Bounds for Value at Risk for Asymptotically Dependent Assets - the Copula Approach

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Abstract

The theory of copulas provides a useful tool for modeling dependence in risk management. In insurance and finance, as well as in other applications, dependence of extreme events is particularly important, hence there is a need for the detailed study of the tail behaviour of the multivariate copulas. In this paper we investigate the class of copulas having homogeneous lower tails. We show that having only such information on the structure of dependence of returns from assets is enough to get estimates on Value at Risk of the multiasset portfolio in terms of Value at Risk of one-asset portfolios.

Keywords: Copulas, Value at Risk, dependence of extreme events.

1 Introduction

In my presentation I shall deal with the advantages of modeling the dependence between the extremal events with the help of copulas. Let us consider the following case. An investor operating on an emerging market, has in his portfolio several currencies which are highly dependent. Let s_i, $i = 1, \ldots, d$ be the quotients of the currency rates at the end and at the beginning of the investment. Let w_i be the part of the capital invested in the i-th currency, $\sum w_i = 1$, $w_i \geq 0$. So the final value of the investment equals

$$W_1(w) = (w_1 s_1 + \ldots + w_d s_d) \cdot W_0.$$

For portfolio consisting of only one currency (say i -th) we put $w = e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$.

The crucial point is to estimate the risk of keeping the portfolio. As a measure of risk I shall consider "Value at Risk" (VaR), which last years became one of the most popular measures of risk in the "practical" quantitative finance (see for example [2, 20, 5, 6, 17, 16, 10, 19]). Roughly speaking the idea is to determine the biggest amount one can lose on a certain confidence level $1 - \alpha$

$$VaR_{1-\alpha}(w) = sup\{V : P(W_0 - W_1(w) \leq V) < 1 - \alpha\}.$$

Note that this quantity warns risk managers how much of "economic" capital (reserves) is needed to keep the solvency.
In order to determine VaR accurately one has to deal with the complexity of the problem. The extremes hardly follow the normal distribution law. Therefore the main challenge is to describe properly the interdependencies of risk factors (in our case the changes of currency rates). In this presentation, it will be based on copulas, which are scaleless dependency measures of random variables. I will show that sometimes it is enough to have only the partial information on the given copula.

The main result, I would like to present, is the following estimate of the Value at Risk of a given portfolio w in terms of Value at Risk of one-currency portfolios e_i (compare [14] for two dimensional case):

$$\sum w_i VaR_{1-\alpha}(e_i) \geq VaR_{1-\alpha}(w) \geq$$

$$\geq \sum w_i VaR_{1-\alpha'}(e_i),$$

where $\alpha' = \frac{\alpha^2}{C(\alpha,\ldots,\alpha)}$. The above estimate is valid for sufficiently small α under the mild assumptions:

- The lower tail part of the copula C of s_i's is nonzero and homogeneous of degree 1, i.e. for sufficiently small q

$$C(q) = L(q), \quad \forall t > 0 \quad L(tq) = tL(q).$$

- For $i = 1, \ldots, d$, for sufficiently small x, the function $G_i(x) = \frac{1}{F_i(x)}$, where F_i is the distribution function of s_i, is convex.

The first assumption is modelling the asymptotic dependence (compare [13] Th.2). For example it describes very well the behaviour of foreign exchange rates on an emerging market, where the *extremal* changes are usually due to the local factors (compare [14]).

The second one is fulfilled by a wide variety of probability laws. For example it is valid if the distributions of $-\ln s_i$ have the same upper tails as normal, Pareto or Gamma distribution (i.e. if their distribution functions coincide for enough big arguments). Moreover, it is easy to check that if $1 - F_i(-x)$ is a von Mises function, i.e

$$\exists z > 0 \quad \forall 0 < x < z \quad F_i(x) = c \exp\left(-\int_x^z \frac{1}{a(t)} dt\right),$$

where a is absolutely continuous and its density has limit 0 at the origin, then $G_i(x)$ is convex (for small x). Note that the von Mises functions played an important role in the Extreme Value Theory, they are classical examples of distribution functions belonging to the Maximum Domain of Attraction of the Gumbel Distribution (for details see [8] §3.3.3).

