Exploring Dialogue Games as Foundation of Fuzzy Logic

Christian G. Fermüller
Theory and Logic Group 185.2
Technische Universität Wien
chrisf@logic.at

Abstract

A dialogue game based approach to the problem of providing a deeper semantic foundation for t-norm based fuzzy logics is explored. In particular, various versions, extensions and alternatives to Robin Giles's dialogue and betting game for Łukasiewicz logic are re-visited and put in the context of other foundational research in logic. It emerges that dialogue games cover a wide range of topics relevant to approximate reasoning.

Keywords: foundations, dialogue games, vagueness, analytic reasoning

1 Introduction

The adequate formalization of correct reasoning with vague notions and propositions is an important challenge in logic, computer science, as well as in philosophy. Many experts agree that modeling vagueness triggers reference to *degrees of truth* (but should strictly be distinguished from *degrees of belief*, and therefore requires methods different from probability theory and from modal logic). *Fuzzy Logics*, taken here in Zadeh's 'narrow sense' [18], are based on the extension of the two classical truth values by infinitely many intermediary degrees of truth. Formal deduction systems for specific fuzzy logics abound; however these systems are hardly ever explicitly related to models of correct reasoning with vague information. In other words: the challenge to derive inference systems for fuzzy logics from first principles about approximate reasoning is as yet largely unmet. The reference to general models of reasoning and to theories of vagueness—a prolific discourse in contemporary analytic philosophy—is only implicit, if not simply missing, in most presentations of inference systems for fuzzy logics. Some notable exceptions are: Ruspini's similarity semantics [25]; voting semantics [16]; 're-randomizing semantics' [15]; measurement-theoretic justifications [4]; the Ulam-Rényi game based interpretation of D. Mundici [20]; and 'acceptability semantics' of J. Paris [22]. As we have argued elsewhere [8], these formal semantics for various t-norm based logics (in particular Łukasiewicz logic) should be placed in the wider discourse on adequate *theories of vagueness*, a prolific subfield of analytic philosophy.

Here, we focus on a specific approach to derive logics from fundamental reasoning principles that was initiated by Robin Giles already in the 1970s [12]. This concept combines Paul Lorenzen's attempt to provide a dialogical foundation of logic in general (explained, e.g., in [17] and [2]) with a 'risk based' evaluation of atomic propositions that is specific to the context of vagueness understood as a phenomenon implying 'dispersion'. By the latter notion Giles refers to the fact that binary (yes/no)-experiments set up to test the acceptability of vague atomic assertions may show different outcomes when repeated. As we will see in Section 8, this concept allows to relate two seemingly very different theories of vagueness. Namely the familiar degree based, truth functional, approach of t-norm based fuzzy logic, on the one hand side, and 'supervaluationism', introduced by Kit Fine [11] and currently very popular among philosophers of vagueness, on the other hand side.

We will briefly review Giles's characterization of Łukasiewicz logic and provide an overview over more recent results covering a wider range of logics. As a sort of conclusion, we will indicate connections of the dialogical paradigm to other important foundational research programs in contemporary logic.

For brevity we restrict our attention to propositional logics, here. We assume the reader to be familiar with the basic concepts of t-norm based fuzzy logics as presented, e.g., in [14]. On the other hand we aim at a self-contained presentation as far as the (presumably less well known) concept of dialogue games as formal foundation of logical reasoning is concerned.

2 Giles's game for Łukasiewicz logic

Giles's analysis [12] of approximate reasoning originally referred to the phenomenon of 'dispersion' in the context of physical theories. Later Giles [13] explicitly applied the same concept to the problem of providing 'tangible meanings' to (logically complex) fuzzy propositions. For this purpose he introduces a game that consists of two independent components:

(1) Betting for positive results of experiments.

Two players—say: *me* and *you*—agree to pay 1€ to the opponent player for every false statement they assert. By $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ we denote an *elementary state* of the game, where I assert each of the q_i in the multiset $\{q_1, \ldots, q_n\}$ of atomic statements (represented by propositional variables), and you, likewise, assert each atomic statement $p_i \in \{p_1, \ldots, p_m\}$.

