AN AGGREGATE CLAIMS MODEL BETWEEN INDEPENDENCE AND COMONOTONE DEPENDENCE

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Abstract

We introduce a simple aggregate claims model, which is able to take into account a continuous range of positive dependence between independence and comonotone dependence. It is based on a multivariate extension of the one-parameter bivariate Fréchet copula, which finds a justification as follows. The chosen model uses only one additional dependence parameter, which is chosen such that it yields the most conservative model of aggregate claims with respect to the concordance order for the bivariate margins of this model. A possible numerical implementation of the aggregate claims distribution of the constructed model is proposed.

Keywords: multivariate Fréchet copula, independence, comonotone dependence, aggregate claims model

1 Mean and variance in aggregate claims models

The principal goal of the present study is the construction of a simple aggregate claims model, which is able to take into account a continuous range of positive dependence between independence and comonotone dependence. This is of great practical importance because the gap between independence and comonotone dependence is known to be significant. As a first preliminary step, we determine the minimum and maximum standard deviation in the aggregate model of a risky business with different positively dependent subunits, which are each subdivided into several product categories.

As in [6] suppose an insurance risk business consists of n different *subunits* $U_i, \quad i=1,...,n$, which are each subdivided in m_i *product categories* $C_{ij}, \quad j=1,...,m_i$. The *insurance risks* during some insurance period, which are associated to all these different business lines, are measured by the random variables $S_i, \quad i=1,...,n$, $S_{ij}, \quad j=1,...,m_i, \quad i=1,...,n$, which denote the aggregate claims random variables, which are associated to U_i, C_{ij}. The main risk characteristics of the aggregate claims risk process are described as follows. The aggregate claims random variables are random sums of the type

$$S_{ij} = \sum_{k=1}^{N_{ij}} Y_{ij}^{(k)}, \quad S_i = \sum_{k=1}^{N_i} Y_i^{(k)}, \qquad (1.1)$$

where N_i, N_{ij} are claim number random variables describing the frequency in the planning units and product categories, and $Y_i^{(k)}, Y_{ij}^{(k)}$ are claim size random variables describing severity given the k-th claim has occurred in the planning units and product categories. It is assumed that the $Y_i^{(k)} \sim Y_i$ respectively $Y_{ij}^{(k)} \sim Y_{ij}$ are identically distributed random variables.

Given are the first two moments of the frequency and severity random variables associated to the product categories, that is

μN_{ij} : mean number of claims of the product categories C_{ij}

σN_{ij} : standard deviation of the number of claims of the product categories C_{ij}

μY_{ij} : mean claim size of the product categories C_{ij}

σY_{ij} : standard deviation of the claim size of the product categories C_{ij}

It follows that the mean and variance of the aggregate claims random variables of the product categories are given by

$$\mu S_{ij} = \mu N_{ij} \cdot \mu Y_{ij}, \quad (\sigma S_{ij})^2 = \mu N_{ij} \cdot (\sigma Y_{ij})^2 + (\sigma N_{ij})^2 \cdot (\mu Y_{ij})^2 \tag{1.2}$$

Since $S_i = \sum_{j=1}^{m_i} S_{ij}$ the mean associated to frequency, severity and aggregate claims random variables of the subunits are obtained from the formulas:

$$\mu N_i = \sum_{j=1}^{m_i} \mu N_{ij}, \quad \mu Y_i = \sum_{j=1}^{m_i} \left(\frac{\mu N_{ij}}{\mu N_i} \right) \mu Y_{ij}, \quad \mu S_i = \sum_{j=1}^{m_i} \mu S_{ij} \tag{1.3}$$

To obtain the standard deviation (or variance) of the aggregate claims random variables of the subunits for positively dependent product categories, one notes that there are 4 basic models related to the extremal situations of independence and comonotone dependence of the claim number and claim size random variables.

