Representations of Archimedean t-norms in interval-valued fuzzy set theory

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Abstract

In this paper the Archimedean property and the nilpotency of t-norms on the lattice $\mathcal{L}^I$ is investigated, where $\mathcal{L}^I$ is the underlying lattice of interval-valued fuzzy set theory (Sambuc, 1975) and intuitionistic fuzzy set theory (Atanassov, 1983). We give some characterizations of continuous t-norms on $\mathcal{L}^I$ which satisfy the residuation principle, $\mathcal{T}(D,D) \subseteq D$, the Archimedean property and nilpotency. **Keywords:** interval-valued fuzzy set, intuitionistic fuzzy set, t-norm, Archimedean, nilpotent, strict, representation.

1 Introduction

Triangular norms on $[0,1]$ were introduced in [18] and play an important role in fuzzy set theory (see e.g. [9, 12, 13] for more details). One of the most important properties that can be satisfied by t-norms on the unit interval is the Archimedean property, for example continuous t-norms can be fully characterized by means of Archimedean t-norms, the Archimedean property is closely related to additive and multiplicative generators [13, 15, 16].

Interval-valued fuzzy set theory [11, 17] is an extension of fuzzy theory in which to each element of the universe a closed subinterval of the unit interval is assigned which approximates the unknown membership degree. Another extension of fuzzy set theory is intuitionistic fuzzy set theory introduced by Atanassov [1]. Intuitionistic fuzzy sets assign to each element of the universe not only a membership degree, but also a non-membership degree which is less than or equal to 1 minus the membership degree (in fuzzy set theory the non-membership degree is always equal to 1 minus the membership degree). In [7] it is shown that intuitionistic fuzzy set theory is equivalent to interval-valued fuzzy set theory and that both are equivalent to L-fuzzy set theory in the sense of Goguen [10] w.r.t. a special lattice $\mathcal{L}^I$.

In this paper we will investigate the nilpotency property and we will give some characterizations of continuous t-norms on $\mathcal{L}^I$ which satisfy the residuation principle, $\mathcal{T}(D,D) \subseteq D$ and the Archimedean property.

2 Preliminary definitions

The underlying lattice $\mathcal{L}^I$ of interval-valued fuzzy set theory is given as follows.

Definition 2.1 *We define $\mathcal{L}^I = (L^I, \leq_{L^I})$, where*

$$L^I = \{[x_1, x_2] \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 \leq x_2\},$$
$$[x_1, x_2] \leq_{L^I} [y_1, y_2] \iff (x_1 \leq y_1 \text{ and } x_2 \leq y_2),$$
$$\text{for all } [x_1, x_2], [y_1, y_2] \text{ in } L^I.$$

Similarly as Lemma 2.1 in [7] it is shown that $\mathcal{L}^I$ is a complete lattice.

Let U be a universal set.

Definition 2.2 [11, 17] *An interval-valued fuzzy set on U is a mapping $A : U \to L^I$.*

Definition 2.3 [1] *An intuitionistic fuzzy set on U is a set*

$$A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\},$$

where $\mu_A(u) \in [0,1]$ *denotes the membership degree and* $\nu_A(u) \in [0,1]$ *the non-membership degree of* u *in* A *and where for all* $u \in U$, $\mu_A(u) + \nu_A(u) \le 1$.

An intuitionistic fuzzy set A on U can be represented by the $\mathcal{L}^I$-fuzzy set A given by

$$A : U \to L^I : \quad u \mapsto [\mu_A(u), 1 - \nu_A(u)], \quad \forall u \in U$$

In Figure 1 the set L^I is shown. Note that to any element $x = [x_1, x_2]$ of L^I there corresponds a point $(x_1, x_2) \in \mathbb{R}^2$.

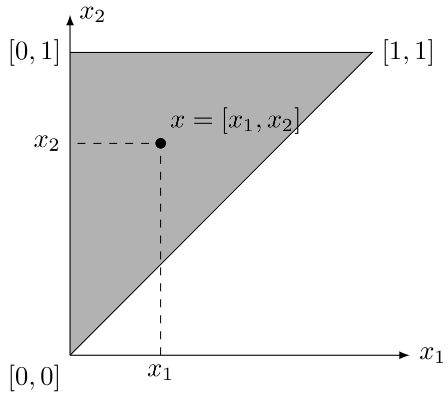

Figure 1: The grey area is L^I.