Each propositional variable q refers to an experiment E_q with binary (yes/no) result. The statement q can be read as 'E_q yields a positive result'. Things get interesting as the experiments may show dispersion; i.e., the same experiment may yield different results when repeated. However, the results are not completely arbitrary: for every run of the game, a fixed *risk value* $\langle q \rangle^r \in [0, 1]$ is associated with q, denoting the probability that E_q yields a negative result. For the special atomic formula $\bot$ (*falsum*) we define $\langle \bot \rangle^r = 1$. The risk associated with a multiset $\{p_1, \ldots, p_m\}$ of atomic formulas is defined as $\langle p_1, \ldots, p_m \rangle^r = \sum_{i=1}^{m} \langle p_i \rangle^r$. The risk $\langle \rangle^r$ associated with the empty multiset is defined as 0. The risk associated with an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ is calculated from my point of view. Therefore the condition $\langle p_1, \ldots, p_m \rangle^r \geq \langle q_1, \ldots, q_n \rangle^r$ expresses that I do not expect any loss (but possibly some gain) when betting on the truth of atomic statements, as explained above.

(2) A dialogue game for the reduction of compound formulas.

Giles follows ideas of Paul Lorenzen and his school that date back already to the 1950s (see, e.g., [17]) and constrains the meaning of logical connectives by reference to rules of a dialogue game that proceeds by systematically reducing arguments about compound formulas to arguments about their subformulas.

We at first assume that formulas are built up from propositional variables, the falsity constant $\bot$, and the connective $\rightarrow$ only.[1] The central dialogue rule can then be stated as follows:

($R_\rightarrow$) If I assert $A \rightarrow B$ then, whenever you choose to attack this statement by asserting A, I have to assert also B. (And vice versa, i.e., for the roles of me and you switched.)

This rule reflects the idea that the meaning of implication is specified by the principle that an assertion of 'if A, then B' ($A \rightarrow B$) obliges one to assert B, if A is granted.

In contrast to dialogue games for intuitionistic logic [17, 7], no special regulation on the succession of moves in a dialogue is required here. However, we assume that each assertion is attacked at most once: this is reflected by the removal of $A \rightarrow B$ from the multiset of all formulas asserted by a player during a run of the game, as soon as the other player has either attacked by asserting A, or has indicated that she will not attack $A \rightarrow B$ at all. Note that every run of the dialogue game ends in an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$. Given an assignment $\langle \cdot \rangle^r$ of risk values to all p_i and q_i we say that I *win* the corresponding run of the game if I do not expect any loss, i.e., if $\langle p_1, \ldots, p_m \rangle^r \geq \langle q_1, \ldots, q_n \rangle^r$.

As an almost trivial example consider the game where I initially assert $p \rightarrow q$ for some atomic formulas p and q; i.e., the initial state is $[\|p \rightarrow q]$. In response, you can either assert p in order to force me to assert q, or explicitly refuse to attack $p \rightarrow q$. In the first case, the game ends in the elementary state $[p\|q]$; in the second case it ends in state $[\|]$. If an assignment $\langle \cdot \rangle^r$ of risk values gives $\langle p \rangle^r \geq \langle q \rangle^r$, then I win, whatever move you choose to make. In other words: I have a winning strategy for $p \rightarrow q$ in all assignments of risk values where $\langle p \rangle^r \geq \langle q \rangle^r$.

Recall that a *valuation* v for Łukasiewicz logic Ł is a function assigning values $\in [0, 1]$ to the propositional variables and 0 to $\bot$, extended to compound formulas using the truth function $x \Rightarrow_{Ł} y = \inf\{1, 1 - x + y\}$.

Theorem. (R. Giles [12])
Every assignment $\langle \cdot \rangle^r$ of risk values to atomic formulas occurring in a formula F induces a valuation $v_{\langle \cdot \rangle^r}$ for Ł such that $v_{\langle \cdot \rangle^r}(F) = 1$ iff I have a winning strategy for F in the game presented above.

Corollary.
F is valid in Ł iff, for all assignments of risk values to atomic formulas occurring in F, I have a winning strategy for F.

3 Other connectives

Although all other connectives can be defined in Łukasiewicz logic from $\rightarrow$ and $\bot$ alone, it will be helpful to illustrate the idea that the meaning of *all* rele-

[1] Remember that in Ł all other connectives can be defined from $\rightarrow$ and $\bot$ alone. (See, e.g., [14].)

vant connectives can be specified directly by intuitively plausible dialogue rules. Interestingly, for conjunction *two* different rules seem to be plausible candidates at a first glance:

($R_\wedge$) If I assert $A_1 \wedge A_2$ then I have to assert also A_i for any $i \in \{1, 2\}$ that you may choose.

($R_{\wedge'}$) If I assert $A_1 \wedge' A_2$ then I have to assert also A_1 as well as A_2.

Of course, both rules turn into rules referring to *your* claims of a conjunctive formula by simply switching the roles of the players ('I' and 'you').