2 Notation

2.1 Copulas

We recall that a function

$$C : [0,1]^d \longrightarrow [0,1]$$

is called a copula (see [18] §2.10, [4] §4.1, [1] §4.4) if for every $u = (u_1, \ldots, u_d)$ and $v = (v_1, \ldots, v_d)$ $(u_i, v_i \in [0,1])$ and every $j \in \{1, \ldots, d\}$

$$\begin{array}{rl} i) & u_j = 0 \Rightarrow C(u) = 0; \\ ii) & (\forall i \neq j \;\; u_i = 1) \Rightarrow C(u) = u_j; \\ iii) & u \preceq v \Rightarrow V_C(u,v) \geq 0, \end{array}$$

where $u \preceq v$ denotes the partial ordering on $\mathbb{R}^d$,

$$u \preceq v \Leftrightarrow \forall i \quad u_i \leq v_i,$$

and $V_C(u,v)$ is the C-volume of the rectangle $I(u,v)$, the one with lower vertex u and upper vertex v.

$$V_C(u,v) = \sum_{j_1=1}^{2} \cdots \sum_{j_d=1}^{2} (-1)^{j_1+\ldots+j_d} C(a_{1,j_1}, \ldots, a_{d,j_d}),$$

where $a_{i,1} = u_i$ and $a_{i,2} = v_i$ for $i = 1, \ldots, d$.
The functions with the last property are called n-nondecreasing. Those which fulfill the first one are called grounded.

Note that every copula is nondecreasing not only with respect to each variable but also with respect to the partial ordering $\preceq$. Moreover it is continuous and even Lipschitz ([18], Theorem 2.10.7, [4], Lemma 4.2)

$$|C(v) - C(u)| \leq \sum_{i=1}^{d} |v_i - u_i|.$$

REMARK **2.1** *(cf. [3], Th. 12.5) Every continuous, grounded, n-nondecreasing function*

$$H : [0,a]^d \longrightarrow \mathbb{R}$$

is a distribution function of a Borel measure μ_H on $[0,a]^d$

$$H(u) = \mu_H(I(0,u)),$$

$$\mu_H(I(u,v)) = \mu_H(int(I(u,v))) = V_H(u,v).$$

Due to the second condition every copula is a distribution function of a probability measure on the unit rectangle $[0,1]^d$ with uniform margins (compare [15], §1.6). Furthermore, the above remark remains true if H is defined on the whole multioctant $[0,+\infty)^d$.

Let $\mathcal{X}_i$, $i = 1, \ldots, d$ be random variables defined on the same probability space $(\Omega, \mathcal{M}, \mathbb{P})$. Their joint cumulative distribution $F_{\mathcal{X}}$ can be described using an appropriate copula $C_{\mathcal{X}}$ ("Sklar Theorem" see [18], Theorem 2.10.11, [4], Theorem 4.5):

$$F_{\mathcal{X}}(x) = C_{\mathcal{X}}(F_{\mathcal{X}_1}(x_1), \ldots, F_{\mathcal{X}_d}(x_d)),$$

where $F_{\mathcal{X}_i}$ are cumulative distributions of $\mathcal{X}_i$. Note that the strictly increasing transformations of random variables $\mathcal{X}_i$ do not affect the copula. Indeed, if

$$\mathcal{X}'_i = f_i(\mathcal{X}_i), \;\; i = 1, \ldots, d,$$

where f_i are strictly increasing (and so invertible), then

$$F_{\mathcal{X}'}(x) = F_{\mathcal{X}}(f_1^{-1}(x_1), \ldots, f_d^{-1}(x_d)) =$$

$$= C_{\mathcal{X}}(F_{\mathcal{X}_1}(f_1^{-1}(x)), \ldots, F_{\mathcal{X}_d}(f_d^{-1}(x_d))) =$$

$$= C_{\mathcal{X}}(F_{\mathcal{X}'_1}(x_1), \ldots, F_{\mathcal{X}'_d}(x_d)).$$

Therefore if one is interested in tail dependence of random variables rather than in their *individual* distribution, then the proper choice is to study the copula. The more so, since the copula is uniquely determined at every point u such, that the equations $F_{\mathcal{X}_i}(x_i) = u_i$ have solutions.

