Introduction to Asymptotically Idempotent Aggregation Operators

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Abstract

This paper deals with aggregation operators. A new class of aggregation operators, called asymptotically idempotent, is introduced. A generalization of the basic notion of aggregation operator is provided, with an in-depth discussion of the notion of idempotency. Some general contruction methods of commutative, asymptotically idempotent aggregation operators admitting a neutral element are illustrated.

Keywords: Aggregation Operator, Idempotency, Commutativity, Neutral Element.

1 Introduction

The mathematical process of fusion of several input real values into a single output real value is crucial in many fields. According to the various applications, different properties are requested to the aggregation. In particular, dealing with problems of ranking, the basic characteristics requested of the aggregation are *anonymity*, which occurs when the knowledge of the order of the input values is irrelevant, and *unanimity*, which reads as follows: when all the partial scores are equal to a certain value, this must be also the global score. Anonymity and unanimity are mathematically translated into commutativity and idempotency of the aggregation operator. However, especially when the number of the inputs is very large, less common aspects or demands assume a certain importance: many data could be devoid of significance (problematical aspect) or the possibility that a group of many positive scores may have a greater weight than one of a few positive scores (a refined request of the aggregation). In order to take into account the problematical aspect, it might be useful to introduce a neutral element, i.e. an element which has the same effect as its omitting. The presence of a neutral element contravenes the idempotency, but, if we relax idempotency in a suitable way, quite surprisingly, we can meet also the above mentioned refined request. In this work, we introduce a new class of aggregation operators, called asympotically idempotent, which, in presence of a neutral element, permit the output value to be sensitive to the number of input values, so undoubtedly improving the quality of ranking. We discuss the properties of this class and provide some general methods of construction of these operators.

2 Basic Concepts

In this work, we are interested in aggregation of input values, as well as outputs, belonging to some closed interval $[a, b] \subset \mathbb{R}$.

Definition 1 *A mapping*

$$g : [a, b]^n \to [a, b],\ n \in \mathbb{N}$$

is called an n-ary aggregation function (AF) acting on $[a, b]$ if it is non-decreasing monotone in its components, that is

$$g(x_1, ..., x_n) \leq g(x'_1, ..., x'_n), \tag{1}$$

whenever $a \leq x_i \leq x'_i \leq b$ for all $i \in \{1, ..., n\}$. Moreover, g is strict if (1) holds with the strict inequality provided that $(x_1, ..., x_n) \neq (x'_1, ..., x'_n)$. Finally, g is commutative if

$$g(x_1, ..., x_n) = g(x^*_1, ..., x^*_n) \tag{2}$$

for any permutation $(x^*_1, ..., x^*_n)$ of an arbitrary tuple $(x_1, ..., x_n) \in [a, b]^n$.

Remark 1 Regarding the property of continuity, according to (1), any AF is continuous if and only if it is continuous in its components.

Definition 2 *Let g be an n-ary AF acting on $[a,b]$. Fixed any $x \in [a,b]$, an element $(x,...,x) \in [a,b]^n$ is called an idempotent element for g if*

$$g(x,...,x) = x. \tag{3}$$

The n-ary AF g is idempotent, if g fulfills (3) for any $x \in [a,b]$.

Definition 3 *A sequence $\boldsymbol{G}$=$\{G_n\}_n$ of n-ary AFs acting on $[a,b]$ is called an aggregation operator on $[a,b]$ (briefly, AO on $[a,b]$).*

Definition 4 *An AO $\boldsymbol{G}$=$\{G_n\}_n$ on $[a,b]$ is called asymptotically idempotent (AI) if*

$$\lim_{n\to\infty} G_n(x,...,x) = x \quad \text{for all } x \in [a,b]. \tag{4}$$

