Enriched Generic Algebras of Fuzzy Relations

Ladislav J. Kohout
Dept of Computer Science, Florida State University,
Tallahassee, Florida 32306-4530, U.S.A.
kohout@cs.fsu.edu

Abstract

The paper presents an overview of a computation friendly calculus of fuzzy relations. It is presented within the framework of an enriched generic algebra of relations that we have been developing since 1979. It is enriched with BK-non-associative products of relations of three kinds: triangle subproduct $\lhd$, superproduct $\rhd$ and the square product $\Box$. The BK-products have been useful in concise formulation of new theorems in relational mathematics as well as in a number of practical applications in medicine, engineering, information retrieval and elsewhere. The concise algebraic manipulation is also advantageous in symbolic computing. Currently we are engaged in developing an equational theorem checker in which the BK-products play a substantial role. In this endeavour, Tarski style relational calculi play an essential role.

Keywords: Fuzzy relations, non-associative compositions, Fuzzy BK-products, relational calculus, BL fuzzy logic.

1 Enriched Calculus of Fuzzy Relations

1.1 Enhancing Expressive Power of Calculus of Relations

There are six distinguishing features of the BK-product systems of relations that facilitate the unification of different many-valued systems of fuzzy relations and enhance their practical applicability.

1 Non-associative BK-products are introduced and used in definitions of relational properties and in computations. These products are defined for both homogeneous and heterogeneous relations.

2 Homomorphisms between relations are extended from mappings used in the literature to general relations. This yields *generalized morphisms* important for practical solving of relational inequalities and equations.

3 Relational properties are not only global but also local (important for applications).

4 The unified treatment of computational algorithms by means of matrix notation is used.

5 The theory unifying crisp and fuzzy relations in some distinguished logics makes it possible to represent a whole *finite nested family* of crisp relations with special properties as a *single* cutworthy [11] fuzzy relation for the purpose of computation. After completing the computations, the resulting fuzzy relation is again converted by α-cuts to a nested family of crisp relations, thus increasing the computing performance considerably.

6 Relations in their predicate forms are distinguished from their satisfaction sets; foresets and aftersets of relations are used in addition to relational predicates. This makes it possible to introduce interpretable linguistic labels (semiotic descriptors) that have a clearly defined meaning within the domains of their applications. Then one can develop an algebra of meaning defined by equations and inequalities that provides a computational basis for forming ontologies in knowledge engineering applications as well as in *computing with words.*

1.2 Basic Notions

Propositional Form. A *binary relation from A to B* is given by an open predicate ____P____ with two empty slots; when the first is filled with the name a of an element of A and the second with the name b of an element of B, there results a proposition. If aPb is true, we write aR_Pb and say that *a is R_P-related to*

b[1]. The lattice of all binary (two-place, 2-argument) relations from A to B is denoted by $\mathcal{R}(A \rightsquigarrow B)$. Relations of this kind are called *heterogeneous*. When the set B happens to be the same as A, we speak of relations **within a set** or *in a set*, or *on a set*, and call these *homogeneous*.

The **Satisfaction Set** The *satisfaction set* or *extension set* of a relation $R \in \mathcal{R}(A \rightsquigarrow B)$ is the set of all those pairs $(a, b) \in A \times B$ for which it holds:

$$R_S = \{(a, b) \in A \times B \mid aRb\}$$

Clearly R_S is a subset of the Cartesian product $A \times B$. We have $R_P = R'_P \implies R_S = R'_S$. That means, knowing R_P, we know R_S; knowing R_S, we know everything about R_P except the wording of its "name" ____P____. Because in symbolic computing with relations we need to deal with names of predicates R_P, it is essential distinguish notationally by R_P the cases when we do not deal just with satisfaction sets R_S(i.e. extensions) of relations.

The **Extensionality Convention** says that, regardless of their propositional wordings, two relations should be regarded as the same if they hold, or fail to hold between exactly the same pairs: $R_S = R'_S \implies R_P = R'_P$. Once the extensionality convention has been adopted, then there is a one-to-one correspondence between the subsets R_S of $A \times B$ and the (distinguishable) relations R_P in $\mathcal{R}(A \rightsquigarrow B)$. Since R_S and R_P now uniquely determine each other, the current fashion for set-theoretical parsimony suggests that they be identified. We, however, maintain the distinction in principle.

1.3 Crisp and Fuzzy Nonassociative Compositions of Relations

In 1977 Bandler and Kohout [2] introduced nonassociative relational compositions $\lhd, \rhd, \square$ that further extended the crisp relational calculus [26],[27],[23],[12],[13],[14]. The fuzzy version of these products was first published in [3], for succinct surveys see [9], [23].