Rule ($R_\wedge$) is dual to the following natural candidate for a disjunction rule:

($R_\vee$) If I assert $A_1 \vee A_2$ then I have to assert also A_i for some $i \in \{1, 2\}$ that I myself may choose.

Moreover, it is clear how ($R_\wedge$) generalizes to a rule for universal quantification.

It follows already from results in [12] that rules ($R_\wedge$) and ($R_\vee$) are adequate for 'weak' conjunction and disjunction in Ł, respectively. $\wedge$ and $\vee$ are also called 'lattice connectives' in the context of fuzzy logics, since their truth functions are given by $v(A \wedge B) = \inf\{v(A), v(B)\}$ and $v(A \vee B) = \sup\{v(A), v(B)\}$.

The question arises, whether one can use the remaining rule ($R_{\wedge'}$) to characterize strong conjunction (&) which corresponds to the t-norm $x *_Ł y = \sup\{0, x + y - 1\}$.

However, rule ($R_{\wedge'}$) is inadequate in the context of our betting scheme for random evaluation in a precisification space. The reason for this is that we have to ensure that for each (not necessarily atomic) assertion that we make, we risk a *maximal* loss of 1€ only. It is easy to see that rules ($R_\to$), ($R_\wedge$), and ($R_\vee$) comply with this 'principle of limited liability'. However, if I assert $p \wedge' q$ and we proceed according to ($R_{\wedge'}$), then I end up with a loss of 2€, in case both experiments E_p and E_q fail. There is a simply way to redress this situation to obtain a rule that is adequate for (&): Allow any player who asserts $A_1 \,\&\, A_2$ to hedge her possible loss by asserting $\bot$ instead of A_1 and A_2, if whished. Asserting $\bot$, of course, corresponds to the obligation to pay 1€ (but not more) in the resulting final state. We obtain the following rule for strong conjunction:

($R_\&$) If I assert $A_1 \,\&\, A_2$ then I either have to assert A_1 as well as A_2, or else I have to assert $\bot$.

In a similar way, also dialogue rules for negation, 'strong' disjunction, and equivalence can be formulated directly, instead of just derived from ($R_\to$).

4 Beyond Łukasiewicz logic

There is an interesting ambiguity in the phrase 'betting for positive results of (a multiset of) experiments' that describes the evaluation of elementary states of the dialogue game. As explained above, Giles identifies the combined risk for such a bet with the *sum* of risks associated with the single experiments. However, other ways of interpreting the combined risk are worth exploring. In [6] we have considered a second version of the game, where an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ corresponds to my single bet that *all* experiments associated with the q_i, where $1 \leq i \leq n$, show a positive result, against your single bet that *all* experiments associated with the p_i ($1 \leq i \leq m$) show a positive result. A third form of the game arises (again, see [6]) if one decides to perform only *one* experiment for each of the two players, where the relevant experiment is chosen by the opponent.

It turns out that these three betting schemes constitute three versions of Giles's game that are adequate for the three fundamental logics Ł (Łukasiewicz logic), P (Product logic), and G (Gödel logic), respectively. To understand this result it is convenient to invert risk values into probabilities of *positive* results (yes-answers) of the associated experiments. More formally, the *value* of an atomic formula q is defined as $\langle q \rangle = 1 - \langle q \rangle^r$; in particular, $\langle \bot \rangle = 0$.

My expected gain in the elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ in Giles's game for Ł is the sum of money that I expect you to have to pay me minus the sum that I expect to have to pay you. This amounts to $\sum_{i=1}^{m}(1 - \langle p_i \rangle) - \sum_{i=1}^{n}(1 - \langle q_i \rangle)$ €. Therefore, my expected gain is greater or equal to zero iff $1 + \sum_{i=1}^{m}(\langle p_i \rangle - 1) \leq 1 + \sum_{i=1}^{n}(\langle q_i \rangle - 1)$ holds. The latter condition is called winning condition $W_\sum$. (Note that 'winning' here refers to *expected* gain: although, in this sense, I 'win' in state $[p \| p]$, I still loose 1€ in those concrete runs of the game, where the instance of the experiment E_p referring to *my* assertion of p results in 'no', but where the instance of E_p referring to *your* assertion of p end positively (answer 'yes').

In the second version of the game, you have to pay me 1€ unless all experiments associated with the p_i test positively, and I have to pay you 1€ unless all experiments associated with the q_i test positively. My expected gain is therefore $1 - \prod_{i=1}^{m}\langle p_i \rangle - (1 - \prod_{i=1}^{n}\langle q_i \rangle)$ €; the corresponding winning condition $W_\prod$ is $\prod_{i=1}^{m}\langle p_i \rangle \leq \prod_{i=1}^{n}\langle q_i \rangle$.