Remark 2 It is our opinion that a sequence of n-ary AFs satisfying (4) is qualified for deserving the "title" of AO. In fact, from the theoretical point of view, the idempotent, "standard" AOs are a particular case of the AI ones; from the practical point of view, condition (4) assures the sensitivity of the output to the number of inputs, a refined property which is recommended in many applications, as told in the introduction. However, on the one hand, the AI AOs could not form a subclass of AOs, as to be expected, if we maintained the classical, commonly used in literature, definition of AO on a real closed interval $[a,b]$, which, in addition, requires the following two conditions:

$$G_n(a,...,a) = a,\ G_n(b,...,b) = b \tag{5}$$

and

$$G_1(x) = x \quad \text{for all } x \in [a,b]. \tag{6}$$

Indeed, there exist AI AOs which do not meet (5) and (6), as shown in the following example, where $[a,b] = [0,1]$ and the n-ary AF is

$$G_n(x_1,...,x_n) = \max_{i=1,...,n}\{x_i\} \cdot \frac{\sqrt{\sum\limits_{i=1}^{n} {x_i}^2}}{1+\sqrt{\sum\limits_{i=1}^{n} {x_i}^2}}. \tag{7}$$

On the other hand, there exist AOs, under the classical definition, which do not satify (4), as shown in the following example, where $[a,b] = [0,1]$ and the n-ary AF is

$$G_n(x_1,...,x_n) = \begin{cases} x_1, & \text{if } n = 1; \\ 0, & \text{if } x_1 = \cdots = x_n = 0 \\ & \text{for all } n; \\ 1, & \text{if } x_1 = \cdots = x_n = 1 \\ & \text{for all } n; \\ 0, & \text{otherwise} \\ & \text{and } n \text{ is even;} \\ 1, & \text{otherwise} \\ & \text{and } n > 2 \text{ is odd.} \end{cases}$$

A way which seems to be reasonable for bypassing this "cul-de-sac" is to weaken the definition of AO, omitting conditions (5) and (6).

Definition 5 *Let $\boldsymbol{G}$=$\{G_n\}_n$ be an AO on $[a,b]$. We say that $\boldsymbol{G}$ is commutative, idempotent, strict or continuous if, for each $n \in \mathbb{N}\,(n \geq 2$ in case of commutativity), any G_n is commutative, idempotent, strict or continuous respectively.*

Definition 6 *An AO $\boldsymbol{G}$=$\{G_n\}_n$ on $[a,b]$ is called associative if for all $m,n \in \mathbb{N}$ and for all tuples $(x_1,...,x_m) \in [a,b]^m$ and $(y_1,...,y_n) \in [a,b]^n$*

$$G_{n+m}(x_1,...,x_m,y_1,...,y_n) =$$
$$= G_2(G_m(x_1,...,x_m), G_n(y_1,...,y_n)).$$

From the structural point of view, an associative AO $\boldsymbol{G}$ is uniquely determined by the corresponding binary AF G_2, hence, with abuse of notation, we will use the same symbol for $\boldsymbol{G}$ and G_2.

Definition 7 *Let $\boldsymbol{G}$=$\{G_n\}_n$ be an AO on $[a,b]$. Then an element $e \in [a,b]$ is called a neutral element (NE) for $\boldsymbol{G}$ if, for each $n \geq 2$, for each $k \in \{1,2,...,n\}$ and for all $x_1,...,x_{k-1},x_{k+1},...,x_n \in [a,b]$, we have*

$$G_n(x_1,...,x_{k-1},e,x_{k+1},...,x_n) =$$
$$= G_{n-1}(x_1,...,x_{k-1},x_{k+1},...,x_n). \tag{8}$$

Remark 3 Observe that if $\boldsymbol{G}$=$\{G_n\}_n$ is an AO on $[a,b]$ which admits $e \in [a,b]$ as NE, the binary AF G_2, according to (8), satisfies

$$G_2(e,x) = G_2(x,e) = G_1(x),$$

which reduces to the standard form when (6) holds. What is interesting is that unicity of neutral elements is preserved also in case of AI AOs, as stated by the following lemma

Proposition 1 *Let $\boldsymbol{G}$=$\{G_n\}_n$ be an AI AO on $[a,b]$ admitting $e \in [a,b]$ as NE. Then, e is the unique NE for $\boldsymbol{G}$.*

The next is a well-known result regarding the transformations of AOs by means of a strictly monotone bijection.