2.2 Model assumptions

We assume that for $t > 0$ the distribution function of each $s_i - F_i(t)$ is positive and the joint probability distribution of s_i's is continuous with respect to Lebesgue measure and is determined by a copula C

$$F_s(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

Furthermore there is a constant $\delta \in (0,1)$ such that:
A1. For $q = (q_1, \ldots q_d)$ and $0 \leq q_i \leq \delta$, $C(q) = L(q)$, where L is some nonzero positive homogeneous function of degree one ($\forall t > 0 \quad L(tq) = tL(q)$).
A2. For $i = 1, \ldots, d$ the function $G_i(t) = \frac{1}{F_i(t)}$ restricted to $t \in F_i^{-1}((0, \delta])$ is convex.

The second assumption implies that the preimage of δ consists of just one point and F_i restricted to $[0, F_i^{-1}(\delta)]$ is strictly increasing. Moreover we get a simpler formula for Value at Risk of one-asset portfolios.

Corollary **2.1** *For $\alpha \in (0, \delta]$,*

$$VaR_{1-\alpha}(e_i) = W_0 \cdot (1 - F_i^{-1}(\alpha)), \quad i = 1, \ldots, d.$$

In [11, 13] we showed that there is a large class of copulas which tails can be approximated by a homogeneous function L. We recall the basics about L's. Comparing [13] Theorem 3 and the construction from the proof of Proposition 6 (also [13]) one gets:

Theorem **2.1** *For a homogeneous of degree 1 function L, $L : [0, +\infty)^d \to \mathbb{R}$, the following conditions are equivalent:*
1. L is equal to the lower tail of some copula C.
2. L is d-nondecreasing and

$$0 \leq L(u) \leq min(u_1, \ldots, u_d) \quad \text{for} \quad u \succeq 0.$$

3. L is continuous, grounded, d-nondecreasing and $\mu_L = m \times \mu_\Delta$, where m is the Lebesgue measure on the real halfline and μ_Δ is a measure on the unit simplex $\Delta = \{q \in \mathbb{R}_+^d : q_1 + \ldots + q_d = 1\}$ such that

$$\int_\Delta \frac{1}{q_i} d\mu_\Delta(q) \leq 1 \quad \text{for} \quad i = 1, \ldots, d.$$

3 Upper estimate

We assume, that $\forall i \;\; w_i > 0$.

Theorem **3.1** *For positive α such that*

$$\sum_{i=1}^d w_i F_i^{-1}(\alpha) \leq \min\{w_j F_j^{-1}(\delta) : j = 1, \ldots, d\}$$

the following inequality holds

$$VaR_{1-\alpha}(w) \leq w_1 VaR_{1-\alpha}(e_1) + \ldots + w_d VaR_{1-\alpha}(e_d).$$

For $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\lambda_i > 0$, we put

$$Y_\lambda = \{q \in \mathbb{R}_+^d \; : \; \sum_{i=1}^d \frac{\lambda_i}{q_i} \geq 1\}.$$

Lemma **3.1** .

$$\mu_L(Y_\lambda) \leq \sum \lambda_i.$$

Proof.
We base on the fact, that the multioctant is the Cartesian product of a halfline and simplex

$$\mathbb{R}_+^d = \mathbb{R}_+ \times \Delta.$$

Since L is homogeneous,

$$\mu_L(Y_\lambda) = \int_\Delta m(\mathbb{R}_+\xi \cap Y_\lambda) d\mu_\Delta(\xi).$$

The intersection of Y_λ and the halfline given by the vector ξ is a segment of length $\sum \frac{\lambda_i}{\xi_i}$,

$$\mathbb{R}_+\xi \cap Y_\lambda = \{t : \sum \frac{\lambda_i}{t\xi_i} \geq 1\} = \{t : 0 \leq t \leq \sum \frac{\lambda_i}{\xi_i}\}.$$

