Power Sets and Implication Operators Revisited: A Retrospective Look at the Foundational and Conceptual Issues in Bandler and Kohout's Paper After 29 Years

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Abstract

In our case study we look at one of the early papers that interrelates the concept of fuzzy set inclusion, power set and many-valued implication operators, namely the paper of Bandler and Kohout [3]. This is followed by discussion of the subsequent related work by the fuzzy community.

Keywords: Fuzzy set inclusion, power set, many-valued implication operators. Fuzziness, crispness, fuzzy disjointness, fuzzy equality.

1 Introduction

It is an interesting story to look at the development of new concepts in fuzzy set theory. One looks at the motivation, the first formulation and the subsequent development.

We shall take as our case study the one of the early papers that look at the relationship of the concept of fuzzy set inclusion, power set and many-valued implication operators. It is the paper of Bandler and Kohout [3] that was presented at an international workshop organised by Professor Mamdani in 1978. The participants of the workshop were people from different disciplines, pure mathematicians, scientists from various fields including brain modelling, and understandably, with strong representation of people from the fuzzy control community. The fuzzy control community had that time strong interest in investigating different types of implications. That established Bandler and Kohout's paper as a repository of useful information about various implication operators and about the bootstrap of their properties into fuzzy sets. The extended version was submitted to Fuzzy Sets and Systems, but the manuscript was considered to be too large, so the Editor-in-Chief Professor Zimmerman recommended to be limited to discussion of different kinds of fuzzy set inclusions and of their link to many-valued logic operators. This paper [5] has become well known in the fuzzy community and was reprinted in a collection edited by Dubois, Prade and Yager. Other parts of the 1978 paper were extended and published as separate papers [7],[6].

We shall examine the historical trace of the development of concept of fuzzy inclusion in the following ways:

1. What part of the original approach has been retained;
2. how was it used in the further development of the concepts involved;
3. what aspects of the paper were understood well, which have been neglected, and what was misunderstood or misinterpreted.
4. We shall also look at the links to other concepts that were only in the original [3] but unfortunately did not appear in the reduced version [5] published in Fuzzy Sets and Systems.

2 Paper of 1978: 'Fuzzy Relational Products and Fuzzy Implication Operators"

The first paper was entitled *Fuzzy Relational Products and Fuzzy Implication Operators.* The list of contents of this paper listed the following topics:

1. Various Products of Crisp Relations.
2. Towards a Theory of Fuzzy Power-Sets.
3. Possibilistic Notation.
4. Comparative Semantics of Fuzzy Implication Operators.
5. Height, Plinth and Crispness of Fuzzy Sets.

6. Fuzzy Set-Inclusions arid Equalities.
7. Disjointness of Fuzzy Sets.
8. A Fuzzy Set and its Complement.
9. Choice of System and Further Aspects.

The main motivation for fuzzification of Zadeh's set inclusion predicate the membership function of which is given by the formula

$$\mu_{A \subseteq B}(x) = \mu_A(x) \leq \mu_B(x)$$

came from the need to fuzzify the crisp BK-products of relations [2]. That is clearly stated in the abstract of the 1978 paper [3]:

> *Besides the usual circlet product of crisp relations, there are three others which are natural and of interest and of use. Their fuzzification requires the choice of a fuzzy implication operator, and will vary with the choice made (Section 1). The reason why this is so leads the problem back to a fundamental and hitherto neglected aspect of fuzzy set theory: the appropriate definition of a fuzzy power-set; thus the motivation for choosing a suitable* **internal** *implication operator is much deepened, and by the use of a possibilistic notation is also somewhat broadened (Sections 2 and 3).* (From the abstract of [3]).

Because the paper links various concepts, it has became the seminal ground for other work of Bandler and Kohout. In particular, the paper [3] introduced the fuzzy non-associative products ($\lhd, \rhd, \square$) also called BK-products in the literature. That was a successful fuzzification of crisp BK-products introduced by Bandler and Kohout earlier [2]. That paved way to development of Enriched theory of fuzzy relations (ETFR) which successfully extended the crisp enriched theory of relations of [2] into the fuzzy realm.

2.1 Power Sets, Inclusion Predicate and Implication Operators

We shall briefly survey the key concepts of [3] section by section first, and then look at subsequent developments.

