Representation of two bipolar decision strategies with generalized Choquet integrals

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Abstract

We study the notion of conditional relative importance in a quantitative framework. This is an important notion in the context of the Choquet integral since this latter is usually motivated by this decisional behavior. A systematic investigation on generalizations of the Choquet integral is performed. The determination of the utility function is central in this analysis.

Keywords: Multiple criteria analysis, Choquet integral, bi-capacity, reference point.

1 Introduction

In Multiple Criteria Decision Aiding (MCDA), it is very seldom that an option dominates the other ones over all or most of the decision criteria. The selection of the *best* option is often performed by prioritizing the criteria. However, the semantics of the prioritization depend on the framework that is considered. We are interested here in options that are a good compromise between the criteria. More precisely, we look for *compensation* between criteria, that is that well-satisfied criteria can compensate ill-satisfied ones. It is well-known that quantitative Multi-Attribute Utility Theory (MAUT) [11] models such as weighted sums are the best suited for representing compensation. The prioritization of the criteria is then performed, based on the notion of "relative importance" of one or several criteria compared to other ones.

Yet, simple models such as the weighted sum are not always sufficient. The decision model has indeed to be rich enough to model the decisional behaviors of the decision maker. This view tends to imply the use of very elaborate models. Two trends have been recently observed. Linear aggregation functions are often not satisfactory, in particular due to the existence of interaction between criteria. Veto or favor are examples of extreme interaction. This has lead to the introduction of the Choquet integral as an aggregation function in MCDA. Another complexity is the possible existence of a special element on the scale, called neutral element, such that the decision behaviors are quite different for values above or below this element. Bipolar aggregation functions must then be used in such situations.

Bi-capacities have been introduced in MCDA for taking into account both interaction between criteria and bipolarity. More generally, bi-capacities are supposed to represent bipolar decision strategies, i.e. decision strategies which are not the same when the value of an attribute is above or below the neutral element.

We investigate in this paper two particular decision behaviors: bipolar conditional relative importance, and bipolar veto. Such strategies are very interesting for two reasons. The first one is that these behaviors are usually easier to obtain from a decision maker than quantitative preferential information. In some cases, the decision maker can naturally express statements of this kind. The second one is that we want to make an in-depth analysis of bipolar conditional relative importance and investigate its close link with interaction between criteria.

The two decision strategies that are considered here are described in Section 2 and motivated by examples. The qualitative and quantitative notion of independence and conditional relative importance are given in Section 3. The definition of bipolarity and of the Choquet integral is recalled in Section 4. The representation of the first bipolar behavior, which is conditional relative importance, is investigated in Section 5. A general analysis of this behavior under piecewise linear aggregation is performed. We present a general framework for the construction of utility functions. It is shown that the bipolar conditional relative importance cannot be strictly satisfied by any piecewise linear aggregation function. However, by relaxing this decision behavior, the non-representation result disappears. It turns out that the Choquet integral w.r.t. a bi-capacity can represent that relaxed decision behavior, whereas the Choquet integral w.r.t. a usual capacity cannot. Finally, Section 6 presents an analysis of the bipolar veto decision behavior.

2 Two examples of decision strategies

2.1 Bipolar conditional relative importance

Consider the situation of financial analysts in a bank that would like to decide on customers credit granting. Three factors have been retained: the debt ratio (denoted by D) given in %, an indicator P reflecting the behavior of the customer in the past (for instance the number of late payments) and the capital contribution ratio (denoted by C) given in %. The main factor (i.e. the most important one) is the debt ratio D. However, the preference of the analysts over P and C is not so simple. For a customer that has an attractive debt ratio, the contribution rate is not so important so that P is more important than C. On the other hand, for a customer that has a bad debt ratio, there is relative substitutability between D and P so that the analysts hopes at least that the contribution ratio is good. Hence, C becomes more important than P.

Let us give another example. Consider the director of a university that decides on students who are applying for graduate studies in Economics on the basis of an assessment of their skills in Mathematics (M), Statistics (S) and Languages(L). The director feels that Mathematics is the most important criteria. However, his preference over S and L is not that simple. There is a relative substitutability between M and S. Hence, for an applicant good in Mathematics, the director prefers if he is furthermore good in L than if he is also good in S, so that L is more important than S in this case. On the contrary, for candidates bad in Mathematics, the director hopes they are at least good in S since the director basically looks for applicants with strong scientific background. Hence, S becomes more important than L.

The previous two examples exhibit the same decisional pattern, in which the relative importance of one attribute k^+ compared to another one k^- may depend on the value of a third attribute k being good or bad. This type of behavior is called "*conditional relative importance*". One can express the statement of this behavior as follows.