Therefore

$$\mu_L(Y_\lambda) = \int_\Delta \sum \frac{\lambda_i}{\xi_i} d\mu_\Delta(\xi) =$$

$$= \sum \lambda_i \int_\Delta \sum \frac{1}{\xi_i} d\mu_\Delta(\xi) \leq \sum \lambda_i.$$

□

For $r > 0$ we put

$$V_r = \{q \in \mathbb{R}_+^d \; : \; \sum_{i=1}^d w_i F_i^{-1}(q_i) \leq r\}.$$

Lemma **3.2** *For positive r and α such, that*

$$r = \sum_{i=1}^d w_i F_i^{-1}(\alpha) \leq \min\{w_j F_j^{-1}(\delta) : j = 1, \ldots, d\}$$

the following inclusions hold

$$V_r \subset [0, \delta]^d, \quad V_r \subset Y_\lambda,$$

where

$$\lambda_i = \alpha \frac{w_i c_i^{-1}}{\sum w_j c_j^{-1}}; \quad c_j = F_j'(F_j^{-1}(\alpha)).$$

Proof.
If q belongs to V_r then

$$\sum_{i=1}^d w_i F_i^{-1}(q_i) \leq r = \sum_{i=1}^d w_i F_i^{-1}(\alpha) \leq \min\{w_j F_j^{-1}(\delta)\}.$$

Therefore

$$w_i F_i^{-1}(q_i) \le w_i F_i^{-1}(\delta),$$

and $q_i \le \delta$.

To proof the second inclusion, we use the convexity of $G_i = \frac{1}{F_i}$.

$$\frac{1}{q_i} - \frac{1}{\alpha} = \frac{1}{F_i(F_i^{-1}(q_i))} - \frac{1}{F_i(F_i^{-1}(\alpha))} =$$

$$= G_i(F_i^{-1}(q_i)) - G_i(F_i^{-1}(\alpha)) \ge$$

$$\ge G_i'(F_i^{-1}(\alpha)) \cdot \left(F_i^{-1}(q_i) - F_i^{-1}(\alpha)\right) =$$

$$= \frac{-F_i'(F_i^{-1}(\alpha))}{(F_i(F_i^{-1}(\alpha)))^2} \cdot \left(F_i^{-1}(q_i) - F_i^{-1}(\alpha)\right) =$$

$$= -\frac{c_i}{\alpha^2}\left(F_i^{-1}(q_i) - F_i^{-1}(\alpha)\right)$$

thus

$$F_i^{-1}(q_i) - F_i^{-1}(\alpha) \ge -\frac{\alpha^2}{c_i}\left(\frac{1}{q_i} - \frac{1}{\alpha}\right).$$

If q belongs to V_r then

$$0 \ge \sum_{i=1}^d w_i F_i^{-1}(q_i) - r =$$

$$= \sum_{i=1}^d w_i F_i^{-1}(q_i) - \sum_{i=1}^d w_i F_i^{-1}(\alpha) \ge$$

$$\ge -\sum_{i=1}^d \frac{w_i \alpha^2}{c_i}\left(\frac{1}{q_i} - \frac{1}{\alpha}\right) =$$

$$= -\alpha\left(\sum_{i=1}^d \frac{\lambda_i}{q_i} \sum_{j=1}^d \frac{w_j}{c_j} - \sum_{i=1}^d \frac{w_i}{c_i}\right) =$$

$$= -\alpha \sum_{j=1}^d \frac{w_j}{c_j}\left(\sum_{i=1}^d \frac{\lambda_i}{q_i} - 1\right).$$

So

$$0 \le \sum \frac{\lambda_i}{q_i} - 1,$$

and therefore q belongs to Y_λ.