2.1.1 Section 2: Towards a Theory of Fuzzy Power-Sets

The situation where sets B and A are both crisp subsets of some universe U is considered first. The standard definition of the subset relation between them is

$$A \subseteq B \quad \text{means } (b \in A \rightarrow b \in B).$$

This is the connection between $\subseteq$ and the implication operator $\rightarrow$. Now, the subset relation itself is expressible in terms of the *belonging relation* and the *power-set* $\mathcal{P}(B)$ of B:

$$A \subseteq B \quad \text{means} \quad A \in \mathcal{P}(B).$$

$$\text{Thus} \qquad A \in \mathcal{P}(B) \quad \text{means } (b \in A \rightarrow b \in B).$$

This formulation is subject to immediate fuzzification as follows:

Definition 1 ([3], Def. 2.1) *Given a fuzzy implication operator $\rightarrow$, and a fuzzy subset B of a crisp universe U, the* ***fuzzy power-set*** *$\mathcal{P}(B)$ of B is given by the membership function with*

$$\mu_{\mathcal{P}B} A = \bigwedge_{x \in U} (\mu_A x \rightarrow \mu_B x).$$

This is well defined in terms of each suitable $\rightarrow$ operator, for every argument $A \in (F)U$.

Hence, the degree to which A is a subset of B is

$$\pi(A \subseteq B) = (\mu_A x \rightarrow \mu_B x).$$

The symbol π indicates that, in fact, that the degree assigned to the statement $(A \subseteq B)$ is *degree of possibility.*

Note that, where I is the unit real closed interval, the fuzzy set B is an element of I^U while its power-set B is an element of I^{U^U} (Otherwise put, $B \in \mathcal{F}(U)$, while $\mathcal{P}(B) \in \mathcal{F}(\mathcal{F}(U)$.)

The Mean Inclusion Bandler and Kohout also introduced the mean inclusion in [3] (Prop. 3.2) replacing *inf* by the *mean* value:

$$\pi_m(A \subseteq B) = \frac{\sum_{x \in U} (\mu_A \rightarrow \mu_B)}{card(supp\ A \cup supp\ B)}$$

Properties of these have been investigated by Willmott later.

2.1.2 Section 3: Possibilistic Notation

Following Zadeh (1971) in using π for "possibility" in comparison to p for "probability", Bandler and Kohout extend the analogy-or-contrast by enclosing statements in brackets after π to indicate their degree of possibility. On the interpretation of π they say the following:

> *One (but not the only) interpretation of this is, "the degree to which the bracketed statement is true." In particular, the previous section shows that we will wish to have $\pi(A \subseteq B) = (\mu_A x \rightarrow \mu_B x)$. "the degree to which A is a subset of B."*

2.1.3 Section 4: Comparative Semantics of Fuzzy Implication Operators

Implication operators play crucial role in linking sets with their power sets as well as with the inclusion predicate. In order to investigate the properties of fuzzy set operations we need to start with examining the properties of logic formulas on which specific set theories are based. The properties of logic connectives are then reflected in the properties of fuzzy sets and set operations as shown in sections 6 – 9 of [3]. The criteria for evaluation outlined in [3]are as follows:

Does an implication operator used in a formula yield

1. a strong, or moderate tautology for $a \to a$?
2. the flat contradiction, or a moderate contradiction for $a \to \neg a$?
3. is the implication operator contrapositive?
4. is the implication operator continuous?

Such questions are, however, meaningful and unambiguous if and only if they are asked in an appropriate context. Bandler and Kohout point out that two entirely different contexts are often not sufficiently distinguished [3],[5].

> *Logic has long been beset with the often-muddied distinction between*
>
> 1. *inferences made in a meta-language from statements in an object-language, on the one hand,*
> 2. *and on the other, the formation in the object-language itself of an implicative combination of its c statements.*
>
> *Both the need for this distinction and the difficulty of keeping to it become more acute in the fuzzy environment.*

They continue [3],[5]

> *Our present need is for a "suitable" generalisation of the second of the distinguenda, the internal implication operator in the object language*[1].

In order to detach this notion from the first one, that of (meta-)reasoning with fuzzy premises, they use the unemotive term favoured by Curry: *PLY operator*; the arrow itself is then read a "ply".