Proposition 2 *Let $\boldsymbol{G}$=$\{G_n\}_n$ be an AO on $[a,b]$ and $\varphi : [c,d] \to [a,b]$ a strictly monotone bijection, where $c, d \in \mathbb{R}$, with $c < d$. Then $\boldsymbol{G}^\varphi = \{G_n^\varphi\}_n$, where the n-ary AF G_n^φ acting on $[c,d]$ is defined by*

$$G_n^\varphi(u_1, ..., u_n) = \varphi^{-1}\Big(G_n(\varphi(u_1), ..., \varphi(u_n))\Big) \quad (9)$$

for all $u_1, ..., u_n \in [c,d]$, is an AO on $[c,d]$. Moreover, if $e \in [a,b]$ is a NE for $\boldsymbol{G}$, then $\varphi^{-1}(e) \in [c,d]$ is a NE for $\boldsymbol{G}^\varphi$. Finally, if $\boldsymbol{G}$ is commutative or continuous, then $\boldsymbol{G}^\varphi$ is commutative or continuous respectively.

This result suggests us to introduce a notion of *isomorphism* between AOs acting on the same interval.

Definition 8 *Let $\boldsymbol{G}$=$\{G_n\}_n$ and $\boldsymbol{G}^* = \{G_n^*\}_n$ be a pair of AOs on $[a,b]$. Then, we will say that $\boldsymbol{G}$ and $\boldsymbol{G}^*$ are isomorphic if there exists a strictly monotone bijection $\varphi : [a,b] \to [a,b]$ such that*

$$\boldsymbol{G}^* \equiv \boldsymbol{G}^\varphi.$$

Finally, we conclude this section with a generalized notion of idempotency for an AO.

Definition 9 *Let $\boldsymbol{G}$=$\{G_n\}_n$ be an AO on $[a,b]$. We will say that $\boldsymbol{G}$ is quasi-idempotent if for each $x_0 \in [a,b]$ there exists an $n_0 = n_0(x_0) \in \mathbb{N}$ such that $G_n(x_0, ..., x_0) = x_0$ for all $n \geq n_0$.*

3 Commutative AOs with a NE

Let us denote by $\mathcal{A}$ and $\mathcal{B}$ the families of AOs on $[a,b]$, for any pair $(a,b) \in \mathbb{R}^2$ with $a < b$, admitting $e \in \{a,b\}$ and $e \in]a,b[$ as NE, respectively. Proposition 1 and Definition 8 allow us, without loss of generality and up to isomorphisms, to fix for both classes $e = 0$ as NE and the domains $[0,1]$ and $[-1,1]$ respectively. Then, we set $\mathcal{E}$:=$\mathcal{A} \cup \mathcal{B}$ and we will denote by $\boldsymbol{A}$=$\{A_n\}_n$, $\boldsymbol{B}$=$\{B_n\}_n$ or $\boldsymbol{E}$=$\{E_n\}_n$ an arbitrary element of $\mathcal{A}$, $\mathcal{B}$ or $\mathcal{E}$ respectively: in the last case, we will denote by D the domain, where D may be indifferently $[0,1]$ or $[-1,1]$. Finally, given any $\boldsymbol{E}$=$\{E_n\}_n \in \mathcal{E}$, we set $d_n(x) := E_n(x, ..., x)$, where $d_n : D \to D$ for all $n \in \mathbb{N}$.