(V1) N_{ij} independent, Y_{ij} independent

(V2) N_{ij} independent, Y_{ij} comonotone dependent

(V3) N_{ij} comonotone dependent, Y_{ij} independent

(V4) N_{ij} comonotone dependent, Y_{ij} comonotone dependent

The standard deviation (or variance) in these 4 basic cases are calculated using the following well-known relationships:

N_{ij} independent =>

$$(\sigma N_i)^2 = \sum_{j=1}^{m_i} (\sigma N_{ij})^2 \tag{1.4}$$

N_{ij} comonotone dependent =>

$$\sigma N_i = \sum_{j=1}^{m_i} \sigma N_{ij} \tag{1.5}$$

Y_{ij} independent =>

$$(\sigma Y_i)^2 = \sum_{j=1}^{m_i} \left(\frac{\mu N_{ij}}{\mu N_i} \right)^2 (\sigma Y_{ij})^2 \tag{1.6}$$

Y_{ij} comonotone dependent =>

$$\sigma Y_i = \sum_{j=1}^{m_i} \left(\frac{\mu N_{ij}}{\mu N_i} \right) \sigma Y_{ij} \tag{1.7}$$

The variance of the aggregate claims in these basic models is given by

$$(\sigma S_i)^2 = \mu N_i \cdot (\sigma Y_i)^2 + (\sigma N_i)^2 \cdot (\mu Y_i)^2 \tag{1.8}$$

The main focus is now on the aggregate claims of the whole insurance risk business, which is described by the random variable

$$S = \sum_{i=1}^{n} S_i \, . \qquad (1.9)$$

First of all it is clear that the means of frequency, severity and aggregate claims satisfy the relationships

$$\mu N = \sum_{i=1}^{n} \mu N_i \, ,$$
$$\mu Y = \sum_{i=1}^{n} \left(\frac{\mu N_i}{\mu N} \right) \cdot \mu Y_i \, , \qquad (1.10)$$
$$\mu S = \sum_{i=1}^{n} \mu S_i$$

where one uses that $N = \sum_{i=1}^{n} N_i$ and $Y = \sum_{i=1}^{n} \left(\frac{\mu N_i}{\mu N} \right) \cdot Y_i$. Second, if the random variables S_i are positively dependent between independence and comonotone dependence, the variance will be bounded by the two values

$$(\sigma S_{\min})^2 = \mu N \cdot \left(\sum_{i=1}^{n} \left(\frac{\mu N_i}{\mu N} \right)^2 \cdot (\sigma Y_i)^2 \right) + \left(\sum_{i=1}^{n} (\sigma N_i)^2 \right) \cdot (\mu Y)^2 \qquad (1.11)$$

$$(\sigma S_{\max})^2 = \mu N \cdot \left(\sum_{i=1}^{n} \left(\frac{\mu N_i}{\mu N} \right) \cdot \sigma Y_i \right)^2 + \left(\sum_{i=1}^{n} \sigma N_i \right)^2 \cdot (\mu Y)^2 \qquad (1.12)$$

2 A one-parameter multivariate Fréchet copula

The more detailed study of the aggregate claims model of a risky business requires the specification of a multivariate distribution for the risk of its subunits.

A natural framework for the construction of multivariate non-normal distributions is the method of copulas, justified by the theorem of Sklar in [12]. It permits a separate study and modeling of the marginal distributions and the dependence structure. According to [10], Section 4.1, a parametric family of distributions should satisfy four desirable properties:

a) There should exist an interpretation like a mixture or other stochastic representation.
b) The margins, at least the univariate and bivariate ones, should belong to the same parametric family and numerical evaluation should be possible.
c) The bivariate dependence between the margins should be described by a parameter and cover a wide range of dependence.
d) The multivariate distribution and density should preferably have a closed-form representation, at least numerical evaluation should be possible.

In general, these desirable properties cannot be fulfilled simultaneously. For example, multivariate normal distributions satisfy properties a), b) and c) but not d). The method of copulas satisfies property c) but implies only partial closedness under the taking of margins, and can lead to computational complexity as the dimension increases. In fact, parametric families of copulas that satisfy all of the desirable properties are seldom. In [7] such a parametric family, called *multivariate linear Spearman copula*, has been constructed (formula (4.9)). It is based on the method of mixtures of independent conditional distributions, a variant of [10], Section 4.5, which is described as follows.