In the sequel, if $x \in L^I$, then we denote its bounds by x_1 and x_2, i.e. $x = [x_1, x_2]$. The smallest and the largest element of $\mathcal{L}^I$ are given by $0_{\mathcal{L}^I} = [0,0]$ and $1_{\mathcal{L}^I} = [1,1]$. We define for further usage the set $D = \{[x_1, x_1] \mid x_1 \in [0,1]\}$. Note that, for x, y in L^I, $x <_{L^I} y$ is equivalent to "$x \le_{L^I} y$ and $x \ne y$", i.e. either $x_1 < y_1$ and $x_2 \le y_2$, or $x_1 \le y_1$ and $x_2 < y_2$. We denote by $x \ll_{L^I} y$: $x_1 < y_1$ and $x_2 < y_2$.

Definition 2.4 *A t-norm on a complete lattice* $\mathcal{L} = (L, \le_L)$ *is a commutative, associative, increasing mapping* $\mathcal{T} : L^2 \to L$ *which satisfies* $\mathcal{T}(1_{\mathcal{L}}, x) = x$, *for all* $x \in L$.

Let $\mathcal{T}$ be a t-norm on a complete lattice $\mathcal{L} = (L, \le_L)$ and $x \in L$, then we denote $x^{(n)_\mathcal{T}} = \mathcal{T}(x, x^{(n-1)_\mathcal{T}})$, for $n \in \mathbb{N} \setminus \{0,1\}$, and $x^{(1)_\mathcal{T}} = x$.

We say that a t-norm $\mathcal{T}$ on $\mathcal{L}$ satisfies the residuation principle if and only if, for all x, y, z in L,

$$\mathcal{T}(x, y) \le_L z \iff y \le_L \mathcal{I}_\mathcal{T}(x, z),$$

where $\mathcal{I}_\mathcal{T}$ is the residual implication of $\mathcal{T}$ defined by $\mathcal{I}_\mathcal{T}(x,z) = \sup\{\gamma \mid \gamma \in L \text{ and } \mathcal{T}(x,\gamma) \le_L z\}$, for all x, z in L.

For t-norms on $\mathcal{L}^I$, we consider the following special classes.

Definition 2.5 *A t-norm* $\mathcal{T}$ *on* $\mathcal{L}^I$ *is called t-representable if there exist t-norms* T_1 *and* T_2 *on* $([0,1], \le)$ *such that* $T_1(x,y) \le T_2(x,y)$, *for all* x, y *in* $[0,1]$, *and such that, for all* x, y *in* L^I,

$$\mathcal{T}(x, y) = [T_1(x_1, y_1), T_2(x_2, y_2)].$$

Then T_1 *and* T_2 *are called the representants of* $\mathcal{T}$ *and we denote* $\mathcal{T}$ *by* $\mathcal{T}_{T_1,T_2}$.

A t-norm $\mathcal{T}$ *on* $\mathcal{L}^I$ *is called pseudo-t-representable if there exists a t-norm* T *on* $([0,1], \le)$ *such that, for all* x, y *in* L^I,

$$\mathcal{T}(x,y) = [T(x_1,y_1), \max(T(x_1,y_2), T(x_2,y_1))].$$

Then T *is called the representant of* $\mathcal{T}$ *and we denote* $\mathcal{T}$ *by* $\mathcal{T}_T$.