The problem is posed very explicitely [3],[5]:

1. We are working in a Multi-Valued System V, which for present purposes is all or some of the real interval I = [0, 1]. The rationals there are more than ample for their purposes (so: cardinality at most $\aleph_0$).

2. Whatever V is, it is furnished with the uncontroversial operators $\wedge$ and $\vee$, and with the accepted negation $\neg$, with $\neg a = 1 - a$.

3. One seeks within this system a ply operator $\to$, that is, a mapping from $V \times V$ to V, suitable for the concepts of the previous sections, which is to say chiefly for defining the degree to which one fuzzy set is to be said to be a subset of another.

4. The fuzzy sets themselves are mappings from some crisp universe U into our V, that is, the membership degrees of elements are numbers in V.

5. The ply operator will take two such degrees and make another out of them. The natural anticipation, is that the fuzziness will not thereby be diminished in this process.

For further investigations of specific power set theories and inclusion predicates, Bandler and Kohout [3],[5] chose five representative ply operators 1–6.

1. $S^{\#}$ Rescher [19] (p. 344).
$a \to_1 b = 1$ if $a \neq 1$ or $b = 1$, 0 otherwise.

2. S The "standard sequence" of Rescher [19] (pp. 46–52, 343–344), Gaines [10] formula (40).
$a \to_b = 1$ if $a \leq b, 0$ otherwise.

3. S^* Gödel. $a \to_3 b = 1$ if $a \leq b, b$ otherwise.

4. G43 Goguen–Gaines. Recommended by Gaines, formula (43), for further investigation.
$a \to_4 b = min(1, b/a)$.

5. Ł Łukasiewicz. $a \to_5 b = min(1, 1 - a + b)$

6. KD Kleene–Dienes. $a \to_6 b = max(1 - a, b)$

7. EZ Zadeh [27]. $a \to_7 b = max(min(a, b), (1 - a))$

8. Wm Willmott [24], [25]
min(max(1-a, b), min(max(a, 1-b), max(b, 1-a))).

[1] When [3] was written, most of the considerations of fuzzy ply operators in the literature had been from quite a different point of view: an operator suitable for the first of the distinguenda had been sought, a means of meta reasoning from fuzzy data.

2.1.4 Section 5: Interrelating Height, Plinth, Crispness and Fuzziness of Fuzzy Sets

Connected with semantics of PLY is the natural notion of natural crispness and fuzziness of an MVL proposition and of a fuzzy set. This was utilised for assessing PLY by their degrees of crispness and investigating how this bootstraps onto the various constructs made of fuzzy sets by set operations.

In [3] Bandler and Kohout introduced the *crispness* of a proposition $a \in V$ as $\kappa a = a \vee (1-a)$. The *fuzziness* $\phi a = 1 - \kappa a$ as its *dual.* The crispness of a fuzzy set is then

$$\kappa B = \bigwedge_U \kappa(\mu_B x).$$

The *mean* crispness of a fuzzy set

$$\kappa_m B = \frac{\sum_U \kappa(\mu_B x)}{card\ suppB}.$$

Later these concepts played a role in assessing the width of intervals in interval fuzzy logic, thus evaluating quality of inference as a function of data on which inference is performed.

2.2 Diversification As a Result Of Fuzzification: Split of a crisp concept into several fuzzy concepts

The paper [3] clearly demonstrates that from a mathematical viewpoint the important feature of fuzzy set theory is the replacement of the two valued logic by a multiple-valued logic. Since every mathematical notion can be written as a formula in a formal language, we have only to internalise, i.e. to interpret these expressions by the given multiple-valued logic. For that reason, it was important to "internalise", i.e. to form in the object language itself an implicative combination of its statements as pointed out above in section 2.1.3.

One important aspect of fuzzification that 1978 paper and 1980 paper demonstrated was the fact that two or more equivalent crisp definitions are not any more equivalent for their fuzzy counterparts. For example the Definition 5.1 [5] provides 2 formulas for disjointness of two sets that are different in the fuzzy case, despite the fact that they are equivalent in the crisp case.

2.2.1 Section 6. Fuzzy Set-Inclusions and Equalities

In addition to fuzzy set inclusions and equalities, this section also looks at the degree to which a fuzzy set is empty.