□

Proof of theorem 3.1 .
In order to estimate $VaR_{1-\alpha}(w)$ we consider

$$1 - P\left(W_0 - W_1(w) \le \sum w_i VaR_{1-\alpha}(e_i)\right) =$$

$$= P\left(W_0 - W_1(w) \ge \sum w_i VaR_{1-\alpha}(e_i)\right) =$$

$$= P\left(1 - \sum w_i s_i \ge \sum w_i (1 - F_i^{-1}(\alpha))\right) =$$

$$= P\left(\sum w_i s_i \le \sum w_i F_i^{-1}(\alpha)\right) =$$

$$= P(\sum w_i s_i \le r) = \mu_C(V_r) =$$

$$= \mu_L(V_r) \le \mu_L(Y_\lambda) \le \sum \lambda_i = \alpha.$$

So

$$P\left(W_0 - W_1(w) \le \sum w_i VaR_{1-\alpha}(e_i)\right) \ge 1 - \alpha.$$

In such a way we obtain the estimate

$$VaR_{1-\alpha}(w) \le \sum w_i VaR_{1-\alpha}(e_i).$$

□

4 Lower estimate

LEMMA **4.1** *If $r = \sum_{i=1}^d w_i F_i^{-1}(\alpha)$ then the multicube $[0,\alpha]^d$ is contained in V_r.*

Proof. If $q_i \le \alpha$ then

$$F_i^{-1}(q_i) \le F_i^{-1}(\alpha).$$

Therefore

$$w_1 F_1^{-1}(q_1) + \ldots + w_d F_d^{-1}(q_d) \le$$

$$\le w_1 F_1^{-1}(\alpha) + \ldots w_d F_d^{-1}(\alpha) = r.$$

□

THEOREM **4.1** *For $\alpha < \delta$*

$$VaR_{1-L(1,\ldots,1)\alpha}(w) \ge \sum w_i VaR_{1-\alpha}(e_i).$$

Proof. (compare [7])
Due to the homogeneity we get

$$\mu_C([0,\alpha]^d) = C(\alpha,\ldots,\alpha) = L(1,\ldots,1)\alpha.$$

Let $r = \sum_{i=1}^d w_i F_i^{-1}(\alpha) = 1 - (w_1 VaR_{1-\alpha}(e_1) + \ldots + w_d VaR_{1-\alpha}(e_d))/W_0$. Since the cube $[0,\alpha]^d$ is contained in V_r, we have

$$\mu_C(V_r) \ge C(\alpha,\ldots,\alpha) = L(1,\ldots,1)\alpha.$$

Therefore the $L(1,\ldots,1)\alpha$ quantile of $\frac{W_1(w)}{W_0}$ is smaller than r. Thus

$$VaR_{1-L(1,\ldots,1)\alpha}(w) \ge (1-r)W_0 =$$

$$= w_1 VaR_{1-\alpha}(e_1) + \ldots + \ldots + w_d VaR_{1-\alpha}(e_d).$$

□

By substitution $\alpha := \alpha/L(\ldots)$ we get

COROLLARY **4.1** *For $\alpha < L(1,\ldots,1)\delta$*

$$VaR_{1-\alpha}(w) \ge \sum w_i VaR_{1-\alpha'}(e_i),$$

where $\alpha' = \frac{\alpha}{L(1,\ldots,1)} = \frac{\alpha^2}{C(\alpha,\ldots,\alpha)}$.

5 Final Remarks

The estimates obtained in theorems 3.1 and 4.1 are exact i.e. there is a copula fulfilling assumption A1 such that in both estimates we get equalities.

LEMMA **5.1** *If* $C(q_1, \ldots, q_d) = min(q_1, \ldots, q_d)$ *then* $L(1, \ldots, 1) = 1$ *and*

$$VaR_{1-\alpha}(w) = \sum w_i VaR_{1-\alpha}(e_i).$$

Proof.
Indeed, if $C(q) = min(q)$ then the measure μ_C is singular with mass uniformly distributed on the diagonal $\{q = (t, \ldots, t) : t \in [0, 1]\}$. Therefore if $r = \sum_{i=1}^{d} w_i F_i^{-1}(\alpha)$ then

$$\mu_C(V_r) = \mu_C([0, \alpha]^d) = \alpha.$$

Hence

$$VaR_{1-\alpha}(w) = (1 - r)W_0 =$$

$$= w_1 VaR_{1-\alpha}(e_1) + \ldots + \ldots + w_d VaR_{1-\alpha}(e_d).$$

□

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