathcal{F} = \{(f,g) \;;\; f, g \in \mathcal{T},\; f + g \leq 1\},$$

where $\mathcal{T}$ is a Lukasiewicz tribe. We formulated the version of conditional IF-probability on this family, too. In this paper we introduce the notion of lower and upper limits and show two concepts of almost everywhere convergence. In *Section 2* we introduce the operations on $\mathcal{F}$ and $\mathcal{J}$, where $\mathcal{J}$ is the family of all closed intervals $\langle a, b\rangle$ such that $0 \leq a \leq b \leq 1$.

2 Basic notions

Now we introduce operations on $\mathcal{F}$. Let $A = (a_1, a_2)$, $B = (b_1, b_2)$. Then we define

$$A \oplus B = (a_1 \oplus b_1, a_2 \odot b_2) = \big((a_1+b_1) \wedge 1, (a_2+b_2-1) \vee 0\big),$$

$$A \odot B = (a_1 \odot b_1, a_2 \oplus b_2) = \big((a_1+b_1-1) \vee 0, (a_2+b_2) \wedge 1\big).$$

If $A_n = (a_{n1}, a_{n2})$, then we write

$$A_n \nearrow A \Longleftrightarrow a_{n1} \nearrow a_1, \ a_{n2} \searrow a_2.$$

IF-probability $\mathcal{P}$ on $\mathcal{F}$ is a mapping from $\mathcal{F}$ to the family $\mathcal{J}$ of all closed intervals $\langle a, b \rangle$ such that $0 \leq a \leq b \leq 1$. Here we define

$$\langle a, b \rangle + \langle c, d \rangle = \langle a + c, b + d \rangle,$$

$$\langle a_n, b_n \rangle \nearrow \langle a, b \rangle \Longleftrightarrow a_n \nearrow a, \ b_n \nearrow b.$$

By an **IF-probability on** $\mathcal{F}$ we understand any function $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ satisfying the following properties:

(i) $\mathcal{P}((1, 0)) = \langle 1, 1 \rangle = 1$; $\mathcal{P}((0, 1)) = \langle 0, 0 \rangle = 0$;

(ii) if $A \odot B = (0, 1)$ and $A, B \in \mathcal{F}$, then $\mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B)$;

(iii) if $A_n \nearrow A$, then $\mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.

IF-probability $\mathcal{P}$ is called **separating**, if

$$\mathcal{P}\big((a_1, a_2)\big) = \langle \mathcal{P}^\flat(a_1), 1 - \mathcal{P}^\sharp(a_2) \rangle,$$

where the functions $\mathcal{P}^\flat, \mathcal{P}^\sharp : \mathcal{T} \to [0, 1]$ are probabilities.

Example 2.1 *Let $\mathcal{T}$ be the tribe of all $\mathcal{S}$-measurable functions $f : \Omega \to [0, 1]$. Let m be an IF-state on $\mathcal{F}$, $m : \mathcal{F} \to [0, 1]$. Evidently $(f, 1 - f) \in \mathcal{F}$, hence we can define the mapping $\overline{m} : \mathcal{T} \to [0, 1]$ by the formula*

$$\overline{m}(f) = m((f, 1 - f)).$$

We can see that $\overline{m}$ is a state on $\mathcal{T}$. Then by the Butnariu - Klement theorem ([2]) there exists a probability $P : \mathcal{S} \to [0, 1]$ such that

$$\overline{m}(f) = \int_\Omega f dP$$

for any $f \in \mathcal{T}$. Of course, the previous formula does not imply the representation of IF-state given by formula

$$m(A) = (1 - \alpha) \int_\Omega \mu_A dP + \alpha(1 - \int_\Omega \nu_A dP)$$

for each $A = (\mu_A, \nu_A) \in \mathcal{F}$, where $\alpha \in [0, 1]$ and $P : \mathcal{S} \to [0, 1]$ be a probability (see [14]). We see that the IF-approach cannot be coordinatwisely reduced to the fuzzy approach.

The next important notion is notion of IF-observable.