Proposition 3 *Given any $\boldsymbol{E}$=$\{E_n\}_n \in \mathcal{E}$ and fixed any $x \in D$, the sequence $\{d_n(x)\}_n$ is monotone (strictly monotone if $\boldsymbol{E}$ is strict and $x \neq 0$).*

Proof If $x = 0$, we have that $d_n(0) = E_n(0, ..., 0) =$ (according to (8)) $E_{n+1}(0, ..., 0) = d_{n+1}(0)$, so that $d_n(0) = d_1(0)$ for all $n \in \mathbb{N}$. If $x > 0$, $d_n(x) = E_n(x, ..., x) =$ (according to (8)) $E_{n+1}(x, ..., x, 0) \leq$ (according to (1), with the strict inequality, if the operator is strict) $E_{n+1}(x, ..., x, x) = d_{n+1}(x)$ for all $n \in \mathbb{N}$. The case $x < 0$ may be shown in complete analogy. □

An immediate consequence is that the sequence $\{d_n(x)\}_n$ converges on D to a function we will denote by $d(x)$, where $d : D \to D$. Hence, it is clear that any $\boldsymbol{E} \in \mathcal{E}$ is AI if and only if $d(x) = x$ for all $x \in D$.

Proposition 4 *Given any $\boldsymbol{E}$=$\{E_n\}_n \in \mathcal{E}$, we have that $E_n(x_1, ..., x_n) \geq 0$ (or > 0 if $\boldsymbol{E}$ is strict) for all $x_1, ..., x_n \geq 0$ (and $x_j > 0$ for some $j \in \{1, ..., n\}$), while $E_n(x_1, ..., x_n) \leq 0$ (or < 0 if $\boldsymbol{E}$ is strict) for all $x_1, ..., x_n \leq 0$ (and $x_j < 0$ for some $j \in \{1, ..., n\}$).*

Remark 4 Note that, starting from an arbitrary $\boldsymbol{B} \in \mathcal{B}$, we can always generate an AO $\boldsymbol{A}^{\boldsymbol{B}} \in \mathcal{A}$, simply restricting the domain of $\boldsymbol{B}$ to the real unit interval, i.e. $\boldsymbol{A}^{\boldsymbol{B}}$:=$\boldsymbol{B}|_{[0,1]}$.

Definition 10 *We will say that $\boldsymbol{B}$=$\{B_n\}_n \in \mathcal{B}$ is symmetrical with respect to $e = 0$ (e-symm, for short) if, for each $n \in \mathbb{N}$, we have*

$$B_n(x_1, ..., x_n) = -B_n(|x_1|, ..., |x_n|)$$

for all $x_1, ..., x_n \in [-1, 0[$.

Now, we set $\mathcal{CA} := \{\boldsymbol{A} \in \mathcal{A} : \boldsymbol{A}$ is commutative$\}$ and $\mathcal{CB} := \{\boldsymbol{B} \in \mathcal{B} : \boldsymbol{B}$ is commutative$\}$.

Remark 5 Observe that, starting from an arbitrary $\boldsymbol{A} \in \mathcal{CA}$, we can always generate an AO $\boldsymbol{B}^{\boldsymbol{A}}$=$\{B_n^A\}_n \in \mathcal{CB}$, where the n-ary AF is defined as follows:

$$B_n^A(x_1, ..., x_n) = B_n^A(x_1^*, ..., x_k^*, x_{k+1}^*, ..., x_n^*) =$$

$$= A_k(x_1^*, ..., x_k^*) - A_{n-k}(|x_{k+1}^*|, ..., |x_n^*|),$$

where $(x_1^*, ..., x_k^*, x_{k+1}^*, ..., x_n^*)$ is any permutation of an arbitrary tuple $(x_1, ..., x_n) \in [-1,1]^n$ such that $x_1^*, ..., x_k^* \geq 0$, while $x_{k+1}^*, ..., x_n^* < 0$, for some $k \in \{0, ..., n\}$, with the convention $A_0 = 0$. The only point deserving to be shown is the monotonicity of the arbitrary n-ary AF: given any $(x_1, ..., x_n) \in [-1,1]^n$, without loss of generality, we can suppose that $x_1, ..., x_k \geq 0$ and $x_{k+1}, ..., x_n < 0$, where $k \in \{0, ..., n-1\}$. Now, fixed any $i \in \{1, ..., n\}$, we have to prove that