2.2.2 Section 7. Disjointness of Fuzzy Sets

In ordinary set theory

$$A \cap B = \emptyset \text{ iff } A \subseteq A^c \text{ iff } A \cap B^c = A.$$

The first two characterisations were investigated by Bandler and Kohout [3][5] while the last characterisation leads to a third possible definition of the degree of disjointness between sets which gives investigated by Willmott.

2.2.3 Section 8. A Fuzzy Set and its Complement

The three distinct concepts of disjointness are also reflected in the issue to what extend a set is disjoint from its complement.

2.2.4 Section 9. Choice of System and Further Aspects

The fuzzier implication operators exhibit a certain property of invariance that has been called by Bandler and Kohout "the conservation of crispness". This is useful in deciding which system to adopt for a particular purpose.

3 Response to the Paper Within the Fuzzy Community

3.1 Connectives

The paper deals with a family of implicational fragments of logics, the properties of which are bootstrapped into the properties of sets. While Willmott extends this by two more PLY operators, Weber looks at link of implications to other connectives.

The six operators of Bandler and Kohout are ordered by them 'in decreasing order of rigidity' or in increasing order of fuzziness, i.e. the later ones give decreasingly many crisp, or increasingly many fuzzy results in the case of non-crisp or fuzzy antecedents.

3.1.1 Willmott [24]

The investigation of Bandler and Kohout in [3] is repeated by Willmott [24] for two more implication operators (EZ, Wm) which follow the above six in this ordering (see Sec. 2.1.3 above, $\leftarrow_7, \leftarrow_8$.). Both EZ and Wm are fuzzier than any considered in [3]. The first was suggested by Zadeh [27] previously. The second is new and probably represents the extreme in fuzziness for a usable operator of this kind, according to Willmott "realising natural anticipation that the fuzziness (the value of an implication compared to that of its components) will not be diminished". In terms of fuzzy sets, while using this operator, the degree of possibility of any relation between two fuzzy sets cannot be larger than the crispness of the less crisp of the two. The operator retains virtually all of the favourable features of the sixth operator (i. e. Kleene-Dienes) investigated by Bandler and Kohout in Sec. 4 of their paper.

Willmott states [24],[25] *This note should be considered as an addendum to the paper by Bandler and Kohout. It assumes all their notation, definitions and results and uses the same section and item numbering, so that some section numbers are here missing.*

3.1.2 Weber [23]

In function of connectives and negation three types of fuzzy implication operators are introduced, which include almost all known implications, and that of type I using AND only; of type II using OR/NON; of type lII using AND/NON. In the non-strict Archimedean cases the formulas become particularly lucid (Section 5). Finally the different types are compared with respect to some logical properties (Section 6).

The rest of Weber's paper was motivated by Bandler and Kohout [5]. In Section 5 Weber presents the construction of three types of implication operators, which contain the known ones.

Section 6.1 compares the three types of implication operators concerning contrapositive symmetry and contradiction. Remarks on natural crispness and fuzziness that was introduced by Bandler and Kohout in [3],[5] conclude the paper of Weber.

3.2 Power Set

Although [3],[5] introduce power sets, the importance of it has been overlooked by most of the papers that quote Bandler and Kohout. The reason for this is clearly indicated in comments of Höhle and Stout [14]:

> *For fuzzy mathematics we would like to have a foundation for higher order structures as well as for the propositional logic of fuzzy sets. To develop such a foundation we need to ask to what extent it makes sense to talk about a fuzzy power object.*
>
> *This can be internal (in which case an individual could have fuzzy membership in such a power set) or external as a construction in classical mathematics (the usual practice in current fuzzy topology). Indeed we claim that the first fifteen years of fuzzy set theory was dominated by the fuzzy power-set problem.*
>
> *In L.A. Zadeh's pioneering paper of 1965 it is obvious that he defines intersection, union, and complement of fuzzy subsets, but he hesitates to specify the fuzzy power set of a given fuzzy set.*

Indeed, as we remarked above, Zadeh's theory is a theory of fuzzy subsets of a crisp set, not a theory of fuzzy sets.

Bandler and Kohout clearly state that they are *looking for an internal implication operator within an object language."* This yields an individual that has fuzzy membership in a power set.