By **IF-observable on** $\mathcal{F}$ we understand any mapping $x : \mathcal{B}(\mathbf{R}) \to \mathcal{F}$ satisfying the following conditions:

(i) $x(\mathbf{R}) = (1, 0)$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = (0, 1)$ and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

If we denote $x(A) = \big(x^\flat(A), 1 - x^\sharp(A)\big)$ for each $A \in \mathcal{B}(\mathbf{R})$, then $x^\flat, x^\sharp : \mathcal{B}(\mathbf{R}) \to \mathcal{T}$ are observables, see [11].

By **product operation on** $\mathcal{F}$ we understand any binary operation $\cdot$ satisfying the following conditions:

(i) $(1, 0) \cdot (a_1, a_2) = (a_1, a_2)$ for each $(a_1, a_2) \in \mathcal{F}$;

(ii) the operation $\cdot$ is commutative and associative;

(iii) if $(a_1, a_2) \odot (b_1, b_2) = (0, 1)$ and $(a_1, a_2), (b_1, b_2) \in \mathcal{F}$, then
$(c_1, c_2) \cdot \big((a_1, a_2) \oplus (b_1, b_2)\big) = \big((c_1, c_2) \cdot (a_1, a_2)\big) \oplus \big((c_1, c_2) \cdot (b_1, b_2)\big)$ and
$\big((c_1, c_2) \cdot (a_1, a_2)\big) \odot \big((c_1, c_2) \cdot (b_1, b_2)\big) = (0, 1)$ for each $(c_1, c_2) \in \mathcal{F}$;

(iv) if $(a_{1n}, a_{2n}) \searrow (0, 1)$, $(b_{1n}, b_{2n}) \searrow (0, 1)$ and $(a_{1n}, a_{2n}), (b_{1n}, b_{2n}) \in \mathcal{F}$, then $(a_{1n}, a_{2n}) \cdot (b_{1n}, b_{2n}) \searrow (0, 1)$.

The operation $\cdot$ defined by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 \cdot x_2, y_1 + y_2 - y_1 \cdot y_2)$$

for each $(x_1, y_1), (x_2, y_2) \in \mathcal{F}$ is a product operation on $\mathcal{F}$ (see [8]).

3 Lower and upper limits

The aim of this section is a modification of almost everywhere convergence with help of lim sup and lim inf and the construction of translation formulas for these limits. First we define max-min connectives

$$A \vee B = (a_1 \vee b_1, a_2 \wedge b_2),$$

$$A \wedge B = (a_1 \wedge b_1, a_2 \vee b_2)$$

and IF-ordering

$$A \leq B \Longleftrightarrow a_1 \leq b_1 \text{ and } a_2 \geq b_2$$

Definition 3.1 *Given a sequence $x_1, x_2, \ldots$ of IF-observables in a family of IF-events $\mathcal{F}$ with IF-probability $\mathcal{P}$. We write $\overline{x}_{IF} = \limsup_{n \to \infty} x_n$ if $\overline{x}_{IF}$:*

$\mathcal{B}(\mathbf{R}) \to \mathcal{F}$ *is an IF-observable having the following property*

$$\overline{x}_{IF}((-\infty,t)) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)$$

for all $t \in \mathbf{R}$.

Note that if another IF-observable y *satisfies the above condition then* $\mathcal{P}\circ y = \mathcal{P}\circ \overline{x}_{IF}$.

Similarly we write $\underline{x}_{IF} = \liminf_{n\to\infty} x_n$ *if* $\underline{x}_{IF} : \mathcal{B}(\mathbf{R}) \to \mathcal{F}$ *is an IF-observable satisfying the condition*

$$\underline{x}_{IF}((-\infty,t)) = \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)$$

for all $t \in \mathbf{R}$.