To satisfy property b), let us focus on the n Fréchet classes $FC_i := FC_i(F_{ij}, \, j \neq i), \quad i = 1,...,n$, of n-variate distributions for which the bivariate margins

$$F_{ij}(x_i, x_j) = F_{(X_i, X_j)}(x_i, x_j) = C_{ij}\left[F_i(x_i), F_j(x_j)\right], \quad j \neq i,$$

belong to a given parametric family of copulas $C_{ij}\left[u_i, u_j\right]$. Assume that the conditional distributions

$$F_{j|i}(x_j | x_i) = \frac{\partial C_{ij}}{\partial u_i}\left[F_i(x_i), F_j(x_j)\right] \qquad (2.1)$$

are well-defined. The n-variate distribution such that the random variables $X_j, \, j \neq i$, are conditionally independent given X_i , is contained in FC_i and is defined by

$$F^{(i)}(x) = \int_{-\infty}^{x_i} \left[\prod_{j \neq i} F_{j|i}(x_j | t) \right] \cdot dF_i(t) \qquad (2.2)$$

Choosing appropriately the bivariate copulas $C_{ij}\lfloor u_i, u_j \rfloor$, it is possible to construct n-variate copulas $C^{(i)}(u_1,...,u_n)$, $i = 1,...,n$, such that $F^{(i)}$ belongs to $C^{(i)}$ and the bivariate margins F_{ij}, $j \neq i$, belong to C_{ij}. Moreover, any convex combination of the $C^{(i)}$'s , that is

$$\begin{aligned} &C(u_1,...,u_n) = \sum_{i=1}^{n} \lambda_i C^{(i)}(u_1,...,u_n), \\ &0 \leq \lambda_i \leq 1, \quad \sum_{i=1}^{n} \lambda_i = 1 \end{aligned} \qquad (2.3)$$

is again a n-variate copula, which by appropriate choice may satisfy the desirable properties. In the following, the class (2.3) of all convex combinations of mixtures of independent conditional distributions with given bivariate margins is denoted by CIC.

In [7], the subclass of CIC, denoted here CICF, which is generated by the bivariate margins with Fréchet copulas

$$\begin{aligned} &C_{ij}(u_i, u_j) \\ &= (1-\theta_{ij})u_i u_j + \theta_{ij} \min(u_i, u_j) \end{aligned} \qquad (2.4)$$

where $\theta_{ij} \in [0,1]$, has been studied in detail. It is well-known that the parameter θ_{ij} of the copula (2.4) coincides with the grade correlation coefficient introduced in [13], simply called Spearman coefficient.

Though presumably possible, the evaluation of the full distribution of aggregate claims using the multivariate distributions in CICF should be rather technical. Instead of this, we consider the simple *multivariate Fréchet copula*

$$\begin{aligned} &C(u_1,...,u_n) = (1-\theta) \cdot \left[\prod_{i=1}^{n} u_i \right] \\ &+ \theta \cdot \min_{1 \leq i \leq n}(u_i), \quad \theta \in [0,1]. \end{aligned} \qquad (2.5)$$

Intuitively, the copula (2.5) models the whole range of possible dependence structure between independence and comonotone dependence. It would also be possible to model similarly the possible dependence structure between independence and "minimal" dependence.

The practical application of this model is motivated as follows. Guided by the concern of "prudent" valuation, we require that the bivariate margins of (2.5) are at least as positively dependent as those obtained from the bivariate model (2.4) in the concordance ordering (e.g. [1], [14], [3]). Consider the bivariate copulas of the bivariate margins of (2.5), which are denoted and given by

$$\begin{aligned} &C^{ij}(u_i, u_j) \\ &= (1-\theta)u_i u_j + \theta \min(u_i, u_j) \end{aligned} \qquad (2.6)$$

Then the stated condition means that $C_{ij}(u_i, u_j) \leq C^{ij}(u_i, u_j)$, and this is equivalent to

$$\begin{aligned} &(\theta - \theta_{ij}) \cdot u_i u_j \\ &\leq (\theta - \theta_{ij}) \cdot \min(u_i, u_j) \end{aligned}. \qquad (2.7)$$

or

$$\theta \geq \theta_{ij}, \quad \text{for all } (i, j). \qquad (2.8)$$

Therefore, for prudent aggregate claims evaluation, it is reasonable to set the dependence parameter of the model (2.5) equal to

$$\theta = \max_{(i,j)} \{\theta_{ij}\}. \qquad (2.9)$$

This model choice yields the most conservative model for aggregate claims with respect to the concordance order for the bivariate margins of this model.