Stout [21] says about attempts to handling the power set problem:

> *In fuzzy set theory the approach has been more external, at least for second order theories, in part because there is no single fully satisfactory fuzzy power set operator. For example, in fuzzy topology one approach which has been developed at length (Wong uses crisp sets together with a topology which is a crisp set of fuzzy subsets). This uses only the properties of the sub-object lattice, an external propositional level approach. Several attempts have been made by Pultr [18], Bandler and Kohout [5], Gottwald [12],[13] to provide a suitable theory of* **fuzzy power sets.**

Bandler and Kohout [3] approach the problem of power set "top down", and algebraically. Note that in [3],[5], where I is the unit real closed interval, the fuzzy set B is an element of I^U while its power-set B is an element of I^{U^U} (Otherwise put, $B \in \mathcal{F}(U)$, while $\mathcal{P}(B) \in \mathcal{F}(\mathcal{F}(U).)$

The axiomatic approach within a logic on the other hand was provided by other authors. There have been two set-theoretic approaches to the foundations of fuzzy sets with a power set: one by Gottwald [13], and Klaua [15],[16], based on Lukasiewicz connectives. For summaries of this work see also [14], [11]. The work of Takeuti and Titani [22] is based on intuitionistic connectives. They abandon the Lukasiewicz connectives because of problems with extensionality resulting from the fact that $(p * (p \leftarrow q)) \leftarrow q)$ need not be valid.

Gottwald parallels the construction of Boolean-valued models to get a hierarchical system of fuzzy sets with membership values in a ruler by giving an inductive definition. There is a sense in which each fuzzy set x in his hierarchy has a natural ordinal rank given by the smallest a such that $x \in R(\alpha)$). Gottwald calls the elements with rank > 0 fuzzy sets to distinguish them from the ur-elements with rank 0. The empty fuzzy set has rank 1, as do other fuzzy subsets in the sense of Zadeh of the set of ur-elements.

The question of ur-elements needs to be revisited, as these may be important for applications. Unfortunately, almost no attention has been given to this important aspect of Gottwald's paper.

The power set problem has been resolved only comparatively recently within the setting of monoidal and other kinds of categories (cf. work of Höhle, Stout, Rodabaugh and others). Approach using the apparatus of (mathematical) categories is useful from the foundational point of view.

The algebraic approach of Bandler and Kohout that uses the many-valued logic connectives directly, is more suitable for development of calculi of fuzzy relations, interval fuzzy logics, and knowledge elicitation. From categorial point of view it is related to *esomathematical* use of category theory pioneered by Bandler [1].

3.3 Fuzzy Set Inclusion

There is rather large literature on this aspect. This unfortunately most papers in this category develop idea in isolation from the power set concept. The papers divide into two groups:

1. papers taking the inclusion predicate just as an index of subsetness, and

2. papers that provide axioms for various "desirable" properties of inclusion predicate.

In the first group are papers by Young [26], Kosko, Bustince [9], Bodenhofer, [8] and others. In the second group (axioms of desirable properties) are papers by Singha and Dougherty [20], Pappis et al, important paper by Kitainik, etc. The extended version of this paper will provide a detailed survey with bibliography which for lack of space cannot be provided here. There are several hundred of quotations of [5], mostly related to the viewing set inclusion as a measure, a subsetness indicator; or quoting Bandler and Kohout's work as a useful repertory of properties of implication operators[2]. Only scant attention is paid to other, equally important aspects of the papers that have been surveyed in greater detail in the previous sections.

[2]None of the authors quoted in this section seem to realize that we have also provided the definition of the mean subsethood [3] and used it extensively in applications since 1979 [4],[7]. Even for crisp sets the mean subsetness yields fuzzy values [7]. Willmott's interest in mean inclusion was triggered by [3],[4] while visited us at Essex. His visit was supported by an SERC grant that was obtained by Bandler for this visit.

4 Summary of Responses

The response in the literature and the influence of the paper on the subsequent work can be summarised as follows.

The paper of Bandler and Kohout [3] presented new concepts and stated also their mutual relationships. The results of investigation of properties of various implication operators $\rightarrow$, of various inclusion predicates $\subseteq$, and of various constructs made from fuzzy sets by fuzzy operators have been recognised and quoted. On the other hand, the important links between these have scarcely been noticed, and the relation of these to the concept of the power set have been almost completely overlooked.