$$B_n^A(x_1, ..., x_i, ..., x_n) \leq B_n^A(x_1, ..., x_i', ..., x_n) \quad (10)$$

for all $x_i' \in [x_i, 1]$. The case $i \in \{1, ..., k\}$ for $k \neq 0$ is trivial, so assume that $i \in \{k+1, ..., n\}$ for any $k \in \{0, ..., n-1\}$. If $x_i' \geq 0$, (10) becomes

$$A_k(x_1, ..., x_k) -$$

$$-A_{n-k}(|x_{k+1}|,...,|x_{i-1}|,|x_i|,|x_{i+1}|,...,|x_n|) \leq$$
$$\leq A_{k+1}(x_1,...,x_k,x_i')-$$
$$-A_{n-k-1}(|x_{k+1}|,...,|x_{i-1}|,|x_{i+1}|,...,|x_n|). \quad (11)$$

Note that $A_k(x_1,...,x_k) = A_{k+1}(x_1,...,x_k,0) \leq A_{k+1}(x_1,...,x_k,x_i')$ by (8) and (1) respectively, and, by the same reasons, $A_{n-k-1}(|x_{k+1}|,...,|x_{i-1}|,|x_{i+1}|,...,|x_n|) = A_{n-k}(|x_{k+1}|,...,|x_{i-1}|,0,|x_{i+1}|,...,|x_n|) \leq A_{n-k}(|x_{k+1}|,...,|x_{i-1}|,|x_i|,|x_{i+1}|,...,|x_n|)$, so that (11) is assured. Finally, if $x_i' < 0$, (10) becomes

$$A_{n-k}(|x_{k+1}|,...,|x_i|,...,|x_n|) \geq$$
$$\geq A_{n-k}(|x_{k+1}|,...,|x_i'|,...,|x_n|),$$

which follows from (1) and the fact that $|x_i| \geq |x_i'|$.

Note that any $\boldsymbol{B^A}$ is $e-symm$ and further it fulfills the following stronger property:

$$B^A_{2n}(x_1,...,x_n,-x_1,...,-x_n) = 0 \quad (12)$$

for all $x_1,...,x_n \in [-1,0[\, \cup\,]0,1]$ and for each $n \in \mathbb{N}$.

Remark 6 Observe that, starting from an arbitrary $\boldsymbol{A} \in \mathcal{CA}$, if we generate, as shown in the previous remark, $\boldsymbol{B^A} \in \mathcal{CB}$ and subsequently $\boldsymbol{A^{B^A}} \in \mathcal{CA}$, as illustrated in Remark 4, we easily get $\boldsymbol{A} \equiv \boldsymbol{A^{B^A}}$. This is equivalent to saying that $\mathcal{CB}|_{[0,1]} = \mathcal{CA}$, where $\mathcal{CB}|_{[0,1]} := \{\ \boldsymbol{A^B}\!:\ \boldsymbol{B} \in \mathcal{CB}\}$. The same does not occur if, starting from an arbitrary $\boldsymbol{B} \in \mathcal{CB}$, we consider first $\boldsymbol{A^B}$ and then $\boldsymbol{B^{A^B}}$, because we cannot generally conclude that $\boldsymbol{B} = \boldsymbol{B^{A^B}}$. Indeed, if this were true, $\boldsymbol{B}$ would necessarily satisfy (12), but, as we will see in the next section, there exist AOs belonging to $\mathcal{CB}$ which do not verify (12).