The family $\{\lhd, \rhd, \circ\}$ is based on the positive fragment of logic, avoiding carefully the negation. Hence many algebraic properties of this triad of relational products carry over from the crisp relational calculus to fuzzy calculi for residuated systems, e.g. up to the monoidal logics based relational calculi[2].

[1]When it is unnecessary to emphasize the propositional form the subscript is dropped in R_P, writing: R, aRb.

[2]Other calculi are investigated (distinct from the monoidal logic based ones) which are based on non-commutative and/or.

Table 1: Definitions of Relational Products

Product Type	Set-based Definition
Zadeh's Circle product:	$x(R \circ S)z \Leftrightarrow$ xR intersects Sz
BK-Triangle Subproduct:	$x(R \lhd S)z \Leftrightarrow xR \,\tilde{\subseteq}\, Sz$
BK-Triangle Superprod.:	$x(R \rhd S)z \Leftrightarrow xR \,\tilde{\supseteq}\, Sz$
BK-Square product:	$x(R \square S)z \Leftrightarrow xR \cong Sz$

Many-Valued Logic DEF.	Tensor Notation
$(R \circ S)_{ik} = \bigvee_j (R_{ij} \& S_{jk})$	$(R \circ S)_{ik} = R_{ij} \circ S^{jk}$
$(R \lhd S)_{ik} = \bigwedge_j (R_{ij} \to S_{jk})$	$(R \lhd S)_{ik} = R_{ij} \lhd S^{jk}$
$(R \rhd S)_{ik} = \bigwedge_j (R_{ij} \leftarrow S_{jk})$	$(R \rhd S)_{ik} = R_{ij} \rhd S^{jk}$
$(R \square S)_{ik} = \bigwedge_j (R_{ij} \equiv S_{jk})$	$(R \square S)_{ik} = R_{ij} \square S^{jk}$

Table 1 of definitions three different notational forms for BK-products:

1. the notation using the concept of set inclusion and equality [4, 5].
2. many-valued logic (MVL) based notation[3], which uses the quantifier $\bigwedge$ and a connective $\to$, or $\leftarrow$, or $\equiv$.
3. the tensor notation[4].

1.4 Granular Form of BK-Products

The set based-definition uses fuzzy granules: subsets of the elements of the relations composed. Conceptually, and also for computational reasons, it is essential to distinguish two different kinds of granules: foresets and aftersets.

The fuzzy **afterset** of $x \in X$ is the *fuzzy subset of* Y consisting of the elements $y \in Y$ to which x is related by R (where $\mu_A x = \delta\{xRy\}$, the degree to which x and y are R-related):

$$xR = \{y \mid y \in Y \text{ and } \delta\{xRy\} > 0\}.$$

The fuzzy **foreset** of $y/\delta\{xRy\} \in Y$ is the *fuzzy subset of* X consisting of all the elements $x \in X$ which are related by R to y (where $\mu_A y = \delta\{xRy\}$, the degree to which x and y are R-related):

$$Ry = \{x/\delta\{xRy\} \mid x \in X \text{ and } \delta\{xRy\} > 0\}.$$

[3]With appropriately defined fuzzy power set and fuzzy set inclusion, the notational forms (1) and (2) of these relational compositions are algebraically equivalent.

[4]The tensor notation preserves in addition the inner structure of the composition when the right hand side of the form (3) is used in the formulas.

The notions of afterset and foreset of an element can be extended to afterset and foreset of a set in (at least) two distinct but equally important ways: an *inclusive* or *exclusive* afterset / foreset (see [8]).

The **inclusive** after- and foresets are given by

$$A'R = A'_R \circ R, \quad RB' = R \circ B'_R$$

The **exclusive** after- and foresets are given by

$$[A']R = A'_R \lhd R, \quad R[B'] = R \rhd B'_R$$

More explicitly, its component-wise definition involves indexed elements.

$$C_k = (A \circ R) = \bigvee_i (A_i \;\&\; R_{ik})$$

$$C_k = (A \lhd R) = \bigwedge_i (A_i \to R_{ik})$$

$$C_k = (A \rhd R) = \bigwedge_i (A_i \leftarrow R_{ik})$$

where A_i is the membership (characteristic function) giving the degree to which the predicate $a_i \in A$ is TRUE; and R_{ij} is the degree to which the predicate $R_{ij} \in R$ is TRUE, where R_{ij} is an element of R.

1.5 N-ary Relations

Operations can also be defined on n-ary relations for $n > 2$.