Theorem 3.2 *The IF-observables* $\overline{x}_{IF}$, $\underline{x}_{IF}$ *from Definition 3.1 can be expressed in the following form*

$$\overline{x}_{IF}(A) = \Big(\overline{x^\flat}(A), 1-\overline{x^\sharp}(A)\Big),$$

$$\underline{x}_{IF}(A) = \Big(\underline{x^\flat}(A), 1-\underline{x^\sharp}(A)\Big),$$

for each $A \in \mathcal{B}(\mathbf{R})$. *Here* $\overline{x^\flat}$, $\underline{x^\flat}$ *are upper and lower limits of sequence* $(x_n^\flat)_1^\infty$ *of observables in tribe* $\mathcal{T}$ *and* $\overline{x^\sharp}$, $\underline{x^\sharp}$ *are upper and lower limits of sequence* $(x_n^\sharp)_1^\infty$ *of observables in tribe* $\mathcal{T}$ *(see [18]).*

Proof. Let $(x_n)_1^\infty$ be a sequence of IF-observables. Denote

$$x_n(A) = \big(x_n^\flat(A), 1-x_n^\sharp(A)\big)$$

for each $A \in \mathcal{B}(\mathbf{R})$, then $x_n^\flat, x_n^\sharp : \mathcal{B}(\mathbf{R}) \to \mathcal{T}$ are observables for every $n \in N$.

Therefore using definition of max-min connectives $\vee, \wedge$ we obtain

$$\overline{x}_{IF}(-\infty,t) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(-\infty, t-\tfrac{1}{p}\Big) =$$
$$= \Bigg(\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n^\flat\Big(-\infty,t-\tfrac{1}{p}\Big), 1-\bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n^\sharp\Big(-\infty, t-\tfrac{1}{p}\Big)\Bigg) = \Big(\overline{x^\flat}(-\infty,t), 1-\overline{x^\sharp}(-\infty,t)\Big)$$

and

$$\underline{x}_{IF}(-\infty,t) = \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(-\infty, t-\tfrac{1}{p}\Big) =$$
$$= \Bigg(\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n^\flat\Big(-\infty,t-\tfrac{1}{p}\Big), 1-\bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n^\sharp\Big(-\infty, t-\tfrac{1}{p}\Big)\Bigg) = \Big(\underline{x^\flat}(-\infty,t), 1-\underline{x^\sharp}(-\infty,t)\Big)$$

for every $t \in \mathbf{R}$. □

Proposition 3.1 *Let* $x_1, x_2, \dots$ *be a sequence of IF-observables in a family* $\mathcal{F}$ *of IF-events with IF-probability* $\mathcal{P}$. *Suppose that both IF-observables* $\overline{x}_{IF}$ *and* $\underline{x}_{IF}$ *exist. Then for every* $t \in \mathbf{R}$

$$\overline{x}_{IF}((-\infty,t)) \le \underline{x}_{IF}((-\infty,t)).$$

Proof. Let $x_1, x_2, \dots$ be a sequence of IF-observables. Suppose that both IF-observables $\overline{x}_{IF}$ and $\underline{x}_{IF}$ exist, then by *Theorem 3.2* there exist observables $\overline{x^\flat}$, $\underline{x^\flat}$, $\overline{x^\sharp}$ and $\underline{x^\sharp}$ in tribe $\mathcal{T}$.

Since from *Proposition 8.6.4* in [18] the following inequalities hold for every $t \in \mathbf{R}$

$$\overline{x^\flat}((-\infty,t)) \le \underline{x^\flat}((-\infty,t)), \tag{1}$$

$$\overline{x^\sharp}((-\infty,t)) \le \underline{x^\sharp}((-\infty,t)),$$

then we obtain

$$1-\overline{x^\sharp}((-\infty,t)) \ge 1-\underline{x^\sharp}((-\infty,t)) \tag{2}$$

by simply modifications.