Definition 2.6 *A negation on a complete lattice* $\mathcal{L} = (L, \le_L)$ *is a decreasing mapping* $\mathcal{N} : L \to L$ *for which* $\mathcal{N}(0_\mathcal{L}) = 1_\mathcal{L}$ *and* $\mathcal{N}(1_\mathcal{L}) = 0_\mathcal{L}$. *If* $\mathcal{N}(\mathcal{N}(x)) = x$, *for all* $x \in L$, *then* $\mathcal{N}$ *is called involutive.*

Let $\mathcal{I}_\mathcal{T}$ be the residual implication of a t-norm $\mathcal{T}$ on $\mathcal{L}$. The mapping $\mathcal{N}_{\mathcal{I}_\mathcal{T}} : L \to L$ defined by $\mathcal{N}_{\mathcal{I}_\mathcal{T}}(x) = \mathcal{I}_\mathcal{T}(x, 0_\mathcal{L})$, for all $x \in L$, is a negation on $\mathcal{L}$, called the negation generated by $\mathcal{I}_\mathcal{T}$.

Theorem 2.1 [6] *Let* $\mathcal{N}$ *be a negation on* $\mathcal{L}^I$. *Then* $\mathcal{N}$ *is involutive if and only if there exists an involutive negation* N *on* $([0,1], \le)$ *such that, for all* $x \in L^I$, $\mathcal{N}(x) = [N(x_2), N(x_1)]$.

Let $n \in \mathbb{N} \setminus \{0\}$. If for a mapping $f : [0,1]^n \to [0,1]$ and a mapping $F : (L^I)^n \to L^I$ it holds that $F(D^n) \subseteq D$, and $F([a_1,a_1], \ldots, [a_n,a_n]) = [f(a_1,\ldots,a_n), f(a_1,\ldots,a_n)]$, for all $a_1, \ldots, a_n$ in L^I, then we say that F is a natural extension of f to L^I. E.g. for any t-norm T on $([0,1], \le)$, the t-norms $\mathcal{T}_{T,T}$ and $\mathcal{T}_T$ are natural extensions of T to L^I; if $\mathcal{N}$ is an involutive negation on $\mathcal{L}^I$, then from Theorem 2.1 it follows that there exists an involutive negation N on $([0,1], \le)$ such that $\mathcal{N}$ is a natural extension of N.

3 The Archimedean property for t-norms on $([0,1],\leq)$

Denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Then Archimedean t-norms are defined as follows.

Definition 3.1 [13, 14] *Let T be a t-norm on $([0,1],\leq)$. We say that T is Archimedean if*

$$(\forall (x,y) \in]0,1[^2)(\exists n \in \mathbb{N}^*)(x^{(n)_T} < y).$$

An element $x \in]0,1[$ is called a nilpotent element of T if there exists an $n \in \mathbb{N}^*$ such that $x^{(n)_T} = 0$; x is called a zero-divisor of T if there exists an $y \in]0,1[$ such that $T(x,y) = 0$. A t-norm T on $([0,1],\leq)$ is called nilpotent if it is continuous and if each $x \in]0,1[$ is a nilpotent element of T; a t-norm T on $([0,1],\leq)$ is called strict if T is continuous and strictly increasing on $]0,1]^2$.

Theorem 3.1 [13] *Let T be a continuous Archimedean t-norm on $([0,1],\leq)$. Then the following are equivalent:*

(i) T is nilpotent;

(ii) there exists some nilpotent element of $\mathcal{T}$ in $]0,1[$;

(iii) there exists some zero divisor of $\mathcal{T}$ in $]0,1[$;

(iv) T is not strict.

For example, the product t-norm T_P on $([0,1],\leq)$ defined by $T_P(x,y) = xy$, for all x,y in $[0,1]$, is a strict t-norm, and the Łukasiewicz t-norm T_W defined by $T_W(x,y) = \max(0, x+y-1)$, for all x,y in $[0,1]$, is a nilpotent t-norm.

Theorem 3.2 [13] *Let T be a t-norm on $([0,1],\leq)$.*

- *T is continuous, Archimedean and nilpotent if and only if there exists an increasing permutation φ of $([0,1],\leq)$ such that T is the φ-transform of T_W, i.e. $T = \varphi^{-1} \circ T_W \circ (\varphi \times \varphi)$, where $\times$ denotes the product operation* [8].