1.5.1 Intersections, Unions, Inclusions

Intersection, union, complementation and inclusion all have their definitions, and endow the set $\mathcal{R}_F(X_1, ..., X_n)$, of all fuzzy $n-$ary relations on $X_1, ..., X_n$ with the usual lattice, structure within which the crisp relations constitute a Boolean sublattice.

1.5.2 Products:

$\mathcal{R}_F(X_1, ..., X_r) \times \mathcal{R}_F(Y_1, ..., Y_s)$ yields
$\mathcal{R}_F(X_1, ..., \hat{X}_m, ..., X_r, Y_1, ..., \hat{Y}_p, ..., Y_s)$.

Products play an equally important part with $n-$ary relations [7],[9] as they do with binary. There are a number of different kinds, which can be distinguished as follows. As usual, the matrix formulation will be the clearest.

We are given an $r-$ary relation R on $(X_1, ..., X_r)$ and an $s-$ary relation on $(Y_1, ..., Y_s)$ such that (at least) one of the X sets, say X_m is the same as one of the Y sets, say Y_p. There are also given two operations on the numbers in [0, 1], which we symbolize by $\oplus$ and $\odot$. We indicate by circumflex or hat above a set or an index its omission from the list in which it is shown. Then we have the following

Definition 1 *The $(\oplus, m, p, \odot)$-product of R and S is the $(r + s - 2)$-ary relation T on the sets $(X_1, ..., \hat{X}_m, ..., X_r, Y_1, ..., \hat{Y}_p, ..., Y_s)$ given by*

$$T_{i_1 \ldots \hat{i}_m \ldots i_r, j_1 \ldots \hat{j}_p \ldots j_s} =$$

$$\bigoplus_{i_m = i_p} (R_{i_1 \ldots i_m \ldots i_r} \odot S_{j_1 \ldots j_p \ldots j_s}).$$

Where $\vee$ indicates maximum or, where appropriate, supremum, and $\wedge$ indicates minimum or infimum, and & a t-norm. The usual Zadeh's circle product is given by $\circ = (sup, 2, 1, \wedge)$. The harsh and mean subtriangular products are respectively $\lhd_h = (inf, \ 2, 1, \to)$ and $\lhd_m = (1/ \mid x_2 \mid \sum, 2, 1, \to)$ and correspondingly for supertriangular and square products. These same kinds of products play equally important role with $n-$ary relations. The only enrichment is the possible substitutions for the 2 and the 1. It may be worth noting that the ordinary matrix product of conventional linear algebra is $(\sum, 2, 1, \cdot)$; the influence of this on the notation of the definition should be clear.

2 Tarski's Calculus of Crisp Relations

The axioms defining the theory of crisp binary homogeneous relations were given by Tarski in 1941, in his now classical paper [29]. Using the first order predicate logic, he gave twelve axioms (1)–(12). Let us take for example the following axioms:

4 $(\forall x)(\forall y)(\forall z)[(xRy \wedge yEz) \to xRz]$

11 $(\forall x)(\forall z)[x(R \circ S)z \leftrightarrow (\exists y)(xRy \wedge ySz)]$

12 $R = S \leftrightarrow (\forall x)(\forall y)[xRy \leftrightarrow xSy)]$

Axiom (4) determines the algebraic behaviour of E; namely, it tells us that E is the two-sided identity. Axiom (11) defines the conventional relation composition $\circ$ (i.e. *circle product*). Finally, (12) defines the *equality* of two relations.

Using the predicate logic, Tarski has constructed the calculus of relations as a part of more comprehensive logical theory consisting of the following components:

1. Individual variables *x, y, z, ...*;
2. relation variables *R, S, P, Q, ...*.
3. Logical constants of a predicate calculus: e.g. connectives $\wedge, \vee, \&, \to, \equiv, \neg; \forall, \exists$.
4. Rules of inference.

From variables and constants various expressions are formed, namely:

- Elementary sentences, xRy (x is related to y).
- Compound sentences: well-formed formulas composed in the usual way of sentences by means connectives and quantifiers.

This basic system is extended by introducing further constants that are specific to the calculus of relations. Tarski [29] uses the following eleven constants:

1. Relations: U.. universal, O ... null , E identity, D ... diversity.
2. Unary operations: the complement, the converse (transpose).
3. Binary operations: $\sqcup$, $\sqcap$, $\circ$ .... relational composition (product) and its dual $\bullet$.
4. The identity predicate: $=$

Relational designations are the expressions formed from relation variables, relation constants, and operation signs.

Elementary sentences in this extension are expressions of the form 'xRy' and '$R = S$', where 'x' and 'y' stand for any individual variables and 'R' and 'S' for any relational designation.