Hence by (1), (2) and *Theorem 3.2* we have

$$\overline{x}_{IF}((-\infty,t)) \le \underline{x}_{IF}((-\infty,t)).$$

□

Definition 3.3 *A sequence* $x_1, x_2, \dots$ *of IF-observables in family of IF-events* $\mathcal{F}$ *with IF-probability* $\mathcal{P}$ *is said to converge* $\mathcal{P}$*-almost everywhere to an IF-observable* x*, if both IF-observables* $\overline{x}_{IF}, \underline{x}_{IF}$ *exist and for each* $t \in \mathbf{R}$

$$\mathcal{P}(\overline{x}_{IF}(-\infty,t)) = \mathcal{P}(x(-\infty,t)) = \mathcal{P}(\underline{x}_{IF}(-\infty,t)).$$

Proposition 3.2 *A sequence* $(x_n)_1^\infty$ *of IF-observables in family of IF-event with product* $(\mathcal{F},\cdot)$ *converges* $\mathcal{P}$*-almost everywhere to zero IF-observable* $0_\mathcal{F}$ *defined by*

$$0_\mathcal{F}(A) = \begin{cases} (1,0), & \text{if } 0 \in A \\ (0,1), & \text{if } 0 \notin A \end{cases}$$

for each $A \in \mathcal{B}(\mathbf{R})$ *if and only if*

$$\mathcal{P}\Bigg(\bigwedge_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(-\frac{1}{p},\frac{1}{p}\Big)\Bigg) = \langle 1,1\rangle.$$

Proof. "⇒" Let $(x_n)_1^\infty$ converges $\mathcal{P}$-almost everywhere to $0_\mathcal{F}$. Then by *Definition 3.3* there exist the IF-observables $\overline{x}_{IF}, \underline{x}_{IF}$ such that the following equality holds for each $t \in \mathbf{R}$,

$$\begin{aligned}\mathcal{P}(\overline{x}_{IF}(-\infty,t)) &= \mathcal{P}(0_\mathcal{F}(-\infty,t)) = \\ &= \mathcal{P}(\underline{x}_{IF}(-\infty,t)).\end{aligned} \tag{3}$$

Let $t > 0$. Then by (3), *Theorem 3.2* and by definition of IF-observable $0_{\mathcal{F}}$ we have

$$\mathcal{P}\Big(\Big(\overline{x^\flat}(-\infty,t) \quad , \quad 1-\overline{x^\sharp}(-\infty,t)\Big)\Big) = \mathcal{P}\big((1,0)\big) = $$
$$= \mathcal{P}\Big(\Big(\underline{x}^\flat(-\infty,t), 1-\underline{x}^\sharp(-\infty,t)\Big)\Big),$$

$$\Big\langle \mathcal{P}^\flat\big(\overline{x^\flat}(-\infty,t)\big) \quad , \quad \mathcal{P}^\sharp\big(\overline{x^\sharp}(-\infty,t)\big)\Big\rangle = \langle 1,1\rangle =$$
$$= \Big\langle \mathcal{P}^\flat\big(\underline{x}^\flat(-\infty,t)\big), \mathcal{P}^\sharp\big(\underline{x}^\sharp(-\infty,t)\big)\Big\rangle.$$

Therefore

$$\mathcal{P}^\flat\big(\overline{x^\flat}(-\infty,t)\big) = 1 = \mathcal{P}^\flat\big(\underline{x}^\flat(-\infty,t)\big) \tag{4}$$

$$\mathcal{P}^\sharp\big(\overline{x^\sharp}(-\infty,t)\big) = 1 = \mathcal{P}^\sharp\big(\underline{x}^\sharp(-\infty,t)\big). \tag{5}$$

Now let $t \leq 0$. Then by (3), *Theorem 3.2* and by definition of IF-observable $0_{\mathcal{F}}$ we have

$$\mathcal{P}\Big(\Big(\overline{x^\flat}(-\infty,t) \quad , \quad 1-\overline{x^\sharp}(-\infty,t)\Big)\Big) = \mathcal{P}\big((0,1)\big) = $$
$$= \mathcal{P}\Big(\Big(\underline{x}^\flat(-\infty,t), 1-\underline{x}^\sharp(-\infty,t)\Big)\Big),$$

$$\Big\langle \mathcal{P}^\flat\big(\overline{x^\flat}(-\infty,t)\big) \quad , \quad \mathcal{P}^\sharp\big(\overline{x^\sharp}(-\infty,t)\big)\Big\rangle = \langle 0,0\rangle =$$
$$= \Big\langle \mathcal{P}^\flat\big(\underline{x}^\flat(-\infty,t)\big), \mathcal{P}^\sharp\big(\underline{x}^\sharp(-\infty,t)\big)\Big\rangle.$$