To maximize the expected gain in the third version of the game I will choose a $p_i \in \{p_1, \ldots, p_m\}$ where the probability of a positive result of the associated experiment is least; and you will do the same for the

q_i's that I have asserted. Therefore, my expected gain is $(1 - \min_{1\le i\le m}\langle p_i\rangle) - (1 - \min_{1\le i\le n}\langle q_i\rangle)$ €. Hence the corresponding winning condition $W_{\min}$ is $\min_{1\le i\le m}\langle p_i\rangle \le \min_{1\le i\le n}\langle q_i\rangle$.

We thus arrive at the following definitions for the value of a multiset $\{p_1,\ldots,p_n\}$ of atomic formulas, according to the three versions of the game:

$\langle p_1,\ldots,p_n\rangle_{\text{Ł}} = 1 + \sum_{i=1}^{n}(\langle p_i\rangle - 1)$
$\langle p_1,\ldots,p_n\rangle_{\mathsf{P}} = \prod_{i=1}^{n}\langle p_i\rangle$
$\langle p_1,\ldots,p_n\rangle_{\mathsf{G}} = \min_{1\le i\le n}\langle p_i\rangle$.

For the empty multiset we define $\langle\rangle_{\text{Ł}} = \langle\rangle_{\mathsf{P}} = \langle\rangle_{\mathsf{G}} = 1$.

In contrast to Ł, the dialogue game rule (R) does not suffice to characterize P and G. To see this, consider the state $[p \to \bot \| q]$. According to rule (R) I may assert p in order to force you to assert $\bot$. Since $\langle\bot\rangle = 0$, the resulting elementary state $[\bot \| p, q]$ fulfills the winning conditions $\langle\bot\rangle \le \langle p\rangle \cdot \langle q\rangle$ and $\langle\bot\rangle \le \min\{\langle p\rangle, \langle q\rangle\}$, that correspond to P and G, respectively. However, this is at variance with the fact that for assignments where $\langle p\rangle = 0$ and $\langle q\rangle < 1$ you have asserted a statement $(p \to \bot)$ that is definitely true $(v(p \to \bot) = 1)$, whereas my statement q is not definitely true $(v(q) < 1)$.[2]

There are different ways to address the indicated problem. They all seem to imply a break of the symmetry between the roles of the two players (me and you). We have to distinguish between elementary states in which my expected gain is non-negative and those in which my expected is strictly positive. Accordingly, we introduce a (binary) signal or *flag* ¶ into the game which, when raised, announces that I will be declared the winner of the current run of the game, only if the evaluation of the final elementary state yields a *strictly positive* (and not just non-negative) expected gain for me. This allows us to come up with a version of the dialogue rules for implication that can be shown to lead to adequate games for all three logics consider here (Ł, P, G):

($R^{I*}_{\to}$) If I assert $A \to B$ then, whenever you choose to attack this statement by asserting A, I have the following choice: either I assert B in reply or I challenge your attack on $A \to B$ by replacing the current game with a new one in which you assert A and I assert B.

In formulating an adequate rule for my attacks on your assertions of an implicative formulas we have to use the flag signalling the strict case of the winning condition:

($R^{Y*}_{\to}$) If you assert $A \to B$ then, whenever I choose to attack this statement by asserting A, you have the following choice: either you assert B in reply or you challenge my attack on $A \to B$ by replacing the current game with a new one in which the flag ¶ is raised and I assert A while you assert B.

In contrast to Ł, in G and P the other connectives cannot be defined from $\to$ and $\bot$ alone. However, the rules presented in Section 3 turn out to be adequate for G and P, too. In the case of Gödel logic (G), the two versions of conjunction ('strong' and 'weak') coincide. This fact, that is well known from the algebraic view of *t*-norm based logic (see, e.g., [14]) can also be obtained by comparing optimal strategies involving the rules $(R_\wedge)$ and $(R_\&)$, respectively.