- *T is continuous, Archimedean and strict if and only if there exists an increasing permutation φ of $([0,1],\leq)$ such that T is the φ-transform of T_P.*

4 The Archimedean property for t-norms on $\mathcal{L}^I$

We extend the definitions from the previous section to $\mathcal{L}^I$. There are several possible extensions of the Archimedean property, which we call Archimedean, weak Archimedean and strong Archimedean property. Throughout this section we will use the sets $L_1^I = \{x \mid x \in L^I \text{ and } x_1 \in]0,1[\}$ and $L_{12}^I = \{x \mid x \in L^I \text{ and } x_1 > 0 \text{ and } x_2 < 1\}$.

Definition 4.1 [5] *Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. We say that*

- *$\mathcal{T}$ is Archimedean if*
$$(\forall (x,y) \in (L_1^I)^2)(\exists n \in \mathbb{N}^*)(x^{(n)_\mathcal{T}} <_{L^I} y);$$

- *$\mathcal{T}$ is strongly Archimedean if*
$$(\forall (x,y) \in (L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\})^2)$$
$$(\exists n \in \mathbb{N}^*)(x^{(n)_\mathcal{T}} <_{L^I} y);$$

- *$\mathcal{T}$ is weakly Archimedean if*
$$(\forall (x,y) \in (L_{12}^I)^2)(\exists n \in \mathbb{N}^*)(x^{(n)_\mathcal{T}} <_{L^I} y).$$

Obviously, if a t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is Archimedean, then it is weakly Archimedean, and if $\mathcal{T}$ is strongly Archimedean, then it is Archimedean. The converse implications do not hold (counterexamples are given in [5]).

In [2, 3] the Archimedean property is defined for t-norms on a general bounded poset. If we apply this definition to $\mathcal{L}^I$, then we obtain the following condition for a t-norm $\mathcal{T}$ on $\mathcal{L}^I$:

$$(\forall (x,y) \in (L^I)^2)((\forall n \in \mathbb{N}^*)(x^{(n)_\mathcal{T}} \geq_{L^I} y) \implies (x = 1_{\mathcal{L}^I} \text{ or } y = 0_{\mathcal{L}^I})). \quad (1)$$

The following theorem shows that the Archimedean property defined using (1) corresponds to the Archimedean property given in Definition 4.1.

Theorem 4.1 [5] *Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$. Then $\mathcal{T}$ is Archimedean (in the sense of Definition 4.1) if and only if $\mathcal{T}$ satisfies* (1).

Now we generalize nilpotency and related concepts to $\mathcal{L}^I$.

Definition 4.2 *Let $\mathcal{T}$ be a t-norm on $\mathcal{L}^I$.*

(i) An element $a \in L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$ is called a nilpotent element of $\mathcal{T}$ if there exists some $n \in \mathbb{N}^*$ such that $a^{(n)_\mathcal{T}} = 0_{\mathcal{L}^I}$.

(ii) An element $a \in L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$ is called a zero divisor of $\mathcal{T}$ if there exists some $b \in L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$ such that $\mathcal{T}(a, b) = 0_{\mathcal{L}^I}$.

Definition 4.3

(i) A t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is called nilpotent if it is continuous and if each $a \in L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$ is a nilpotent element of $\mathcal{T}$.

(ii) A t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is called weakly nilpotent if it is continuous and if each $a \in L^I_{12}$ is a nilpotent element of $\mathcal{T}$.

(iii) A t-norm $\mathcal{T}$ on $\mathcal{L}^I$ is called strict if it is continuous and strictly increasing on $(L^I \setminus \{0_{\mathcal{L}^I}\})^2$.

Theorem 4.2 [5] *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$. Then $\mathcal{T}$ satisfies the Archimedean property if and only if $\mathcal{T}(x, x) <_{L^I} x$, for all $x \in L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$.*

Theorem 4.3 *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$. Then the following are equivalent:*

(N1) $\mathcal{T}$ is nilpotent;

(N2) each $a \in L^I_1$ is a nilpotent element of $\mathcal{T}$.

5 Representations of continuous t-norms on $\mathcal{L}^I$

We first recall two representation theorems which we will need in order to represent some classes of Archimedean t-norms on $\mathcal{L}^I$.