4 Construction methods of commutative, AI AOs with a NE

In this section, we are interested in providing some procedures for building different models of AI AOs belonging to $\mathcal{CE} := \mathcal{CA} \cup \mathcal{CB}$. First of all, we emphasize the fact that there are not many examples of idempotent AOs $\in \mathcal{CE}$ and generally they have a rather poor structure. For instance, if we restrict to the associative ones, the only associative, idempotent AO $\in \mathcal{CA}$ is the $t-$conorm given by the maximum (and, at the same time, if we fixed $e = 1$ as NE, we would find as unique example the t-norm given by the minimum). Otherwise, the associative, idempotent AOs $\in \mathcal{CB}$ form the more substantial family of idempotent uninorms: however, observe that, if $\boldsymbol{U}$ belongs to this class, necessarily $\boldsymbol{U}|_{[0,1]^2} \equiv \max$ and $\boldsymbol{U}|_{[-1,0]^2} \equiv \min$.

Now we present the first model of AI AO $\in \mathcal{CA}$, in which there is no necessity of permutation, or rearrangement in more general sense, of the input values.

Example 1 Let $\boldsymbol{A}^{\mathbf{H},\psi} = \{A_n^{\mathbf{H},\psi}\}_n$ be a sequence of n-ary AFs so described:

$$A_n^{\mathbf{H},\psi}(x_1,...,x_n) =$$
$$= \max\{x_1,...,x_n\} \cdot \psi\big(h_n(x_1,...,x_n)\big),$$

where $\psi : [0,\infty] \to [0,1]$ is a non-decreasing mapping such that $\psi(0) = 0$ and $\psi(\infty) = 1$, while $\mathbf{H} = \{h_n\}_n$ is a commutative AO acting on the interval $[0,\infty[$, with $e = 0$ as NE, such that $h_1(0) = 0$ and $\sup_{n\in\mathbb{N}} h_n(x,...,x) = \infty$ for any $x > 0$. The fact that $\boldsymbol{A}^{\mathbf{H},\psi}$ actually belongs to $\mathcal{CA}$ is quite easy to show. The set of mappings which behave as ψ is very large; what is more interesting is to investigate some simple ways to construct explicit examples of $\mathbf{H}$. For instance, if we consider any non-decreasing function $\mu : [0,\infty[\to [0,\infty[$ such that $\mu(0) = 0$, and $\mu(t) > 0$ as $t > 0$, it is trivial to see that $\mathbf{H}^\mu = \{h_n^\mu\}_n$, where the n-ary AF is defined

$$h_n^\mu(x_1,...,x_n) = \sum_{i=1}^n \mu(x_i),$$

fulfills all the required properties on $[0,\infty[$. Note that (7) is a particular case of this model, with $\psi(t) = \frac{\sqrt{t}}{1+\sqrt{t}}$ and $\mu(t) = t^2$. Finally, observe that $\boldsymbol{A}^{\mathbf{H},\psi}$ is continuous if ψ and $\mathbf{H}$ are continuous.

The next model of AI AO $\in \mathcal{CA}$ is a sort of generalization of ordered weighted average operator (OWA, for short), which requires a permutation of the input values.

Example 2 Let $\boldsymbol{W}^{\Delta_\infty} = \{W_n^{\Delta_\infty}\}_n$ be a sequence of n-ary AFs so described:

$$W_n^{\Delta_\infty}(x_1,...,x_n) = \sum_{i=1}^n w_i \cdot x_i',$$

where $(x_1',...,x_n')$ is a non-decreasing permutation of any arbitrary input n-tuple $(x_1,...,x_n)$, while $\Delta_\infty = \{w_n\}_n$ is a fixed sequence of real, non negative numbers such that

$$\sum_{n=1}^\infty w_n = 1. \quad (13)$$

It is not difficult to show that $\boldsymbol{W}^{\Delta_\infty}$ is a continuous AO belonging to $\mathcal{CA}$. Observe that $\boldsymbol{W}^{\Delta_\infty}$ is not absolutely reducible to the classical OWA, because, in that case, any n-ary AF has its own weighting triangle $\Delta_n = \{w_{1,n},...,w_{n,n}\}$, where $\sum_{i=1}^n w_{i,n} = 1$. However, no weighting triangle is generally associated with any Δ_∞. Finally, note that, if we choose Δ_∞ such that $w_1 = 1$ and $w_i = 0$ as $i \geq 2$, then $\boldsymbol{W}^{\Delta_\infty} \equiv \max$, but, on the contrary, $\boldsymbol{W}^{\Delta_\infty} \not\equiv \min$, whatever is the Δ_∞ chosen.