Bandler and Kohout's ideas that were first outlined in [3] further branched into fuzzy relational calculi exploring BK-products of relations – here the first bridging papers are [7],[6]. The paper [7] also contained the Checklist Paradigm that has provided the semantics and the tools for interval fuzzy logics [17].

Unfortunately, because in the truncated version of [3] published in Fuzzy Sets and Systems did not contain the section 1 of [3], the fuzzy community views these three branches that stem from unified foundational study presented in [3] as completely unrelated.

The full report on which this brief EUSFLAT 07 paper is based, lists and more fully describes these developments that are related despite of the received view in the fuzzy community that that represent totally different branches of fuzzy set theory. Because the fuzzy field is rapidly dividing into many specialities and fragmenting view examining these old links may help to make new connection that counteract this undesirable fragmentation. It is clear that we need more and deeper foundational studies.

5 The Need for Foundational Studies

Contemporary mathematical logic is conveniently classified into the parts listed below, which extend into the many-valued domain of fuzzy structures by means of judicious fuzzification. It can be seen that Zadeh and his disciples attempted to fuzzify with success some of these, now classical parts of mathematical logic. Some selected references of such attempts are listed together with the classification below.

- Recursive functions.
- Set theory.
- Arithmetics.

- Quantification & identity.
- Propositional logic.

Although the above hierarchy covers what is known as mathematical logic – the logic intimately linked with the foundations of mathematics and computation, other approaches to logic stem from the linguistic philosophy and the linguistic proper. So, in any foundational studies, attention has to be paid also to these.

Höhle and Stout ask a pertinent question in the context of foundational studies, and offer an answer [14]

> What should the study of foundations of fuzzy sets offer? Certainly it should place fuzzy sets in a longer and broader tradition of many- valued mathematics ... but it must also speak to the needs of the practitioner of applied fuzzy set theory. A foundation for fuzzy set theory should provide a rigorous base for the actual practice of those applying the theory. ... people working with fuzzy sets want to use them for practical purposes ... These practitioners need a fuzzy set theory which is robust ... not particularly sensitive to the details of the model and connectives used but flexible enough so that the model can be tuned to provide high levels of performance. Thus a foundation for fuzzy sets needs to provide for a variety of connectives while clarifying the bounds on choices available.
>
> The second property that foundation should have is elegance. ... We can also ask if a foundation can take into account the 'linguistic variables' and experimental, computational approach

The suggestions are a good start, but in my opinion, one has to go even further. One has to build on algebraic strength of many-valued logic also learning from its failure to tap the conceptual and formal resources of contemporary philosophical logic.

The foundations of fuzzy sets, logics and systems contain some general systemic concepts that run across the boundary between theory and methodology. Although the initial motivation came from Systems Science through the important work of Zadeh that predated his first paper on fuzzy sets in 1965, the field has become rather fragmented in the last decade, losing to a great extent its initial cross disciplinary character. There is also a wide gap between mathematical and philosophical formal logic. Mathematical theory of General Systems has some features that may help to bridge this gap by mediating communication between the two disparate logic disciplines. Also the notions of *dynamics, stability, approximation, optimisation* etc. may provide a fertile ground for formalisation employing the notion of many-valuedness; in particular in the form of many-valued logic based algebraic theories of relations.

So, in a foundational analysis we have to distinguish sharply not only

1. mathematical questions,
2. logical questions,
3. ontological, epistemological and metaphysical questions,

but also look at their interrelationship, with particular emphasis on many-valued systems. For example, there are some interesting links between the mathematical and logical features of fuzzy structures of any kind and the ontological and epistemological questions of the foundational concepts. In order to bring these out explicitly, we need to employ an adequate method of conceptual analysis. In (1) we deal with the structure, in (2) we add to the structure the logical form. In (3) we deal with the problem of ontology, epistemology of the primitive concepts and perhaps, some *minimal* metaphysics of the systems involved; and also with with the questions of selection and justification of the appropriate meaning of the concepts employed. We have also to add the problematics of methods of enquiry and problem solving. This provides us with a conceptual framework, on the backcloth of which we should judge the issues dealt with comparative studies of various approaches in the field of fuzzy sets and systems.

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