After eliminating individual variables and logical constants Tarski obtained axioms of pure relational calculus that contain only *relational constants*, relational *unary operations*, relational *binary operations* and the *identity predicate*. In this way Tarski obtained fifteen axioms (I)-(XV) of crisp (Boolean, non-fuzzy) relational calculus [29].

As an illustration, let us look at

Axiom X. $(R \circ (S \circ Q)) = ((R \circ S) \circ Q)$
Axiom XI. $R \circ E = R$
Axiom XIII. $((R \circ S) \sqcap Q^T = O) \rightarrow ((S \circ Q) \sqcap R^T = O)$

Axioms (I) – (XV) of relational calculus can be translated back into the first order logic expressions by means of the first order logic relational axioms (1) – (12). This translation yields axioms (I-C) – (XV-C). For illustration, the axioms IX, X, XI, XIII are translated as follows:

X-C. $(\forall x)(\forall v)((\exists y)(xRy \wedge (\exists z)(ySz \wedge zQv)) \equiv (\exists z)((\exists y)(xRy \wedge ySz) \wedge zQv))$
XI-C. $(\forall x)(\forall y)((xRy \wedge yEy) \equiv xRy)$
XIII-C $(\forall x)(\forall z)(((x(R \circ S)z \wedge xQ^T z) \equiv xOz) \rightarrow$

$$((\forall y)(\forall x)((y(S \circ Q)x \wedge yR^T x) \equiv yOx))$$

Note that whereas axioms (I) – (XV) do not have quantifiers and do not refer to the individual elements of relations, the axioms (I-C) – (XV-C) do. In fact, the latter quantify over the variables denoting individual elements.

3 BK-Calculus in Tarski Style: Crisp Relations

BK-products and all our crisp work was strongly motivated by the work of Otakar Boruvka. The style of mathematical proofs was essentially similar to what we have learned from works of Otakar Boruvka, Eduard Čech, and van Der Waerden[5] Only when the work was substantially developed, and we started looking at strict formalization in predicate calculus we understood the significance of Tarski's paper for our work.

3.1 Characterization of Special Properties of Relations Between Two Sets

Self-inverse circle product is very useful in characterization of *special properties of relations* between two sets. Using the product, one can characterize these properties in purely relational way, without directly referring to individual elements of the relations involved.

Theorem 2 *Special properties of a heterogeneous relation $R \in \mathcal{R}(X \rightsquigarrow Y)$*

1. *R is covering* $\Leftrightarrow E_X \sqsubseteq R \circ R^{-1}$.
2. *R is univalent* $\Leftrightarrow R^{-1} \circ R \sqsubseteq E_Y$.
3. *R is onto* $\Leftrightarrow E_Y \sqsubseteq R^{-1} \circ R$.
4. *R is separating* $\Leftrightarrow R \circ R^{-1} \sqsubseteq E_X$.

where E_X and E_Y are the left and right identities, respectively. Note that the relational inclusion $\sqsubseteq$ is distinguished from the set inclusion $\subseteq$.

3.2 Characterization of Special Properties of Relations On a Set

Defining a relation R on a set (i.e. a homogeneous relation, $R \in \mathcal{R}(X \rightsquigarrow X)$) allows one to work with

[5] Wyllis Bandler's Ph.D. supervisor was van Der Waerden. Indeed, without WB's familiarity with van Der Waerden's methods and my knowledge of Boruvka's and Čech's work our work on crisp relations [2] would not have developed that way. It was a lucky coincidence that it is translatable into the style of Tarski [29]. This provided a bridge to using BL logic of Hájek for rigorous fuzzification.

additional properties. Relational properties of homogeneous relations are well covered in the literature. *Local reflexivity* is an exception. It appeared in [2] and was generalized to fuzzy relations in [6], leading to new computational algorithms for identification of relational properties of data-sets both crisp and fuzzy relations [6],[10].

1. *Covering* $\Leftrightarrow$ every x_i is related by R to something $\Leftrightarrow \forall i \in I, \quad \exists j \in I, \quad$ s.t. $\quad R_{ij} = 1$.
2. Locally reflexive $\Leftrightarrow$ if x_i is related to anything, or if anything is related to x_i, then x_i is related to itself $\Leftrightarrow \forall i \in I, \quad R_{ii} = max_j(R_{ij}, R_{ji})$
3. *Reflexive* $\Leftrightarrow$ covering and locally reflexive $\Leftrightarrow \forall i \in I, \quad R_{ii} = 1$

Unfortunately, it is absent from the textbooks, yet it is extremely important in applications of relational methods to analysis of the real life data (see the notion of participant in the next two subsections).