Theorem 5.1 [4] *Consider a continuous mapping $\mathcal{T} : (L^I)^2 \to L^I$. Then $\mathcal{T}$ is a t-norm on $\mathcal{L}^I$ for which*

(T.1) $\mathcal{T}(x, \sup(y, z)) = \sup(\mathcal{T}(x, y), \mathcal{T}(x, z))$, for all x, y, z in L^I, and

(T.2) $\mathcal{T}(D, D) \subseteq D$,

if and only if there exist an element $t \in [0, 1]$, a continuous t-norm T on $([0, 1], \leq)$ and a continuous increasing mapping $\tilde{g} : [0, 1] \to [0, 1]$ such that

(T'.1) $\tilde{g}(T(y_1, z_1)) = \tilde{g}(T(\tilde{g}^{(-1)}(\tilde{g}(y_1)), z_1))$, for all y_1, z_1 in $[0, 1]$,

(T'.2) $\tilde{g}(T(y_1, z_1)) \leq T(\tilde{g}(y_1), z_1)$, for all y_1, z_1 in $[0, 1]$,

(T'.3) $\tilde{g}(T(\tilde{g}^{(-1)}(t), \tilde{g}^{(-1)}(y_1))) \leq \tilde{g}(y_1)$, for all y_1 in $[0, 1]$,

(T'.4) $\tilde{g}(1) = 1$,

(T'.5) for all x, y in $[0, 1]$,

$$\mathcal{T}(x, y) = \Big[T(x_1, y_1), \max\Big(\tilde{g}\Big(T\Big(\tilde{g}^{(-1)}(t), T(\tilde{g}^{(-1)}(x_2), \tilde{g}^{(-1)}(y_2))\Big)\Big), \tilde{g}(T(\tilde{g}^{(-1)}(x_2), y_1)), \tilde{g}(T(\tilde{g}^{(-1)}(y_2), x_1)), T(x_1, y_1)\Big)\Big],$$

where, for all z_1 in $[0, 1]$,

$$\tilde{g}^{(-1)}(z_1) = \sup\{y_1 \mid y_1 \in [0, 1] \text{ and } \tilde{g}(y_1) = z_1\}.$$

Theorem 5.2 [4] *Consider a continuous mapping $\mathcal{T} : (L^I)^2 \to L^I$. Then $\mathcal{T}$ is a t-norm on $\mathcal{L}^I$ such that*

(T".1) $\mathcal{T}$ satisfies the residuation principle, and

(T.2) $\mathcal{T}(D, D) \subseteq D$,

if and only if there exist an element $t \in [0, 1]$, a continuous t-norm T on $([0, 1], \leq)$ and a continuous increasing mapping $\tilde{g} : [0, 1] \to [0, 1]$ such that

(T'.1) $\tilde{g}(T(y_1, z_1)) = \tilde{g}(T(\tilde{g}^{(-1)}(\tilde{g}(y_1)), z_1))$, for all y_1, z_1 in $[0, 1]$,

(T'.2) $\tilde{g}(T(y_1, z_1)) \leq T(\tilde{g}(y_1), z_1)$, for all y_1, z_1 in $[0, 1]$,

(T'.3) $\tilde{g}(T(\tilde{g}^{(-1)}(t), \tilde{g}^{(-1)}(y_1))) \leq \tilde{g}(y_1)$, for all y_1 in $[0, 1]$,

(T'.4) $\tilde{g}(1) = 1$,

(T'.5) for all x, y in $[0, 1]$,

$$\mathcal{T}(x, y) = \Big[T(x_1, y_1), \max\Big(\tilde{g}\Big(T\Big(\tilde{g}^{(-1)}(t), T(\tilde{g}^{(-1)}(x_2), \tilde{g}^{(-1)}(y_2))\Big)\Big), \tilde{g}(T(\tilde{g}^{(-1)}(x_2), y_1)), \tilde{g}(T(\tilde{g}^{(-1)}(y_2), x_1)), T(x_1, y_1)\Big)\Big],$$

where, for all z_1 in $[0, 1]$,

$$\tilde{g}^{(-1)}(z_1) = \sup\{y_1 \mid y_1 \in [0, 1] \text{ and } \tilde{g}(y_1) = z_1\}.$$