It should be pointed out that the considered very simple multivariate Fréchet copula, which models the whole range of dependence between independence and comonotone dependence, finds important practical applications in the fields of insurance and finance. The interested reader is invited to consult [8] and [9].

3 Multivariate Fréchet compound models of aggregate claims

Given that the volatility, as measured by the variance, varies between two bounds, what is a useful and practical way to get an aggregate claims distribution for the whole insurance risk business? To model the positive dependence between independence and comonotone dependence, it is reasonable to consider a model based on the multivariate Fréchet copula, which has been introduced and motivated in Section 2. Besides notations and standard assumption introduced in Section 1, our *multivariate Fréchet compound model of aggregate claims* is based on the following assumptions:

(A1) The random vector $\boldsymbol{N} = (N_1, ..., N_n)$ is multivariate Fréchet distributed with dependence parameter ρ_N.

(A2) The random vector $\boldsymbol{Y} = (Y_1, ..., Y_n)$ is multivariate Fréchet distributed with dependence parameter ρ_Y.

(A3) The random variables N_i are independent from the Y_i's.

Let $\boldsymbol{N}^{\perp} = (N_1^{\perp}, ..., N_n^{\perp})$ be a version of $\boldsymbol{N}$ with independent components, $\boldsymbol{N}^{+} = (N_1^{+}, ..., N_n^{+})$ a version of $\boldsymbol{N}$ with comonotone dependent components, $\boldsymbol{Y}^{\perp} = (Y_1^{\perp}, ..., Y_n^{\perp})$ a version of $\boldsymbol{Y}$ with independent components, and $\boldsymbol{Y}^{+} = (Y_1^{+}, ..., Y_n^{+})$ a version of $\boldsymbol{Y}$ with comonotone dependent components. Under the multivariate Fréchet assumption (A1), the distribution of the aggregate claim number random variable $N = \sum_{i=1}^{n} N_i$ is then given by

$$F_N(k) = (1-\rho_N) \cdot F_{N^{\perp}}(k) + \rho_N \cdot F_{N^{+}}(k), \quad k = 0,1,2,... \tag{3.1}$$

where $N^{\perp} = \sum_{i=1}^{n} N_i^{\perp}$ and $N^{+} = \sum_{i=1}^{n} N_i^{+}$. In the following let us set $\lambda_i = \mu N_i$, $\lambda = \mu N = \sum_{i=1}^{n} \lambda_i$. Under the multivariate Fréchet assumption (A2) the distribution of $Y = \sum_{i=1}^{n} \frac{\lambda_i}{\lambda} \cdot Y_i$ is given by

$$F_Y(y) = (1-\rho_Y) \cdot F_{Y^{\perp}}(y) + \rho_Y \cdot F_{Y^{+}}(y), \quad y > 0 \tag{3.2}$$

where $Y^{\perp} = \sum_{i=1}^{n} Y_i^{\perp}$ and $Y^{+} = \sum_{i=1}^{n} Y_i^{+}$. The main interest lies in the distribution of the aggregate claims $S = \sum_{i=1}^{n} S_i = \sum_{i=1}^{N} Z_i$, where each Z_i has the distribution of Y. By assumption (A3) each Z_i is independent from N and the compound model of risk theory implies the well-known convolution representation for the aggregate claims distribution

$$F_S(s) = \sum_{k=0}^{\infty} p_N(k) \cdot F_Y^{*(k)}(s) \tag{3.3}$$

with the claim number probabilities

$$p_N(k) = F_N(k) - F_N(k-1), \quad k = 1,2,..., \quad p_N(0) = F_N(0). \tag{3.4}$$

It is not difficult to see that the k-th convolution of Y has the representation

$$F_Y^{*(k)}(y) = \sum_{j=0}^{k} \binom{k}{j} \rho_Y^j (1-\rho_Y)^{k-j} \cdot \left[F_{Y^{+}}^{*(j)} * F_{Y^{\perp}}^{*(k-j)}\right](y) \tag{3.5}$$

It is clear that the extreme cases $\rho_N = \rho_Y = 0$ and $\rho_N = \rho_Y = 1$ yield the minimum and maximum variance of S as given above in (1.11) and (1.12). In the special cases $\rho_Y = 0$ and $\rho_Y = 1$, the aggregate claims distribution takes the simpler form:

$$F_{S^{\perp}}(s) = \sum_{k=0}^{\infty} p_N(k) \cdot F_{Y^{\perp}}^{*(k)}(s) \tag{3.6}$$

$$F_{S^{+}}(s) = \sum_{k=0}^{\infty} p_N(k) \cdot F_{Y^{+}}^{*(k)}(s) \tag{3.7}$$