> **(S1)**: If the value w.r.t. criterion k is very well-satisfied, then criterion k^+ is more important than criterion k^-. If the value w.r.t. criterion k is very ill-satisfied, then criterion k^+ is less important than criterion k^-.

Such statements are often intuitive for actors. The Choquet integral has been shown to represent this type of statement [7]. The relative importance of criteria k^+ and k^- is specified in statement **(S1)** only for the two extreme levels of performance (very good and very bad respectively) on criterion k. One wonders what happens for intermediate values on criterion k. One should whenever possible specify the preferences between criteria k^+ and k^- for all values of criterion k. A generalization of statement **(S1)** is the following one :

> **(S2)**: If value w.r.t. criterion k is "well-satisfied", then criterion k^+ is more important than criterion k^-. If value w.r.t. criterion k is "ill-satisfied", then criterion k^+ is less important than criterion k^-.

In the previous statement, the values judged as well-satisfied and ill-satisfied are supposed to form a partition of the scale. There thus exists on attribute k a particular element called "neutral element" such that better elements are considered as well-satisfied and worse elements are considered as ill-satisfied for the actor. Hence, the scale underlying the criterion k is of *bipolar nature*. The classical Choquet integral fails to represent **(S2)** [8, 14].

2.2 Bipolar veto

The engineering of complex systems is a difficult task since all components of the system interact together in a hard-to-predict way. An analysis of each component separately is not enough. There are many consequences that have to be analyzed when considering a system as a whole. Some of these aspects concern the measure of the performance on the system. This often requires large simulations run on several scenarios. The indicators on which the analysis of the results of the simulations is performed are called *metrics*. They correspond to the so-called *functional* criteria. On the other hand, there are also *non-functional* criteria, i.e. criteria that can be assessed without the use of these simulators. One can mention for instance, acquisition and possession costs, and the technical readiness levels of the components of the system.

The functional criteria often correspond to fuzzy requirements given by the customer. Hence, if the functional criteria are ill-satisfied, then the customer will be ill-satisfied whatever the value on the non-functional attributes. This means that the functional criteria behaves like a veto. Now, when the functional criteria are well-satisfied, the customer now seeks for systems that also have good figures in non-functional parts. Hence there is compensation between the functional and the non-functional criteria. Assuming that the functional criteria have been gathered in one attribute denoted by x_F, and that all non-functional criteria have been gathered in one attribute x_{NF}, we obtain the following rule.

> **(R1)**: If the value w.r.t. criterion k_F is "ill-satisfied", then criterion k_F is a veto. If the value w.r.t. criterion k_F is "well-satisfied", then criterion k_{NF} can compensate k_F.

3 Representation of the preferences

3.1 Construction of the preferences on each attribute

Consider a problem of selecting one option among several, where each option is described by several attributes. $N = \{1, \ldots, n\}$ is the set of attributes and the set of possible values of attribute $i \in N$ is denoted by X_i. Options are thus elements of the product set $X := X_1 \times \cdots \times X_n$. The preferences of the DM over the options can be described by a preference relation $\succeq$ over X. For $x, y \in X$, $x \succeq y$ means that x is at least as good as y according to the DM.

Considering two acts $x, y \in X$ and $S \subseteq N$, we use the notation (x_S, y_{-S}) to denote the compound act $w \in X$ such that $w_i = x_i$ if $i \in S$ and y_i otherwise. Likewise, options $(x_S, y_T, z_{-S\cup T})$ denotes the compound act $w \in X$ such that $w_i = x_i$ if $i \in S$, $w_i = y_i$ if $i \in T$, and $w_i = z_i$ otherwise.

Representing $\succeq$ by a numerical or graphical model demands to address two issues: the preferences over each attribute and the aggregation of these preferences. Speaking of a preference relation focusing only on one attribute implies that the other attributes could have been removed. The existence of a preference relation on each attribute classically relies on *weak separability* [12].

Definition 1 *A relation $\succeq$ is said to be* weakly separable *if for every $i \in N$, and for all $x_i, x'_i \in X_i$, $y_{N\setminus\{i\}}, y'_{N\setminus\{i\}} \in X_{N\setminus\{i\}}$,*

$$(x_i, y_{-i}) \succeq (x'_i, y_{-i}) \iff (x_i, y'_{-i}) \succeq (x'_i, y'_{-i}) \ .$$

Then, for $i \in N$, the marginal preference relation $\succeq_i$ on attribute i is defined on X_i as follows

$$x_i \succeq_i y_i \iff \forall z_{N\setminus\{i\}} \in X_{N\setminus\{i\}} \ , \ (x_i, z_{-i}) \succeq (y_i, z_{-i}) \ .$$

This property can be interpreted as a weak independence between attributes. This assumption is essential for quantitative models based on an overall utility function. From [12], $\succeq$ can be represented by functions $u_i : X_i \to \mathbb{R}$ and $F : \mathbb{R}^n \to \mathbb{R}$:

$$x \succeq y \iff U(x) \geq U(y) \tag{1}$$

where

$$U(x) = F(u_1(x_1), \ldots, u_n(x_n)) \tag{2}$$

iff $\succeq$ is a weak order, is weakly separable and satisfies a technical topological assumption.