Therefore

$$\mathcal{P}^\flat\big(\overline{x^\flat}(-\infty,t)\big) = 0 = \mathcal{P}^\flat\big(\underline{x}^\flat(-\infty,t)\big) \tag{6}$$

$$\mathcal{P}^\sharp\big(\overline{x^\sharp}(-\infty,t)\big) = 0 = \mathcal{P}^\sharp\big(\underline{x}^\sharp(-\infty,t)\big). \tag{7}$$

Finally by (4), (6) and *Proposition 8.6.6* in [18] we obtain that the sequence of observables $(x_n^\flat)_1^\infty$ converges $\mathcal{P}^\flat$-almost everywhere to a observable $0_{\mathcal{T}}$ defined by

$$0_{\mathcal{T}}(A) = \begin{cases} 1, & \text{if } 0 \in A \\ 0, & \text{if } 0 \notin A \end{cases}$$

for each $A \in \mathcal{B}(\mathbf{R})$. Similarly by (5), (7) and *Proposition 8.6.6* in [18] we obtain that the sequence of observables $(x_n^\sharp)_1^\infty$ converges $\mathcal{P}^\sharp$-almost everywhere to the observable $0_{\mathcal{T}}$, too.

Hence the equalities

$$\mathcal{P}^\flat\left(\bigwedge_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n^\flat\Big(-\frac{1}{p},\frac{1}{p}\Big)\right) = 1$$

and

$$\mathcal{P}^\sharp\left(\bigwedge_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n^\sharp\Big(-\frac{1}{p},\frac{1}{p}\Big)\right) = 1$$

hold from *Theorem 3* in [15] and therefore we have

$$\mathcal{P}\left(\bigwedge_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(-\frac{1}{p},\frac{1}{p}\Big)\right) = \langle 1,1\rangle.$$

The proof of "$\Leftarrow$" is a analogue to the proof of "$\Rightarrow$". □

The *Proposition 3.2* says about equality of two kinds of definition of almost everywhere convergence in family of IF-events $\mathcal{F}$ with product.

Now we recall the notion of **IF-distribution function** defined in [6]. It is a mapping $\mathbf{F} : \mathbf{R} \to \mathcal{J}$ given by formula

$$\begin{aligned}\mathbf{F}(t) &= \mathcal{P} \circ x((-\infty,t)) = \\ &= \langle \mathcal{P}^\flat((-\infty,t)), \mathcal{P}^\sharp((-\infty,t))\rangle = \langle \mathbf{F}^\flat(t), \mathbf{F}^\sharp(t)\rangle\end{aligned}$$

for each $t \in \mathbf{R}$, where $\mathbf{F}^\flat, \mathbf{F}^\sharp : \mathbf{R} \to \langle 0,1\rangle$ are the distribution functions.

Proposition 3.3 *Let $\mathbf{F} : \mathbf{R} \to \mathcal{J}$ be an IF-distribution function and $(x_n)_1^\infty$ be a sequence of IF-observables. Let*

$$\varphi_{IF}(t) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big),$$

$$\psi_{IF}(t) = \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big).$$

If $\mathbf{F}(t) = \mathcal{P}\circ\varphi_{IF}(t) = \mathcal{P}\circ\psi_{IF}(t)$ for each $t \in \mathbf{R}$, then there exist IF-observables $\overline{x}_{IF}, \underline{x}_{IF} : \mathcal{B}(\mathbf{R}) \to \mathcal{F}$ such that

$$\overline{x}_{IF}\big((-\infty,t)\big) = \varphi_{IF}(t),$$
$$\underline{x}_{IF}\big((-\infty,t)\big) = \psi_{IF}(t)$$

for each $t \in \mathbf{R}$.