Instead of (3.3) it is reasonable to consider the Fréchet model

$$F_S(s) = (1-\rho_S)\cdot F_{S^\perp}(s) + \rho_S \cdot F_{S^+}(s) \quad (3.8)$$

with some Spearman coefficient $\rho_S \in [0,1]$. This simpler model yields the minimum and maximum variance provided $\rho_N = \rho_S = 0$ respectively $\rho_N = \rho_S = 1$. Each pair $(\rho_N, \rho_S) \in [0,1]^2$ defines another *multivariate Fréchet compound model of aggregate claims*.

4 Numerical evaluation of a multivariate Fréchet compound model

For a numerical evaluation of our multivariate Fréchet compound model, we assume for simplicity that claim numbers are Poisson distributed, that is $N_i \sim Po(\lambda_i)$, $i = 1,...,n$, and that claim sizes are Gamma distributed, that is $Y_i \sim \Gamma(\alpha_i, \beta_i)$, $i = 1,...,n$. Note that the widely used compound Poisson Gamma aggregate claims model has been justified through a characterization result in Mathematical Statistics in [4]. The introduced claim size random variables $Y^\perp$, respectively Y^+, are the independent, respectively comonotone, sums of the gamma distributed random variables $\Gamma\left(\alpha_i, \frac{\lambda}{\lambda_i}\beta_i\right)$, $i = 1,...,n$, whose distribution functions can be determined using the methods presented in [5]. Through approximation it is possible to assume that $Y^\perp$ and Y^+ are also gamma distributed with mean

$$\mu_Y = \frac{1}{\lambda}\sum_{i=1}^{n}\lambda_i \frac{\alpha_i}{\beta_i} \quad (4.1)$$

and standard deviations

$$\sigma_{Y^\perp} = \frac{1}{\lambda}\sqrt{\sum_{i=1}^{n}\lambda_i^2 \frac{\alpha_i}{\beta_i^2}} \quad \text{(independent case)} \quad (4.2)$$

$$\sigma_{Y^+} = \frac{1}{\lambda}\sum_{i=1}^{n}\lambda_i \frac{\sqrt{\alpha_i}}{\beta_i} \quad \text{(comonotone case)} \quad (4.3)$$

In this situation the claim sizes are gamma distributed such that $Y^\perp \sim \Gamma(\alpha^\perp, \beta^\perp)$ and $Y^+ \sim \Gamma(\alpha^+, \beta^+)$, with

$$\alpha^\perp = k_{Y^\perp}^{-2},\ \beta^\perp = \alpha^\perp \cdot \mu_Y^{-1},$$
$$\alpha^+ = k_{Y^+}^{-2},\ \beta^+ = \alpha^+ \cdot \mu_Y^{-1},$$

where $k_{Y^\perp}, k_{Y^+}$ are the coefficients of variation of the claims size random variables $Y^\perp$ and Y^+. In the special case

$$\lambda_i = \frac{\lambda}{n},\ \alpha_i = \alpha,\ \beta_i = \beta,\ i = 1,...,n$$

the claim sizes are gamma distributed with parameters

$$\alpha^\perp = n\alpha,\ \beta^\perp = n\beta,\ \alpha^+ = \alpha,\ \beta^+ = \beta$$

In general, the *k*-th convolution of the claim sizes $Y^\perp$ and Y^+ are gamma distributed such that

$$(Y^\perp)^{*(k)} \sim \Gamma(k\alpha^\perp, \beta^\perp),$$
$$(Y^+)^{*(k)} \sim \Gamma(k\alpha^+, \beta^+)$$

The claim number random variables $N^\perp$ is Poisson distributed with parameter λ. An explicit analytical expression for the claim number distribution N^+ is not known unless $\lambda_i = \frac{\lambda}{n}$, $i = 1,...,n$, in which case the probabilities are

$$\Pr(N^+ = k) = Po\left(\left[\frac{k}{n}\right]; \frac{\lambda}{n}\right),\ k = 0,1,2,...$$