There are representations in which the weak-separability assumption is relaxed. Weak separability implies that the elements of an attribute can be ranked independently of the values w.r.t. the other attributes. One can indeed say that $x_i \in X_i$ is preferred to $x'_i \in X_i$ "*everything else being equal*" (or "*Ceteris Paribus*"), that is if $(x_i, y_{-i}) \succeq (x'_i, y_{-i})$ for any $y_{N\setminus\{i\}} \in X_{N\setminus\{i\}}$. However, it may happen that the partial preferences over an attribute $i \in N$ are conditional on the value of some other attributes. This leads to generalizing Definition 1, yielding the notion of partial preferential independence [11].

Definition 2 *Let $S \subseteq N$. Attribute $i \subseteq N$ is said to be* conditionally preferentially independent *of $N \setminus (S \cup \{i\})$ for S if for all $x_i, x'_i \in X_i$, $y_{N\setminus(S\cup\{i\})}, y'_{N\setminus(S\cup\{i\})} \in X_{N\setminus(S\cup\{i\})}$ and $z_S \in X_S$,*

$$(x_i, z_S, y_{-(S\cup\{i\})}) \succeq (x'_i, z_S, y_{-(S\cup\{i\})}) \iff (x_i, z_S, y'_{-(S\cup\{i\})}) \succeq (x'_i, z_S, y'_{-(S\cup\{i\})}) \ .$$

Given an attribute i, we denoted by $\mathrm{Pa}(i) \subseteq N \setminus \{i\}$ the attributes under which the partial preferences over an attribute $i \in N$ depend. We note $\succeq_i^{z_{\mathrm{Pa}(i)}}$ on X_i as follows

$$x_i \succeq_i^{z_{\mathrm{Pa}(i)}} y_i \iff \forall t_{N\setminus(\mathrm{Pa}(i)\cup\{i\})} \in X_{N\setminus(\mathrm{Pa}(i)\cup\{i\})} \ (x_i, z_{\mathrm{Pa}(i)}, t_{-(\mathrm{Pa}(i)\cup\{i\})}) \succeq (y_i, z_{\mathrm{Pa}(i)}, t_{-(\mathrm{Pa}(i)\cup\{i\})})$$

A *Conditional Preference network* (CP-net in short) is a network of conditional preferences on the attributes. It is defined as follows [1].

Definition 3 *A* CP-net *over attributes N is a directed graph G over N whose nodes are annotated with conditional preference tables. For a node $i \in N$, $\mathrm{Pa}(i)$ is the set of all antecedents of i in the graph. The conditional preference table for i is composed of the conditional preferences $\succeq_i^{z_{\mathrm{Pa}(i)}}$ over X_i for all $z_{Pa(i)} \in X_{\mathrm{Pa}(i)}$.*

This defines a qualitative representation of $\succeq$ that is compact.

3.2 Aggregation of the preferences on each attribute

From a knowledge of only the preferences with respect to each attribute separately, one is interested in all the preference relation compatible with this a priori information. If $\succeq$ is assumed to satisfy weak separability, the *least specific* preference relation compatible with $\succeq_i$ is the Pareto ordering relation

$$x \succeq_{\text{Pareto}} y \iff \forall i \in N \ , \ x_i \succeq_i y_i$$

Most of the interesting couples of options of X are incomparable according to the ordering. In CP-nets, one looks for the preference relations that are compatible with a given CP-net. Depending on the graph structure, usually there are also many couples of options that are incomparable.

In order to reduce the number of incomparabilities in the preference relation, information on how to combine several attributes together must be added. Looking at Definition 2, one sees that it is not sure that an actor can compare two options that are not equal on all attributes except one. This is quite restrictive. This is why CP-nets have been extended to allow more general preference representations than conditionally preferentially independence. The extension is called *Tradeoff CP-nets* (*TCP-nets* in short) [3]. The notion of qualitative importance between two attributes is introduced in this formalism. Attribute $i \in N$ is said to be *more important* than attribute $j \in N$ (we note $i \triangleright j$) iff

$$\begin{aligned} &\forall w_{N\setminus\{i,j\}} \in X_{N\setminus\{i,j\}} \ , \ \forall x_i, y_i \in X_i \ , \ \forall z_j, t_j \in X_j \\ &\text{with } x_i \succeq_i^{w_{N\setminus\{i,j\}}} y_i \ , \\ &\qquad (x_i, z_j, w_{N\setminus\{i,j\}}) \succeq (y_i, t_j, w_{N\setminus\{i,j\}}) \end{aligned}$$

In other words, when i is more important than j, this implies that, for two options differing only on attributes i and j, one prefers the options that has the preferred value on attribute i whatever the value on attribute j.