Proof. Denote

$$\varphi^\flat(t) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n^\flat\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big),$$

$$\psi^\flat(t) = \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n^\flat\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big)$$

and analogously

$$\varphi^\sharp(t) = \bigvee_{p=1}^{\infty}\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty} x_n^\sharp\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big),$$

$$\psi^\sharp(t) = \bigvee_{p=1}^{\infty}\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty} x_n^\sharp\Big(\Big(-\infty, t-\frac{1}{p}\Big)\Big).$$

Then $\varphi_{IF}(t)$, $\psi_{IF}(t)$ can be expressed by formulas

$$\varphi_{IF}(t) = (\varphi^\flat(t), 1 - \varphi^\sharp(t)),$$

$$\psi_{IF}(t) = (\psi^\flat(t), 1 - \psi^\sharp(t)).$$

If $\mathbf{F}(t) = \mathcal{P} \circ \varphi_{IF}(t) = \mathcal{P} \circ \psi_{IF}(t)$ for each $t \in \mathbf{R}$, then

$$\begin{aligned} \langle \mathbf{F}^\flat(t), \mathbf{F}^\sharp(t) \rangle &= \langle \mathcal{P}^\flat \circ \varphi^\flat(t), \mathcal{P}^\sharp \circ \varphi^\sharp(t) \rangle = \\ &= \langle \mathcal{P}^\flat \circ \psi^\flat(t), \mathcal{P}^\sharp \circ \psi^\sharp(t) \rangle. \end{aligned}$$

Hence

$$\mathbf{F}^\flat(t) = \mathcal{P}^\flat \circ \varphi^\flat(t) = \mathcal{P}^\flat \circ \psi^\flat(t), \qquad (8)$$

$$\mathbf{F}^\sharp(t) = \mathcal{P}^\sharp \circ \varphi^\sharp(t) = \mathcal{P}^\sharp \circ \psi^\sharp(t), \qquad (9)$$

where $\mathbf{F}^\flat(t), \mathbf{F}^\sharp(t)$ are the distribution functions.

By (8), (9) and from *Proposition* in [15] there exist observables $\overline{x^\flat}$, $\underline{x^\flat}$, $\overline{x^\sharp}$, $\underline{x^\sharp} : \mathcal{B}(\mathbf{R}) \to \mathcal{T}$ such that

$$\overline{x^\flat}((-\infty, t)) = \varphi^\flat(t), \ \underline{x^\flat}((-\infty, t)) = \psi^\flat(t),$$

$$\overline{x^\sharp}((-\infty, t)) = \varphi^\sharp(t), \ \underline{x^\sharp}((-\infty, t)) = \psi^\sharp(t)$$

for each $t \in \mathbf{R}$.

Therefore there exist IF-observables $\overline{x}_{IF}, \underline{x}_{IF}$ given by formulas

$$\overline{x}_{IF}(A) = \big(\overline{x^\flat}(A), 1 - \overline{x^\sharp}(A)\big),$$

$$\underline{x}_{IF}(A) = \big(\underline{x^\flat}(A), 1 - \underline{x^\sharp}(A)\big)$$

for each $A \in \mathcal{B}(\mathbf{R})$ such that the following equalities hold

$$\begin{aligned} \overline{x}_{IF}((-\infty, t)) &= \big(\overline{x^\flat}((-\infty, t)), 1 - \overline{x^\sharp}((-\infty, t))\big) = \\ &= \big(\varphi^\flat(t), 1 - \varphi^\sharp(t)\big) = \varphi_{IF}(t), \end{aligned}$$

$$\begin{aligned} \underline{x}_{IF}((-\infty, t)) &= \big(\underline{x^\flat}((-\infty, t)), 1 - \underline{x^\sharp}((-\infty, t))\big) = \\ &= \big(\psi^\flat(t), 1 - \psi^\sharp(t)\big) = \psi_{IF}(t) \end{aligned}$$

for each $t \in \mathbf{R}$. □

4 Conclusion

The paper is concerned in the probability theory on IF-events with product. We define the notion of upper and lower limits. We compare two kinds of definition almost everywhere convergence for IF-events with product and we show that they are equivalent.

Acknowledgement

This paper was supported by Grant VEGA 1/2002/05.

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