The latter result suggests the following approximation in the general case. The claim number $N^\perp$ has the same distribution as the claim number random variable

$$\tilde{N}^\perp = \sum_{i=1}^{n}\tilde{N}_i^\perp,\ \tilde{N}_i \sim Po\left(\frac{\lambda}{n}\right)$$

This suggests to approximate the claim number N^+ by the random variable $\tilde{N}^+ = \sum_{i=1}^{n}\tilde{N}_i^+$, which implies that the probability $\Pr(N^+ = k)$ is approximately equal to

$$\Pr(\tilde{N}^+ = k) = Po\left(\left[\frac{k}{n}\right]; \frac{\lambda}{n}\right),\ k = 0,1,2,...$$

Proceeding this way, the distribution of the multivariate Fréchet compound Poisson model of aggregate claims with gamma claim sizes can be evaluated using the following analytical expressions:

$$F_S(s) = (1-\rho_S)\cdot F_{S^\perp}(s) + \rho_S \cdot F_{S^+}(s) \quad (4.4)$$

$$F_{S^\perp}(s) = \sum_{k=0}^{\infty} p_N(k)\cdot G(\beta^\perp s; k\alpha^\perp) \quad (4.5)$$

$$F_{S^+}(s) = \sum_{k=0}^{\infty} p_N(k)\cdot G(\beta^+ s; k\alpha^+) \quad (4.6)$$

$$G(x;\alpha) = \frac{1}{\Gamma(\alpha)}\cdot\int_0^x t^{\alpha-1}e^{-t}dt \quad (4.7)$$

$$p_N(k) = (1-\rho_N)\cdot p_{N^\perp}(k) + \rho_N \cdot p_{N^+}(k), \quad k = 0,1,2,... \quad (4.8)$$

$$p_{N^\perp}(k) = Po(k;\lambda) - Po(k-1;\lambda), \quad k = 1,2,..., \quad p_{N^\perp}(0) = e^{-\lambda} \quad (4.9)$$

$$p_{N^+}(k) = Po\left(\left[\frac{k}{n}\right];\frac{\lambda}{n}\right) - Po\left(\left[\frac{k-1}{n}\right];\frac{\lambda}{n}\right) \quad (4.10)$$
$$k = 1,2,..., \quad p_{N^+}(0) = e^{-\frac{\lambda}{n}}$$

The following Table illustrates the exact calculation of the multivariate Fréchet compound Poisson gamma distribution in the special case $n = 3, \quad \lambda_i = \frac{1}{3}, \alpha_i = \frac{1}{4}, \beta_i = \frac{1}{4}, i = 1,2,3,$ $\rho_N = 0.37895, \rho_S = 0.2$

It is immediately seen that the aggregate claims random variables are stochastically ordered in the dangerousness order, that is one has the inequality $S^\perp \le_D S \le_D S^+$ (e.g. [11] for a definition of dangerousness order). The interest of such inequalities is well-known because actuaries feel that positive dependence reveal a more dangerous situation compared to independence (e.g. [2]).

Table 4.1: Distribution of aggregate claims for a multivariate Fréchet compound model

Aggregate Claims s	$F_{S^\perp}(s)$	$F_S(s)$	$F_{S^+}(s)$
0	0.50000	0.50000	0.50000
1	0.70156	0.71676	0.77757
2	0.81401	0.82055	0.84673
3	0.88544	0.88623	0.88936
4	0.93003	0.92769	0.91835
5	0.95744	0.95374	0.93894
6	0.97415	0.97011	0.95393
7	0.98430	0.98044	0.96502
8	0.99045	0.98702	0.97331
9	0.99418	0.99126	0.97955
10	0.99645	0.99402	0.98429
11	0.99783	0.99585	0.98790
12	0.99868	0.99707	0.99066
13	0.99919	0.99791	0.99278
14	0.99951	0.99849	0.99441
15	0.99970	0.99889	0.99567
16	0.99982	0.99918	0.99664
17	0.99989	0.99939	0.99739
18	0.99993	0.99954	0.99797
19	0.99996	0.99965	0.99842
20	0.99997	0.99973	0.99877
21	0.99998	0.99980	0.99904
22	0.99999	0.99984	0.99925
23	0.99999	0.99988	0.99942
24	1.00000	0.99991	0.99954

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