An attribute can be more important than an another one, depending from the value w.r.t. some other attributes. This leads to the notion of *conditional relative importance* between attributes. Attribute $i \in N$ is said to be *more important* than attribute $j \in N$ given $v_S \in X_S$ (we note $i \triangleright^{v_S} j$), with $S \cap \{i,j\} = \emptyset$, iff

$$\begin{aligned} &\forall w_{N\setminus(S\cup\{i,j\})} \in X_{N\setminus(S\cup\{i,j\})} \ , \ \forall x_i, y_i \in X_i \ , \\ &\forall z_j, t_j \in X_j \text{ with } x_i \succeq_i^{(v_S, w_{N\setminus(S\cup\{i,j\})})} y_i \ , \\ &\qquad (x_i, z_j, v_S, w_{-(S\cup\{i,j\})}) \succeq (y_i, t_j, v_S, w_{-(S\cup\{i,j\})}) \end{aligned}$$

One can have for instance $i \triangleright^{v_S} j$ for some $v_S \in X_S$ and $j \triangleright^{v'_S} i$ for some other $v'_S \in X_S$.

There is no compensation in the previous qualitative notion of relative importance. For instance, if $1 \triangleright 2 \triangleright \cdots \triangleright n$, then $\succeq$ is the lexicographic ordering. In a quantitative setting, the relative importance of a criterion in an aggregation function F is clearly defined when F is a weighted sum $F(s_1, \ldots, s_n) = \sum_{k=1}^n w_k s_k$, where all weights w_k are non-negative and they sum up to one. The importance of criterion k is then its associated weight w_k. More generally, if F is continuous piecewise linear, then there exists a partition of $\mathbb{R}^n$ such that F is a weighted sum on each domain of the partition. The importance of a criterion is thus defined in each domain as its relative weight. Since any smooth function can be approximated by continuous piecewise linear functions, this leads to defining the importance of criterion k at point $s \in \mathbb{R}^n$ as the partial derivative $\frac{\partial F}{\partial s_k}(s)$ (since the weight of criterion k in the continuous piecewise linear approximation converges to $\frac{\partial F}{\partial s_k}(s)$ when the size of the mesh describing the approximation tends to zero). Yet, this interpretation is not so simple for an actor.

So, we will stick to continuous piecewise linear aggregation functions so that the notion of relative importance clearly makes sense to the DM. He can indeed elicit a statement such as **(S2)** only if the notion of relative importance is clear to him. From a measurement standpoint [12], piecewise linearity is derived from the property of invariance to linear changes of the scales [14]. The weight of criterion l at point $s \in \Omega$ is denoted by $w_l(s)$:

$$F(s) = \sum_{l \in N} w_l(s)\, s_l \ ,$$

where $w_l(s)$ is piecewise constant. These weights are said to be *normalized* if, for any s, they are non-negative and sum up to one.

A piecewise linear function F is characterized by a partition $\Phi(F)$ of $\mathbb{R}^n$ on which F is a weighted sum on each element of the partition. Hence

$$\Phi(F) = \left\{ \Phi_F^1, \ldots, \Phi_F^{\phi(F)} \right\}$$

where for all $l \in \{1, \ldots, \phi(F)\}$, there exists normalized weights $w_1^{\Phi_F^l}, \ldots, w_n^{\Phi_F^l}$ such that one has, for all $s \in \Phi_F^l$, $F(s) = \sum_{k\in N} w_k^{\Phi_F^l} s_k$. The partition is unique if each subset Φ_F^l is connected and

$$\overline{\Phi_F^l} \cap \overline{\Phi_F^{l'}} \neq \emptyset \implies \exists k \in N \ , \ w_k^{\Phi_F^l} \neq w_k^{\Phi_F^{l'}} \ . \quad (3)$$

Criterion i is said to be more important than j relatively to point $v \in \mathbb{R}^n$ if

$$w_i^{\Phi_F^l} > w_j^{\Phi_F^l}$$

where $v \in \Phi_F^l$.

4 Bipolarity and Choquet integral

In this paper, we are interested in the quantitative model (1). The notion of relative importance in quantitative models implies that the attributes must be made comparable. Let us consider indeed the following relation

$$(x_i, y'_j, z_{-\{i,j\}}) \sim (x'_i, y_j, z_{-\{i,j\}})$$

where $x_i \neq x'_i$ and $y_j \neq y'_j$. The previous relation expresses that the change from x_i to x'_i on attribute i is similar to the change from y_j to y'_j on attribute j. In order to be able to say that the criteria i and j have the same importance, the change from x_i to x'_i on attribute i shall represent the same difference of satisfaction as the change from y_j to y'_j on attribute j. In other word, one can define the notion of importance of criteria only once the attributes have been make commensurable.