Proposition 5 *The AO $\boldsymbol{A}^{\boldsymbol{H},\psi}$ is quasi-idempotent if and only if there exists $t_0 > 0$ such that $\psi(t_0) = 1$. The AO $\boldsymbol{W}^{\Delta_\infty}$ is quasi-idempotent if and only if there exists a $k \in \mathbb{N}$ such that $w_n = 0$ for all $n > k$.*

The final, general construction method we present regards a class of $e - symm$ AI AOs belonging to $\mathcal{CB}$. The philosophy of this method is that, according to Remark 6, the subdomain $[0,1]^n$ of the n-ary AF we have to define may be covered by the respective AF of any AI AO $\in \mathcal{CA}$, hence, by the symmetry, also the subdomain $[-1,0]^n$ is covered. The most interesting part is the rest of the domain, more precisely $cl([-1,1]^n \setminus I_n)$, i.e.the topological closure of the set $[-1,1]^n \setminus I_n$, where $I_n := [0,1]^n \cup [-1,0]^n$.

Example 3 Given any AI $\boldsymbol{A} \in \mathcal{CA}$, we set

$$\boldsymbol{B^*}|_{[0,1]\cup[-1,0]} := \boldsymbol{B^A}|_{[0,1]\cup[-1,0]},$$

i.e. for each $n \in \mathbb{N}$ we have $B_n^*(x_1,...,x_n) = A_n(x_1,...,x_n)$, if $(x_1,...,x_n) \in [0,1]^n$, while $B_n^*(x_1,...,x_n) = -A_n(|x_1|,...,|x_n|)$, if $(x_1,...,x_n) \in [-1,0]^n$. Let $f : [-1,1] \to [-1,1]$ be an arbitrary strictly increasing bijection such that $f(0) = 0$. Consider then a sequence $\{g_n\}_n$ of mappings from I_n to I_n defined

$$g_n(x_1,...,x_n) = f(B_n^*(x_1,...,x_n))$$

Evidently, any g_n is non-decreasing and commutative: further, for every $n \geq 2$, we have $g_n(x_1,...,x_{n-1},0) = g_{n-1}(x_1,...,x_{n-1})$ for all $(x_1,...,x_{n-1}) \in I_{n-1}$. Now, for each $n \geq 2$, we can define the n-ary AF B_n^* on $cl([-1,1]^n \setminus I_n)$ as follows:

$$B_n^*(x_1,...,x_n) = B_n^*(x_1^*,...,x_k^*,x_{k+1}^*,...,x_n^*) =$$

$$= f^{-1}\big(g_k(x_1^*,...,x_k^*) + g_{n-k}(|x_{k+1}^*|,...,|x_n^*|)\big),$$

recalling that $(x_1^*,...,x_k^*,x_{k+1}^*,...,x_n^*)$ is any permutation of an arbitrary input n-tuple $(x_1,...,x_n)$ such that $x_1^*,...,x_k^* \geq 0$, while $x_{k+1}^*,...,x_n^* < 0$, for some $k \in \{0,...,n\}$. It is not difficult to check that B_n^* is well defined on $cl([-1,1]^n \setminus I_n)$ and the whole $\boldsymbol{B^*}$ so obtained is actually an $e - symm$ AI AO belonging to $\mathcal{CB}$. Note that if f is not symmetrical with respect to zero, unlike B_1^*, also $g_1(x) = f(B_1(x))$ is not symmetrical, hence, for some $x \neq 0$, by definition of B_2, we get $B_2(x,-x) = f^{-1}(g_1(x) + g_1(-x)) \neq 0$, so proving that such a kind of $\boldsymbol{B^*}$ generally does not fulfill (12).

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