5 Truth comparison games

In [9] yet another dialogue game based approach to reasoning in Gödel logic G has been described. It relies on the fact that G is the only *t*-norm based logic, where the validity of formulas depends only on the *relative order* of the values of the involved propositional variables. This observation arguably is of philosophical interest in the context of scepticism concerning the meaning of particular real numbers $\in [0,1]$ understood as 'truth values'. To emphasize that only the *comparison* of degrees of truth, using the standard order relations $<$ and $\le$, is needed in evaluating formals in G, one may refer to a dialogue game which is founded on the idea that any logical connective $\circ$ of G can be characterized via an adequate response by a player **X** to player **Y**'s attack on **X**'s claim that a statement of form $(A \circ B) \lhd C$ or $C \lhd (A \circ B)$ holds, where $\lhd$ is either $<$ or $\le$.

We need the following definitions. An assertion $F \lhd G$ is *atomic* if F and G are either propositional ; otherwise it is a *compound assertion.* Atomic assertions of form $p < p$, $p < \bot$, $\top < p$ or $\top \le \bot$ are called *elementary contradictions.* An *elementary order claim* is a set of two assertions of form $\{E \lhd_1 F, F \lhd_2 G\}$, where E, F, and G are atoms, and $\lhd_1, \lhd_2 \in \{<, \le\}$.

Following traditional terminology, introduced by Paul Lorenzen, we call the player that initially claims the validity of a chosen formula the *Proponent* **P**, and the player that tries to refute this claim the *Opponent* **O**. The dialogue game proceeds in rounds as follows:

1. A dialogue starts with **P**'s claim that a formula F is valid. **O** answers to this move by contradicting this claim with the assertion $F < \top$.
2. Each following round consists in two steps:
   (i) **P** either attacks a compound assertion or an elementary order claim, contained in the set

[2] The problem does not arise in logic Ł, since there the expected gain for state $[\bot \| p, q]$ is $\langle p, q\rangle_{\text{Ł}} - \langle\bot\rangle_{\text{Ł}} = 1 - (\langle p\rangle - 1) - (\langle q\rangle - 1) - (1 - 1) = \langle p\rangle + \langle q\rangle - 1$ and therefore, indeed, negative, as expected, if $\langle p\rangle = 0$ and $\langle q\rangle < 1$.

of assertions that have been made by **O** up to this state of the dialogue, but that have not yet been attacked by **P**.

(ii) **O** answers to the attack by adding a set of assertions according to the rules of Table 1 (for compound assertions) and Table 2 (for elementary order claims).

3. The dialogue ends with **P** as winner if **O** has asserted an elementary contradiction. Otherwise, **O** wins if there is no further possible attack for **P**.

Table 1: Rules for connectives

P attacks:	**O** asserts as answer:
$A\&B \lhd C$	$\{A \lhd C\}$ or $\{B \lhd C\}$
$C \lhd A\&B$	$\{C \lhd A,\ C \lhd B\}$
$A \vee B \lhd C$	$\{A \lhd C,\ B \lhd C\}$
$C \lhd A \vee B$	$\{C \lhd A\}$ or $\{C \lhd B\}$
$A \to B < C$	$\{B < A,\ B < C\}$
$C < A \to B$	$\{C < B\}$ or $\{A \leq B,\ C < \top\}$
$A \to B \leq C$	$\{\top \leq C\}$ or $\{B < A, B \leq C\}$
$C \leq A \to B$	$\{A \leq B\}$ or $\{C \leq B\}$

In the first four lines, $\lhd$ denotes either $<$ or $\leq$, consistently throughout each line. Assertions, which involve a choice of **O** in the answer (indicated by 'or') are called *or-type* assertions. All other assertions are of *and-type*.

Table 2: Rules for elementary order claims

P attacks:	**O** asserts as answer:
$\{A \leq B,\ B \leq C\}$	$\{A \leq C\}$
$\{A < B,\ B \lhd C\}$	$\{A < C\}$
$\{A \lhd B,\ B < C\}$	$\{A < C\}$

where $\lhd$ is either $<$ or $\leq$.

Remark. Instead of considering the rules of Table 1 and 2 as derived from the truth functions for G, one may argue that the dialogue rules are derived from fundamental principles about reasoning in a truth functional, order based fuzzy logic.

Consider the example of conjunction. We contend that anyone who claims '$A\&B$ is at least as true as C' (for any concrete statements A, B, and C) has to be prepared to defend the claim that 'A is at least as true as C' and the claim that 'B is at least as true as C'. On the other hand, the claim that 'C is at least as true as $A\&B$', arguably, should be supported either by 'C is at least as true as A' or by 'C is at least as true as B'. (Likewise, if we replace 'at least as true' by 'truer than'.) One may then go on to argue that this reading of the rules for & in Table 1 completely determines correct reasoning about assertions of this form. Form this assumption, one can *derive* that $v(A\&B) = \min(v(A), v(B))$ is the only adequate definition for the semantics of conjunction in this setting.