3.2.1 Partitions IN and ON a Set

A *partition* **on** a set X is a division of X into non-overlapping (and nonempty) subsets called blocks. A partition **in** a set X is a partition on the subset of X called the *subset of participants*.

There is a one-to-one correspondence between partitions **in** X and *local equivalences* (i.e. locally reflexive, symmetric relations and transitive relations) in $\mathcal{R}(X \rightsquigarrow B)$. The partitions in X (so also the local equivalences in $\mathcal{R}(X \rightsquigarrow B)$) form a lattice with `"___is-finer-than___"` as its ordering relation.

To obtain a global equivalence[6] one needs to add the covering property to the properties of local equivalence, so that local reflexivity turns into a (total) reflexivity. Equivalences have the following universal representation:

Theorem 3 *[8],[2]*
$R = R\Box R^{-1}$ *if and only if* R *is an equivalence.*

3.2.2 Tolerances and Overlapping Classes

- $R \circ R^T$ is always symmetric and locally reflexive
- $R \circ R^T$ is a tolerance iff R is covering.
- $R\Box R^T$ is always a (local) tolerance
- $R\Box R^T \sqsubseteq R$ iff R is reflexive
- $E \sqsubseteq R \sqsubseteq R\Box R^T$ iff R is an equivalence
- $R\Box R^T = R$ iff R is an equivalence
- $R\Box R^T \sqsubseteq R \circ R^T$ iff R is covering.

Tests for tolerance and equivalence.

[6]usually called just "equivalence".

It is not always the case that one manages, or even attempts, to classify participants into non-overlapping blocks. Local *tolerance* (i.e. locally reflexive and symmetric) relations lead to classes which may well overlap, where one participant may belong to more than one class.

4 Fuzzification

Many of the theorems and formulas proven in standard predicate logic discussed in the previous section generalize to BL logics when the negation is not involved. The purpose of this section is to demonstrate that the relational calculus considerably simplifies the proofs and relational computations in BL fuzzy logics.

4.1 Residuated bootstrap of BK-products

Important formulas of the BK-enriched relational calculus are given by the next theorem. This theorem which describes the interrelationship of $\circ, \lhd, \rhd$ plays a substantial role in further development of the theory of crisp and fuzzy relations. Because these formulas depend on residuation, they carry over into relational theories based on t-norms and corresponding residuated implication operators[7] [25],[20].

Theorem 4 *Residuated bootstrap of BK-products [21],[25] For arbitrary* $V \in \mathcal{B}(A \rightsquigarrow C)$,

$$(R \circ S \sqsubseteq V) \;\equiv\; (R \sqsubseteq V \rhd S^T) \;\equiv\; (S \sqsubseteq R^T \lhd V)$$

Proof: [22],[25]
$BL \vdash (\forall x)(\varphi \to \nu) \equiv (\exists x)\varphi \to \nu$ *(by [15], T 5.1.4(2)).*
The substitutions $\varphi := xRy\&ySz, \nu := xVz$ *yield*
$\vdash (\forall y)((xRy\&ySz) \to xVz) \equiv ((\exists y)(xRy\&ySz) \to xVz)$
$\vdash (\forall y)((xRy\&ySz) \to xVz) \to ((\exists y)(xRy\&ySz) \to xVz)$ *by [15], L 2.2.16(25).*
$\vdash (\forall x)(\forall z)(\forall y)((xRy\&ySz) \to xVz) \to (\forall x)(\forall z)((\exists y)(xRy\&ySz) \to xVz))$ *by [15], T 5.1.16(5) and MP.*
$\vdash (\forall x)(\forall y)(xRy \to (\forall z)(xVz \leftarrow zS^Ty)) \to (\forall x)(\forall z)((\exists y)(xRy\&ySz) \to xVz))$ *By [15], T 15.1.14(1).*
$(A) \vdash (\forall x)(\forall y)(xRy \to x(V \rhd S^T)y) \to (\forall x)(\forall z)(x(R \circ S)z \to xVz)$
Similarly, we derive (B)
$(B) \vdash (\forall x)(\forall y)(xRy \to x(V \rhd S^Ty) \leftarrow (\forall x)(\forall z)(x(R \circ S)z \to xVz)$
Substituting $\varphi := (A), \psi := (B)$ *into* $\vdash \varphi \to (\psi \to \varphi\&\psi)$,
applying MP twice and using the definition of $\equiv$ *yields*
$(B) \vdash (\forall x)(\forall y)(xRy \to x(V \rhd S^T)y) \equiv$

[7]In fact, it generalizes even further, to fuzzy Monoidal logics [24].