Since the aggregation functions we use are based on sums, the utility functions must be constructed as interval scales [12].

4.1 Bipolar and unipolar scales

It may exists in X_k a particular element or level $\mathbf{0}_k$, called *neutral level*, such that if $x_k \succ_k \mathbf{0}_k$, then x_k is considered as “good”, while if $x_k \prec_k \mathbf{0}_k$, then x_k is considered as “bad” for the actor.

Such a neutral level exists whenever relation $\succeq_k$ corresponds to two opposite notions of common language. For example, this is the case when $\succeq_k$ means “more attractive than”, “better than”, etc., whose pairs of opposite notions are respectively “attractiveness/repulsiveness”, and “good/bad”. By contrast, relations “more satisfactory than”, “more allowed than”, “belongs more to category C than” do not clearly exhibit a neutral level.

A scale is said to be *bipolar* if X_k contains such a neutral level. A *unipolar scale* has no neutral level. As an example, preference statement **(S2)** implies that criterion i is of bipolar nature.

4.2 Aggregation of unipolar scales

A *capacity* [4], also called *fuzzy measure*, is a set function $\nu : 2^N \to \mathbb{R}$ satisfying $\nu(\emptyset) = 0$, $\nu(N) = 1$, and $A \subset B$ implies $\nu(A) \leq \nu(B)$. In MCDA, $\nu(A)$ is interpreted as the overall assessment of the binary act $(1_A, 0_{-A})$.

The Choquet integral [4] defined w.r.t. a capacity ν has the following expression :

$$\mathcal{C}_\nu(s_1, \ldots, s_n) = s_{\pi(1)}\nu(N) + \sum_{i=2}^{n} \left(s_{\pi(i)} - s_{\pi(i-1)}\right) \nu\left(A_{\pi(i)}\right) \tag{4}$$

where $s_{\pi(1)} \leq s_{\pi(2)} \leq \cdots \leq s_{\pi(n)}$, $A_{\pi(i)} = \{\pi(i), \cdots, \pi(n)\}$ and $s_1, \ldots, s_n \in \mathbb{R}_+$. We also have

$$\mathcal{C}_\nu(s) = \sum_{i=1}^{n} s_{\pi(i)} \left[\nu(A_{\pi(i)}) - \nu(A_{\pi(i+1)})\right] .$$

Clearly, the Choquet integral w.r.t. a capacity is continuous piecewise linear. Moreover, one has

$$\Phi\left(\mathcal{C}_\nu\right) = \{\Omega_\pi \ , \ \pi \text{ permutation on } N\}$$

where $\Omega_\pi = \left\{s \in \mathbb{R}^n \ , \ s_{\pi(1)} \leq s_{\pi(2)} \leq \cdots \leq s_{\pi(n)}\right\}$.

4.3 Aggregation on bipolar scales

The Choquet integral has a natural extension to bipolar scales. The limitation of the Choquet integral w.r.t. capacities is that the overall evaluation at any point is computed from information coming only from the attractive part (i.e. the parameters of the capacity correspond to the overall assessment of the positive binary acts $(1_A, 0_{-A})$). Hence, the notion of capacity is not suited to deal with real bipolar scales. The idea is thus to generalize the notion of capacity. Let

$$\mathcal{Q}(N) = \{(A, B) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A \cap B = \emptyset\} .$$

A *bi-capacity* [8, 14] is a function $\mu : \mathcal{Q}(N) \to \mathbb{R}$ satisfying $\mu(\emptyset, \emptyset) = 0$, $\mu(N, \emptyset) = 1$, $\mu(\emptyset, N) = -1$, $A \subseteq A'$ implies $\mu(A, B) \leq \mu(A', B)$, and $B \subseteq B'$ implies $\mu(A, B) \geq \mu(A, B')$. The last two properties depict increasing monotonicity. In MCDA, $\mu(A, B)$ is interpreted as the overall assessment of the ternary act $(1_A, -1_B, 0_{-A\cup B})$.