The case for disjunction is very similar. Implication, as usual, is more controversial. However, it is easy to see that there are hardly any reasonable alternatives to our rules, if the truth of any assertion involving a formula $A \to B$ should only depend on the relative degree of truth of A and B (but should not depend on the result of an arithmetical operations that had to be performed on the values of A and B, respectively).

Formally we may summarize this analysis of Gödel logic as follows:

Theorem. [9]
A formula F is valid in G *iff there exists a winning strategy for* **P** *on F in the presented comparison game.*

6 Pavelka style reasoning

An important paradigm for approximate reasoning has been explored in a series of papers by J. Pavelka [23]. It is sometimes also advocated as 'fuzzy logic with evaluated syntax' (see, e.g., [21]). In this approach one makes the reference to degrees of truth explicit by considering *graded formulas* $r : F$ as basic objects of inference, where r is a rational number $\in [0, 1]$ and F is an Ł-formula, with the intended interpretation that F is evaluated to a value $\geq r$. The resulting logic is called *rational Pavelka logic* RPL in [14].

Inference systems for RPL can be obtained by using the following graded version of modus ponens as rule of derivation:

$$\frac{r : A \quad s : A \to B}{r *_{Ł} s : B}$$

Completeness and soundness of such systems can be stated as the coincidence of the *truth degree* $\|F\|_T$ of F over some theory (set of graded formulas) T with the *provability degree* $|F|_T$ of F over T. Here $\|F\|_T$ is defined as $\inf_{v \in I_T} v(F)$, where I_T is the set of all Ł-valuations satisfying T; and $|F|_T$ is defined as $\sup\{r \mid T \vdash r : F\}$, where $\vdash$ denotes the indicated derivability relation. (See, e.g., [14] for details.)

It is easy to see that Giles's dialogue game for Ł can be adapted to RPL, since a graded formula $r : F$ can be expressed as $\bar{r} \to F$ in Ł if Ł is extended by truth constants $\bar{r}$ for all rationals $\in [0, 1]$, interpreted by stipulating $v(\bar{r}) = r$. The only change in Giles's original dialogue and betting scenario (explained in Section 2) is the additional reference to special elementary experiments $E_{\bar{r}}$ with fixed success probabilities r. Such experiments can easily be defined for all rational p by referring to a certain number of fair coin tosses and an adequate definition of a 'positive result'. According to the dialogue rule $(R_\to)$ of Section 2 an attack on the graded statement $r : F$ $(=\bar{r} \to F)$ indicates the willingness of the attacking player to bet on a positive result of $E_{\bar{r}}$ in exchange for an assertion of F by the other player. Clearly, one can simplify the overall pay-off scheme by stipulating that an attack by player **X** on a graded formula $r : F$ consists in paying $(1-r)$€ to the opponent player **Y** and thereby forcing **Y** to continue the game with an assertion of F.

Since Ł is the only fuzzy logic, where also the residuum $\Rightarrow_{Ł}$ of the underlying t-norm is a continuous function, one cannot readily transfer the concept of provability degrees that match truth degrees to other logics. Nevertheless, it makes sense to enrich the syntax of Gödel logic G and Product logic P not only by rational truth constants, but also by a binary connective ':' with the corresponding truth function $\tilde{:}$ given by

$$x \,\tilde{:}\, y = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Whereas Hilbert-style axiomatizations of such enriched logics, which contain 'evaluated syntax', seem elusive (beyond RPL), it is rather straightforward to define dialogue game rules and pay-off schemes that capture the intended meaning of the thus extended versions of G and P. Moreover, in the case of G, it is also possible to extend the truth comparison game sketched in Section 5 to include evaluated syntax. (Details are left to future work.)

7 Connections to supervaluation

Supervaluation is a widely discussed concept in philosophical logic. Kit Fine has pioneered its application to formal languages that accommodate vague propositions in [11], a paper that remains an important reference point for philosophers of language and logic. The main idea is to evaluate propositions not simply with respect to classical interpretations—i.e., assignments of the truth values 0 ('false') and 1 ('true') to atomic statements—but rather with respect to a whole *space* Π of (possibly) partial interpretations. For every partial interpretation I in Π, Π is required to contain also a classical interpretation I' that extends I. I' is called an *admissible (complete) precisification* of I. A proposition is called *supertrue* in Π if it evaluates to 1 in all admissible precisifications, i.e., in all classical interpretations contained in Π.