$(\forall x)(\forall z)(x(R \circ S)z \rightarrow xVz)$
which is $\vdash (R \sqsubseteq V \triangleright S^T) \equiv (R \circ S \sqsubseteq V)$

Similar proofs can be given for other equivalences of Th. 4.

Theorem 4 has been proven in BL logic. It is an important equality of BK-relational calculus that can be used directly for proofs of other theorems. We shall give some examples in the sections that follow. For further details and discussion of the importance of this theorem as the core statement in axiomatization of relations see [25].

4.1.1 Fuzzy Generalised Morphisms

Very important for distributed knowledge networking [19],[28] is a generalization of conventional homomorphisms defined constructively by BK-products [21].:

Definition 5 *Let F, R, G, S be the relations between the sets A, B, C, D such that $R \in \mathcal{R}(A \rightsquigarrow B)$, $S \in \mathcal{R}(C \rightsquigarrow D)$, $F \in \mathcal{R}(A \rightsquigarrow C)$, $G \in \mathcal{R}(B \rightsquigarrow D)$. The conditions that (for all $a \in A, b \in B, c \in C, d \in D$) aFc and aRb and bGd imply cSd, can be expressed in any of the following ways:*
(i) $FRG : S$ are **forward compatible**
(ii) F, G are **generalized homomorphisms** *from R to S.*

Let $FRG : S$ denote relations that satisfy the forward compatibility criterion $(F^{-1} \circ R \circ G \sqsubseteq S)$. This criterion is fulfilled iff $(R \sqsubseteq F \triangleleft S \triangleright G^{-1})$. Furthermore, we have the following theorem:

Theorem 6

$$(F^{-1} \circ R \circ G \sqsubseteq S) \quad \equiv \quad (R \sqsubseteq F \triangleleft S \triangleright G^{-1})$$

The direct proof in BL logic is more involved than the proof of Theorem 4 (bootstrap). It gets simplified if done in the equalities of Th. 4 [25],[20].
Proof:
Substituting $T := F^{-1}$, $U := R \circ G$, $V := S$ we obtain $(F^{-1} \circ R \circ G \sqsubseteq S) \equiv (R \circ G \sqsubseteq F \triangleleft S)$.
Substituting $T := R$, $U := G$, $V := F \triangleleft S$ we obtain $(R \circ G \sqsubseteq F \triangleleft S) \equiv (R \sqsubseteq F \triangleleft S \triangleright G^{-1})$.
Transitivity of equivalences yields $(F^{-1} \circ R \circ G \sqsubseteq S) \equiv (R \sqsubseteq F \triangleleft S \triangleright G^{-1})$. This completes the proof.

4.2 Characterization of Fuzzy Classivalent Relations

Another important relational property is *classivalence* (partial difunctionality).

Definition 7 *A fuzzy relation $R \in \mathcal{R}(X \rightarrow X)$ in a t-norm residuated logic (BL) is called classivalent (partial difunctional) when it satisfies the following condition: $(\forall a)(\forall b)(\forall a')(\forall b')(aRb \& a'Rb \& a'Rb' \rightarrow aRb')$ where $\&$ is a t-norm and $\rightarrow$ is the residuated implication operator associated with $\&$.*

Theorem 8 *It is provable in BL that a fuzzy relation is classivalent (partial bifunctional) iff $R \circ R^T \circ R \sqsubseteq R$*

Theorem 9 *Let $R \in \mathcal{R}(X \rightsquigarrow X)$ be a classivalent (partial functional) relation a t-norm residuated logic BL. Then the following equivalence holds:*

$$(R \circ R^T \circ R \sqsubseteq R) \equiv (R \circ R^T \sqsubseteq R \triangleright (R^T \circ R))$$
$$\equiv (R^T \circ R \sqsubseteq R^T \triangleleft R)$$

Proof:
Substituting $R := R$, and also $S := R^T \circ R$ into the equivalences of Theorem 4 (Residuated Bootstrap) yields $(R \circ R^T \circ R \sqsubseteq R) \equiv (R \sqsubseteq R \triangleright (R^T \circ R)) \equiv (R^T \circ R \sqsubseteq R^T \triangleleft R)$ ¶

In a similar way, the *residuated bootstrap of BK-products* is used to prove (in BL) other theorems (cf. [25]) concerned with the theory of Generalized Morphisms. A software tool GMorph is described in [17].

4.3 Fuzzification of Tarski's Relational Systems

Axioms I, II, III, V, VII, VIII, IX, X, XIII of the original system of Tarski [29] described in Sec. 2 above carry over to BL completely. The remaining axioms IV, VI, XII, XIV, XV have more alternatives.