The Choquet integral w.r.t. a bi-capacity μ proposed in [8] is now given. For any $A \subseteq N$, $s \in \Sigma_A$,

$$\mathcal{BC}_\mu(s) := \mathcal{C}_\nu\left(s_A, -s_{-A}\right) = \mathcal{C}_\nu\left(|s|\right)$$

where $\nu(C) := \mu\left(C \cap A, C \setminus A\right)$ and $\Sigma_A := \{s \in \mathbb{R}^n \ , s_A \geq 0 \ , s_{-A} < 0\}$. Let τ be a permutation such that $|s_{\tau(1)}| \leq \ldots \leq |s_{\tau(n)}|$, and

$$\begin{aligned} A^+_{\tau(i)} &= \{\tau(i), \cdots, \tau(n)\} \cap A \\ &= \left\{\tau(j) \ , \ j \geq i \text{ and } s_{\tau(j)} \geq 0\right\} , \end{aligned}$$

$$\begin{aligned} A^-_{\tau(i)} &= \{\tau(i), \cdots, \tau(n)\} \cap (N \setminus A) \\ &= \left\{\tau(j) \ , \ j \geq i \text{ and } s_{\tau(j)} < 0\right\} . \end{aligned}$$

Then one can write

$$\mathcal{BC}_\mu(s) = \sum_{i=1}^{n} |s_{\tau(i)}| \left[\mu(A^+_{\tau(i)}, A^-_{\tau(i)}) - \mu(A^+_{\tau(i+1)}, A^-_{\tau(i+1)})\right] \quad (5)$$

Clearly, the Choquet integral w.r.t. a bi-capacity is continuous piecewise linear. Moreover, one has

$$\Phi\left(\mathcal{BC}_\mu\right) = \{\Omega_{A,\tau}\ ,\ A \subseteq N \text{ and } \tau \text{ permutation on } N\}$$

where $\Omega_{A,\tau} = \left\{s \in \Sigma_A\ ,\ |s_{\tau(1)}| \leq \ldots \leq |s_{\tau(n)}|\right\}$.

5 Representation of Statement (S2) by an aggregation function

5.1 Construction of the utility functions

As we have seen in Section 4.1, statement **(S2)** clearly exhibits a bipolar behavior on criterion k. Due to commensurateness between the criteria, all criteria are considered as bipolar. Statement **(S2)** shows that there are interaction between criteria in preference relation over options. The construction of the utility functions u_i is thus more complex than for the case where all criteria are independent (see for instance the *utility independence* assumption [11]). Utility functions u_i have a priori no intrinsic meaning, and only make sense through the overall utility U thanks to (2). When all criteria are independent, $U(x_i, z_{-i}) = F(u_i(x_i), u_{-i}(z_{-i}))$ and $u_i(x_i)$ are two equivalent interval scales for any $z_{-i} \in X_{-i}$ fixed. This relation gives a sense to u_i. In this relation, the presence of z_{-i} is not essential so that u_i can be considered as a utility representation of a preference relation over attribute X_i *all else being equal* (i.e. the value w.r.t. the other attributes being fixed to any value). The utility functions can thus be considered and constructed separately.

When there are some interactions among criteria, the "all else being equal" assumption does not hold anymore. The choice of the reference z_{-i} from which the utility function is constructed becomes essential. For a given z_{-i}, if the following assumption

$$\exists l \in \{1, \ldots, \phi(F)\}\ ,\quad \forall x_i \in X_i \quad (u_i(x_i), u_{-i}(z_{-i})) \in \Phi^l_F \quad (6)$$

holds then the utility function u_i can be constructed as for the weighted sum since the options used in the construction of u_i remain in the same domain Φ^l_F of linearity of F. Hence F does not alter the perception of u_i through $U(x_i, z_{-i})$. For instance, if F is a Choquet integral w.r.t. a capacity, z_{-i} is considered at either the lowest or the highest satisfaction level on each attribute $l \neq i$ [13]. These two extreme values correspond to two reference levels. If F is a Choquet integral w.r.t. a bi-capacity, the attractive and repulsive parts of u_i must necessarily be constructed separately in order to have (6) (see the end of Section 4.3). The neutral level becomes an essential point in the construction of utility functions. Apart from the neutral level, one reference level is required on the attractive part and the repulsive one in order to normalize the scale. Hence three reference levels are necessary [14].

The actor is first asked to identify on each attribute X_i a neutral element $\mathbf{0}_i$ that is considered as neither good nor bad [16]. Since statement **(S2)** clearly relies on the existence of such level on attribute k, one can assume that the actor who provides **(S2)** can identify the value of $\mathbf{0}_k$. We assume here that this neutral element can also be identified on the other attributes. It is assumed furthermore that there exists an element denoted by $\mathbf{1}_i$ that is considered as good and completely satisfactory, even if more attractive elements could exist on this point of view [13]. The existence of such reference level comes from the theory of satisficing bounded rationality [17]. We assume finally that there exists an element denoted by $-\mathbf{1}_i$ that is considered as bad and unsatisfactory. Element $-\mathbf{1}_i$ is somehow symmetric to $\mathbf{1}_i$. More precisely, $-\mathbf{1}_i$ corresponds to the same level of appreciation in the repulsive scale than $\mathbf{1}_i$ in the attractive scale. All levels have the same absolute meaning throughout the criteria, so we impose:

$$\begin{aligned} u_1(-\mathbf{1}_1) = \cdots = u_n(-\mathbf{1}_n) = -1\ , \\ u_1(\mathbf{0}_1) = \cdots = u_n(\mathbf{0}_n) = 0\ , \\ u_1(\mathbf{1}_1) = \cdots = u_n(\mathbf{1}_n) = 1\ . \end{aligned}$$