Supervaluation and fuzzy logics can be viewed as capturing contrasting, but individually coherent intuitions about the role of logical connectives in vague statements. Consider a sentence like

(*) "The sky is blue and is not blue".

When formalized as $b \,\&\, \neg b$, (*) is superfalse in all precisification spaces, since either b or $\neg b$ is evaluated to 0 in each precisification. This fits Kit Fine's motivation in [11] to capture 'penumbral connections' that prevent any mono-colored object from having two colors at the same time. According to Fine's intuition the statement "The sky is blue" absolutely contradicts the statement "The sky is not blue", even if neither statement is definitely true or definitely false. Consequently (*) is judged as definitely false, although admittedly composed of vague sub-statements. On the other hand, by asserting (*) one may intend to convey the information that both component statements are true *only to some degree*, different from 1 but also from 0. Under this reading and certain 'natural' choices of truth functions for & and ¬ the statement $b \,\&\, \neg b$ is *not* definitely false, but receives some intermediary truth value.

In [10], we have worked out a dialogue game based attempt to reconcile supervaluation and t-norm based ('fuzzy') evaluation within a common formal framework. To this aim we interpret 'supertruth' as a modal operator and define a logic SŁ that extends both, Łukasiewicz logic Ł, as well as the classical modal logic S5.

Formulas of SŁ are built up from the propositional variables $p \in V = \{p_1, p_2, \ldots\}$ and the constant $\bot$ using the connectives & and $\rightarrow$. The additional connectives $\neg$, $\wedge$, and $\vee$ are defined as explained above. In accordance with our earlier (informal) semantic considerations, a precisification space is formalized as a triple $\langle W, e, \mu \rangle$, where $W = \{\pi_1, \pi_2, \ldots\}$ is a non-empty (countable) set, whose elements π_i are called *precisification points*, e is a mapping $W \times V \mapsto \{0,1\}$, and μ is a probability measure on the σ-algebra formed by all subsets of W. Given a precisification space $\Pi = \langle W, e, \mu \rangle$ a *local truth value* $\|A\|_\pi$ is defined for every formula A and every precisification point $\pi \in W$ inductively by

$$\begin{aligned}
\|p\|_\pi &= e(\pi, p), \text{ for } p \in V \\
\|\bot\|_\pi &= 0 \\
\|A \,\&\, B\|_\pi &= \begin{cases} 1 & \text{if } \|A\|_\pi = 1 \text{ and } \|B\|_\pi = 1 \\ 0 & \text{otherwise} \end{cases} \\
\|A \rightarrow B\|_\pi &= \begin{cases} 1 & \text{if } \|A\|_\pi = 1 \text{ and } \|B\|_\pi = 0 \\ 0 & \text{otherwise} \end{cases} \\
\|\mathsf{S}A\|_\pi &= \begin{cases} 1 & \text{if } \forall \sigma \in W : \|A\|_\sigma = 1 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Local truth values are classical and do not depend on the underlying t-norm $*_{Ł}$. In contrast, the *global truth value* $\|A\|_\Pi$ of a formula A is defined by

$$\begin{aligned}
\|p\|_\Pi &= \mu(\{\pi \in W | e(\pi, p) = 1\}), \text{ for } p \in V \\
\|\bot\|_\Pi &= 0 \\
\|A \,\&\, B\|_\Pi &= \|A\|_\Pi *_{Ł} \|B\|_\Pi \\
\|A \rightarrow B\|_\Pi &= \|A\|_\Pi \Rightarrow_{Ł} \|B\|_\Pi \\
\|\mathsf{S}A\|_\Pi &= \|\mathsf{S}A\|_\pi \text{ for any } \pi \in W
\end{aligned}$$

Note that $\|\mathsf{S}A\|_\pi$ is the same value (either 0 or 1) for all $\pi \in W$. In other words: 'local' supertruth is in fact already global; which justifies the above clause for $\|\mathsf{S}A\|_\Pi$. Also observe that we could have used the global conditions, referring to $*_Ł$ and $\Rightarrow_Ł$, also to define $\|A \,\&\, B\|_\pi$ and $\|A \rightarrow B\|_\pi$, since the t-norm based truth functions coincide with the (local) classical ones, when restricted to $\{0, 1\}$. (However that presentation might have obscured their intended meaning.)

Most importantly for our current purpose, it has been demonstrated in [10] that the evaluation of formulas of SŁ can be characterized by a dialogue game extending Giles's game for Ł, where 'dispersive elementary experiments' (see Section 2) are replaced by 'indeterministic evaluations' over precisification spaces. The dialogue rule for the supertruth modality involves a relativization to specific precisification points:

(R_S) If I assert $\mathsf{S}A$ then I also have to assert that A holds at any precisification point π that you may choose. (And *vice versa*, i.e., for the roles of me and you switched.)