We have seen that in his 1941 paper, Tarski introduced two different axiomatizations for the fragment of the calculus of relations. The axioms introduced by the first method based on the predicate logic are axioms (1) – (12) mentioned in Sec.2 above[8]. The second method is exemplified by axioms I – XV listed above. The second method led subsequently to the theory of relational algebras.

The first axiomatization of relational algebras appeared in [16] by Jónsson and Tarski.

They added to all basic axioms of Boolean algebra the following formulas:

JT1: $(R^T)^T = R$
JT2: $(R \circ S) \circ T = R \circ (S \circ T)$
JT3: $(R \sqcup S) \circ T = R \circ T \sqcup S \circ T$
JT4: $R \circ (S \sqcup T) = (R \circ S) \sqcup (R \circ T)$
JT5: $(R \sqcup S)^T = R^T \sqcup S^T$
JT6: $R \circ E = R$
JT7: $R^T \circ \overline{(R \circ S)} \sqsubseteq \overline{S}$

[8]For the full list see [29].

The most interesting is JT7. It contains 3 operations: 1-ary complement and transpose; and 2-ary circle product $\circ$.

It is more difficult to prove that the last axiom, namely

$$R^T \circ \overline{R \circ S} \sqsubseteq \overline{S}$$

caries over to BL fuzzification. We shall now prove the theorem to that effect.

The substitutions $R := R^T$; $S := \overline{R \circ S}$; $V := \overline{S}$ into the formula of Theorem 4 yield

Theorem 10

$$R^T \circ \overline{R \circ S} \sqsubseteq \overline{S} \quad \equiv \quad \overline{R \circ S} \sqsubseteq (R^T)^T \lhd \overline{S}$$

Theorem 11

$$\overline{R \circ S} \sqsubseteq (R^T)^T \lhd \overline{S}$$

is a 1-tautology in BL.

5 Conclusions

Taking the Program of Tarski, extending the mathematics of relations – this can be considered as the goal of our programme, This is compatible with the General System theory. Indeed, Klir in his GS publications [18] emphasises the importance of relations in GST. This, of course, should be compatible with the programme of fuzzy mathematics. Already, our theory of extended relations defines relations as systems with foresets and aftersets; relations are presents as predicates, as well as satisfaction sets. Link with map algebras is also important. This has also has to be considered.

Now, the question is, how this link of relational algebra/calculus with fuzzy set theory is to be provided. L-categories of Wyllis Bandler give the lead [1]. There are two different approaches: one is meta–mathematics and the other is eso–mathematical use of category theory[9]

Esomathematical use of category theory already indicates that to distinguish different roles, different contexts[10] we need to compose morphism bunches of different colours, and composition of such bunches is not necessarily associative [1].

[9]This is connected with external and internal – when we try import from the external to the internal language, the connectives. ...This is also connected to formal fuzzy logic, inference rules and implication, when we import the deduction theorem into the system.

[10]the associativity of category theory is harmful there

References

[1] W. Bandler. Some esomathematical uses of category theory. In G. Klir, editor, *Applied General Systems Research: Recent Developments and Trends*, NATO Conference Series II, Vol. 5, pages 243–255. Plenum Press, New York, 1978.

[2] W. Bandler and L. Kohout. *Mathematical Relations, their Products and Generalized Morphisms.* Tech. report, Man-Machine Systems Laboratory, EES-MMS-REL 77-3, Dept. of Electrical Eng., University of Essex, Colchester, Essex, U.K., 1977.

[3] W. Bandler and L. Kohout. Fuzzy relational products and fuzzy implication operators. In *International Workshop on Fuzzy Reasoning Theory and Applications*, London, September 1978. Queen Mary College, University of London. Also available as FRP-1 report.

[4] W. Bandler and L. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:13–30, 1980. Reprinted in: *Readings in Fuzzy Sets for Intelligent Systems*, D. Dubois, H. Prade and R. Yager (eds.), Morgan Kaufmann Publishers, San Mateo, Calif., 1993, pages 88-96.

[5] W. Bandler and L. Kohout. Fuzzy relational products as a tool for analysis and synthesis of the behaviour of complex natural and artificial systems. In P. Wang and S. Chang, editors, *Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems*, pages 341–367. Plenum Press, New York and London, 1980.

[6] W. Bandler and L. Kohout. Fast fuzzy relational algorithms. In A. Ballester, D. Cardús, and E. Trillas, editors, *Proc. of the Second Internat. Conference on Mathematics at the Service of Man*, pages 123–131, Las Palmas, 1982. (Las Palmas, Canary Islands, Spain, 28 June - 3 July), Universidad Politechnica de las Palmas.