Since the attractive and repulsive values refer to different affect stimuli [18], it may be more appropriate to construct separately the positive and the negative parts of the partial utility functions in order to make the actor compare attractive values with repulsive ones. If F satisfies for all $i \in N$

$$\begin{array}{l} \exists l \in \{1, \ldots, \phi(F)\}\ ,\quad \forall s_i \geq 0\ \ (s_i, 0_{-i}) \in \Phi^l_F \\ \exists l' \in \{1, \ldots, \phi(F)\}\ ,\quad \forall s_i \leq 0\ \ (s_i, 0_{-i}) \in \Phi^{l'}_F \end{array} \quad (7)$$

then one can construct the utility function u_i from $U(x_i, \mathbf{0}_{-i})$:

$$\begin{aligned} \forall x_i \succeq_i \mathbf{0}_i\ ,\quad u_i(x_i) = \frac{U(x_i, \mathbf{0}_{-i}) - U(\mathbf{0}_N)}{U(\mathbf{1}_i, \mathbf{0}_{-i}) - U(\mathbf{0}_N)} \\ \forall x_i \preceq_i \mathbf{0}_i\ ,\quad u_i(x_i) = \frac{U(x_i, \mathbf{0}_{-i}) - U(\mathbf{0}_N)}{U(\mathbf{0}_N) - U(-\mathbf{1}_i, \mathbf{0}_{-i})} \end{aligned}$$

Relation (7) is satisfied by the Choquet integral w.r.t. a bi-capacity.

5.2 General analysis of rule (S2)

Rule **(S2)** can be stated in the more precise form

(S2'): If the value w.r.t. criterion k is attractive (i.e. $x_k \succ_k \mathbf{0}_k$) then criterion k^+ is more important than criterion k^-. If the value w.r.t. criterion k is repulsive (i.e. $x_k \prec_k \mathbf{0}_k$) then criterion k^+ is less important than criterion k^-.

Aggregation functions are henceforth assumed to be continuous since a slight modification in the argument shall also result in a slight change in the overall utility [6].

Theorem 1 shows that statement **(S2')** cannot be thoroughly modeled in all situations. Some restrictions will thus be made. One can show the following result.

Theorem 1 *There does not exist any continuous and piecewise linear aggregation function F, for which statement* **(S2')** *is satisfied.*

Let $\mathbb{R}^n_{+i} = \{s \in \mathbb{R}^n \ , \ s_i \geq 0\}$, $\mathbb{R}^n_{-i} = \{s \in \mathbb{R}^n \ , \ s_i \leq 0\}$ and $\mathbb{R}^n_{0\,i} = \{s \in \mathbb{R}^n \ , \ s_i = 0\}$. More precisely, let $\Phi^+, \Phi^- \in \Phi(F)$ such that $\Phi^+ \subseteq \mathbb{R}^n_{+i}$ and $\Phi^- \subseteq \mathbb{R}^n_{-i}$. Assume that $\Gamma := \overline{\Phi^+} \cap \overline{\Phi^-} \neq \emptyset$. If $\Gamma \subseteq \mathbb{R}^n_{0\,i}$ is parallel to the axis of criteria k^+ and k^-, then statement **(S2')** *cannot be satisfied in both Φ^+ and Φ^-.*

Under the condition of the previous theorem, the variables s_{k^+} and s_{k^-} are free on the boundary Γ, even though criterion k vanishes. The following Corollary provides a special case of this relation between criteria k, k^+, k^-.

Corollary 1 *Assume that the domains of $\Phi(F)$ correspond to that of a bi-capacity, i.e. $\Phi(\mathcal{BC})$. If criterion k is the one closest to the neutral level among criteria k, k^+, k^-, then the weights of criteria k^- and k^+ are not conditional on the fact that criterion i is attractive or repulsive (i.e. statement* **(S2')** *cannot be satisfied).*

The result of Theorem 1 is not true when F is not continuous piecewise linear. Consider indeed the following nonlinear aggregation function

$$F(s) = \frac{1+s_k}{2} \times s_{k^+} + \frac{1}{2} \times s_{k^-} \ .$$

Then the weight of criterion j^+ is $\frac{\partial F}{\partial s_{k^+}}(s) = \frac{1+s_k}{2}$ and that of criterion k^- is $\frac{\partial F}{\partial s_{k^-}}(s) = \frac{1}{2}$. Hence, statement **(S2')** is perfectly satisfied by F.