The resulting game is adequate for SŁ:

Theorem. [10]
A formula F is valid in SŁ *iff for every precisification space* Π *I have a winning strategy for the game starting with my assertion of* F.

8 Dialogue games in a wider context

Having sketched the rather varied landscape of dialogue game based approaches to the foundations of fuzzy logic—following Giles's pioneering work in the 1970s—we finally want to hint briefly at some connections with other foundational enterprises in logic. We think that these connections indicate potential benefits that the dialogue game approach might enjoy relative to alternative semantic frameworks mentioned in the introduction (Section 1).

Connections to Lorenzen style constructivism. It is certainly true that reasoning with vague notions and propositions poses challenges to philosophical logic that are different from well known concerns about, e.g., constructive meaning, adequate characterization of entailment ('relevance'), or intentional logics. However, one should not dismiss the possibility that traditional approaches to foundational problems in logic may benefitly be employed to enhance the understanding of fuzzy logics, too. Lorenzen's dialogical paradigm is a widely discussed, flexible tool in such foundational investigations. (See, e.g., [2, 24, 17].) Its philosophical underpinnings can assist in the difficult task to derive mathematical structures that are used in fuzzy logics from more fundamental assumptions about correct reasoning. In this context, the fact that Lorenzen and his collaborators have (somewhat narrowly) focussed on intuitionistic logic, may help to uncover deep connections between constructive reasoning and reasoning under vagueness.

Connections to 'game logics' and 'logic games'. In recent years the logical analysis of games as well as game theoretic approaches to logic emerge as prolific foundational research areas that entail interest in topics like dynamics and interaction of reasoning agents, analysis of strategies and different forms of knowledge. (See, e.g., [3] or `www.illc.uva.nl/lgc/` for further references.) It is clear that dialogue games, like the ones described in this paper, nicely fit in this framework. Foundational research in fuzzy logic, along the lines indicated here, will surely profit from new results about games in logic and logic in games. Moreover, it is not unreasonable to hope that, *vice versa*, fuzzy logic has to offer interesting new perspectives on agent knowledge and interaction that will be taken up by 'game logics' in future research.

Connections to proof search and analytic calculi. The renewed interest in Giles's game for Ł, indicated in our brief survey, above, was in fact triggered by the discovery of relations between corresponding winning strategies, on the one hand side, and cut-free derivations in a so-called hypersequent system for Ł[19], on the other hand side. The logical rules of the uniform analytic proof system for Ł, G, and P introduced in [6], correspond directly the dialogue rules of the modified dialogue game described in Section 4, above. This entails a close correspondence between the systematic construction of winning strategies for the dialogue game and proof search strategies in the uniform analytic calculus. Moreover, this correspondence can be viewed as a general principle for the interpretation of logical rules in analytic (i.e., cut-free) hypersequent or sequent calculi in terms of the options of the two players in a dialogue game. It remains to be seen whether, as a consequence, the dialogue game approach can be used (beyond foundational concerns) also to model and plan efficient proof search.

Connections to substructural logics. Games that have been inspired by Lorenzen's original dialogue game for intuitionistic logic are widely used in the analysis of (fragments of) linear logic and related formalism (see, e.g., [5, 1]). This research field, often simply called 'game semantics', highlights applications of rather abstract forms of dialogue games, where logical connectives are viewed as certain operators on formal games. While the emphasis in dialogical approaches to fuzzy logics, arguably, is closer to philosophical concerns about providing 'tangible meaning' (to use a phrase of Robin Giles), it is nevertheless evident that there are common interests in the search for alternative semantics of linear logic and t-norm based fuzzy logics, respectively. To name just one corresponding problem: How can the feature of 'resource consciousness' of logics be adequately characterized at the level of analytic reasoning? Dialogue semantics clearly aims at a direct model of this and related features of information processing, thus stressing the well known fact that t-norm based fuzzy logics can be viewed as a particular type of substructural logics.

Let us finally point out that this short survey on dialogue games for fuzzy logics is far from complete. Among related topics, pursued elsewhere, we just mention evaluation games, parallel dialogue games for intermediate logics (including G) and connections to Mundici's analysis of the Ulam-Rényi game. However, already the results described here allow us to conclude that the dialogical approach, although originally developed in a quite different philosophical context, bears fruits also in the realm fuzzy logic.

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