[7] W. Bandler and L. Kohout. Mathematical relations. In *Fuzzy Research Project*, University of Essex, Colchester, U.K., August 1982. Dept. of Mathematics. FRP-23 report.

[8] W. Bandler and L. Kohout. On the general theory of relational morphisms. *International Journal of General Systems*, 13:47–66, 1986.

[9] W. Bandler and L. Kohout. Relations, mathematical. In M. Singh, editor, *Systems and Control Encyclopedia*, pages 4000 – 4008. Pergamon Press, Oxford, 1987.

[10] W. Bandler and L. Kohout. Special properties, closures and interiors of crisp and fuzzy relations. *Fuzzy Sets and Systems*, 26(3):317–332, June 1988.

[11] W. Bandler and L. Kohout. Cuts commute with closures. In B. Lowen and M. Roubens, editors, *Fuzzy Logic: State of the Art*, pages 161–167. Kluwer Academic, Boston and Dordrecht, 1993.

[12] R. Bělohlávek. Similarity relations and BK-relational products. *Information Sciences*, 126(1-4):287–295, 2000.

[13] R. Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles.* Kluwer Academic/Plenum Press, New York, 2002.

[14] P. Hájek. A remark on Bandler-Kohout products of relations. *Internat. Journal of General Systems*, 25(2):165–166, 1996.

[15] P. Hájek. *Metamathematics of Fuzzy Logics.* Kluwer, Dordrecht, 1998.

[16] B. Jónsson and A. Tarski. Representation problems for relational algebras. *Bulletin of Amer. Math. Soc.*, 54:80 and 1192, 1948.

[17] E. Kim and L. Kohout. Design of GMORPH by means of Activity Structures. In R. M. et al., editor, *Proc. of IFSA 1997 (The world Congress of International Fuzzy Systems Association, Prague), vol. I.*, pages 84–89. IFSA, June 1997.

[18] G. Klir. *Architecture of Systems Problem Solving.* Plenum Press, New York, 1985.

[19] L. Kohout. Dynamic sharing of cognitive structures by means of generalized morphisms. In P. Wang, editor, *Proc. of the 2nd Joint Conf. on Information Sciences (Sept. 28 - Oct. 1, Wrightsville Beach, NC, USA)*, pages 597–600. Duke University, 1995.

[20] L. Kohout. Generalized morphisms in BL-logics. In *Logic Colloquium 1998 (The 1998 ASL European Summer Meeting, Prague).* Association for Symbolic Logic, August 9-15 1998. (Extended abstract presenting the main mathematical theorems published the Bulletin of ASL No. 1, March 1999, pp. 116-117.).

[21] L. Kohout. Generalized morphisms in BL-logics. *The Bulletin of Symbolic Logic*, 5(1):116–117, 1999.

[22] L. Kohout. Basic algebra of BK products of relations in t-norm fuzzy logics. In R. Langari and J. Yen, editors, *Proc. of FUZZ-IEEE2000*, pages 599–604, Piscataway, NJ, 2000. IEEE Neural Network Council, IEEE.

[23] L. Kohout. Boolean and fuzzy relations. In P. Pardalos and C. Floudas, editors, *The Encyclopedia of Optimization*, pages 189–202. Kluwer, Boston, 2001. vol.I, A-D.

[24] L. Kohout. Fuzzy relational equations in monoidal logics. In *Proc. 2004 IEEE Internat Conference on Fuzzy Systems*, pages 1555–1559, New York, July 2004. IEEE Neural Network Society, IEEE.

[25] L. Kohout. Theory of fuzzy generalized morphisms and relational inequalities. *Internat. J. of General Systems*, 33(4):339–360, 2004.

[26] L. Kohout and W. Bandler. Fuzzy relational products in knowledge engineering. In V. Novák et al., editor, *Fuzzy Approach to Reasoning and Decision Making*, pages 51–66. Academia and Kluwer, Prague and Dordrecht, 1992.

[27] L. Kohout and W. Bandler. Use of fuzzy relations in knowledge representation, acquisition and processing. In L. Zadeh and J. Kacprzyk, editors, *Fuzzy Logic for the Management of Uncertainty*, pages 415–435. John Wiley, New York, 1992.

[28] J. Muhammad and L. Kohout. The role of generalized morphisms in co-operative activities of agents. In P. Wang, editor, *Joint Conference on Information Sciences – JCIS'98 Proceedings*, pages 56–59, Research Triangle Park, N.C., October 1998. Association for Intelligent Machinery. In vol.1.

[29] A. Tarski. Calculus of relations. *Journal of Symbolic Logic*, 6(3):73–89, 1941.