Theorem 1 can be interpreted in the following way. This result states that when x_k is close to the neutral level $\mathbf{0}_k$ (relatively to criteria k^+ and k^-), the relative preference of the actor over criteria k^+ and k^- is not so clear. This is a hesitation area.

Let us show as an example that bi-capacities satisfy to the restriction imposed by the previous corollary. By (5), when the scores w.r.t. criteria are all different in absolute value, the weight of criterion l for an act $s \in \Omega$ for a bi-capacity μ is given by

$$w_l(s) = \\ = \begin{cases} \mu\left(\{l\} \cup C_l^+(s), C_l^-(s)\right) - \mu\left(C_l^+(s), C_l^-(s)\right) \\ \qquad\qquad\qquad \text{if } s_l \geq 0 \\ \mu\left(C_l^+(s), C_l^-(s)\right) - \mu\left(C_l^+(s), \{l\} \cup C_l^-(s)\right) \\ \qquad\qquad\qquad \text{if } s_l < 0 \end{cases}$$

where

$$C_l^+(s) = \{m \neq l \ , \ s_m \geq 0 \text{ and } |s_m| \geq |s_l|\}$$

and

$$C_l^-(s) = \{m \neq l \ , \ s_m < 0 \text{ and } |s_m| \geq |s_l|\} \ .$$

If $|s_i| < |s_l|$ then $i \notin C_l^+(s) \cup C_l^-(s)$. Hence, if criterion k is the one closest to the neutral level among criteria k, k^+, k^-, then $k \notin C_{k^+}^+(s) \cup C_{k^+}^-(s)$ and $k \notin C_{k^-}^+(s) \cup C_{k^-}^-(s)$. This means that $w_{k^+}(s)$ and $w_{k^-}(s)$ do not depend on whether $s_k \geq 0$. Henceforth, **(S2')** cannot be modeled in this case. Now when $|s_i| \geq |s_l|$, then $i \in C_l^+(s)$ if $s_i \geq 0$ and $i \in C_l^-(s)$ if $s_i < 0$. Hence, the weight $w_l(s)$ can change between the two cases $s_i \geq 0$ and $s_i < 0$. As a consequence, one cannot model **(S2')** with a bi-capacity whenever criterion k is the one closest to the neutral level among criteria k, k^+, k^-.

We restrict **(S2')** according to Theorem 1:

> **(S3)**: If the value w.r.t. criterion k is attractive (> 0), and k is not the one closest to the neutral level among criteria k, k^+, k^-, then criterion k^+ is more important than criterion k^-. If the value w.r.t. criterion k is repulsive (< 0), and k is not the one closest to the neutral level among criteria k, k^+, k^-, then criterion k^+ is less important than criterion k^-.

Let us give the requirements on a bi-capacity imposed by the previous statement **(S3)**. One has the following lemma.

Lemma 1 *There exists a bi-capacity such that the corresponding Choquet integral satisfies to* **(S3)**.

We have shown in this section that the general statement **(S2)** cannot be satisfied by a continuous piecewise linear aggregation function. We introduce then a restriction of **(S2)** - namely **(S3)**. Finally we have seen that **(S3)** can be thoroughly fulfilled at least by the Choquet integral w.r.t. some bi-capacity. It turns out that there is no capacity such that its associated Choquet integral satisfies statement **(S2)**.

6 Representation of Statement (R1) by an aggregation function

Consider Rule **(R1)**. Rule **(R1)** becomes

$$w_{NF}(u_F, u_{NF}) = 0 \quad \text{if } u_F < 0 \tag{8}$$

$$w_{NF}(u_F, u_{NF}) > 0 \quad \text{if } u_F > 0 \tag{9}$$

The previous two relations cannot be satisfied when F is the Choquet integral w.r.t. a capacity. Indeed, if (8) is satisfied with a Choquet integral, then (8) is satisfied for all $u_F \in \mathbb{R}$ such that $u_F \leq u_{NF}$. A similar result is obtained with (9).

Theorem 2 *There does not exist any continuous and piecewise linear aggregation function F, for which statement* **(R1)** *is satisfied.*

Moreover, one can show that the Choquet integral w.r.t. a bi-capacity cannot do better than the Choquet integral w.r.t. a capacity. Indeed, if (8) is satisfied with a Choquet integral w.r.t. a bi-capacity, then (8) is satisfied for all $u_F \in \mathbb{R}$ such that $u_F \leq u_{NF}$.

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