In this case the residuum $\mathcal{I}_\mathcal{T}$ of $\mathcal{T}$ is given by, for all x, z in L^I,

$$\begin{aligned}
&\mathcal{I}_\mathcal{T}(x,z)\\
&= \Big[\inf\Big(I_T(x_1,z_1), I_T(\tilde{g}^{(-1)}(x_2), \tilde{g}^{(-1)}(z_2)),\\
&\quad \tilde{g}\Big(I_T\Big(T(\tilde{g}^{(-1)}(t), \tilde{g}^{(-1)}(x_2)), \tilde{g}^{(-1)}(z_2)\Big)\Big),\\
&\quad \tilde{g}(I_T(x_1, \tilde{g}^{(-1)}(z_2)))\Big),\\
&\quad \inf\Big(\tilde{g}\Big(I_T\Big(T(\tilde{g}^{(-1)}(t), \tilde{g}^{(-1)}(x_2)),\\
&\quad \tilde{g}^{(-1)}(z_2)\Big)\Big), \tilde{g}(I_T(x_1, \tilde{g}^{(-1)}(z_2)))\Big)\Big].
\end{aligned}$$

Example 5.1 Let for all $a, b \in [0,1]$ and x, y in L^I, $T(a,b) = \max(0, a+b-1)$,

$$\tilde{g}(a) = \begin{cases} 0, & \text{if } a \leq \frac{1}{2}, \\ 2a-1, & \text{else,} \end{cases}$$

and $\mathcal{T}(x,y) = [\max(0, x_1+y_1-1), \max(0, 2x_1+y_2-2, 2y_1+x_2-2, x_1+y_1-1)]$. Then T, $\tilde{g}$ and $\mathcal{T}$ satisfy the conditions of Theorem 5.1.

Using Theorems 5.1 and 5.2 we obtain the following result.

Theorem 5.3 *If $\mathcal{T}$ is a continuous t-norm on $\mathcal{L}^I$ which satisfies (T.1) and (T.2), then $\mathcal{T}$ satisfies the residuation principle.*

Note that from the fact that $\tilde{g}$ is increasing and continuous, it follows that, for all $z_1 \in [0,1]$,

$$\begin{aligned}
&\tilde{g}^{(-1)}(z_1)\\
&= \sup\{y_1 \mid y_1 \in [0,1] \text{ and } \tilde{g}(y_1) \leq z_1\},\\
&\tilde{g}(\tilde{g}^{(-1)}(z_1))\\
&= \tilde{g}(\sup\{y_1 \mid y_1 \in [0,1] \text{ and } \tilde{g}(y_1) = z_1\}) = z_1.
\end{aligned}$$

Hence, for all x_1, z_1 in $[0,1]$,

$$\begin{aligned}
&\tilde{g}(x_1) \leq z_1\\
&\iff x_1 \in \{y_1 \mid y_1 \in [0,1] \text{ and } \tilde{g}(y_1) \leq z_1\}\\
&\iff x_1 \leq \tilde{g}^{(-1)}(z_1). \qquad (2)
\end{aligned}$$

Theorem 5.4 *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$ which satisfies (T.1) and (T.2). The negation $\mathcal{N}_{\mathcal{I}_\mathcal{T}}$ generated by $\mathcal{I}_\mathcal{T}$ is involutive if and only if $\mathcal{T}$ is pseudo-t-representable and the negation N_{I_T} generated by the residual implication of the representant T of $\mathcal{T}$ is involutive.*

Theorem 5.4 shows that the class of pseudo-t-representable t-norms play an important role if we need t-norms for which the negation generated by the residual implication is involutive. Indeed, in the class of continuous t-norms which satisfy the residuation principle and which are a natural extension of a t-norm on the unit interval, the only t-norms for which the negation generated by their residual implication is involutive, are pseudo-t-representable.

6 Properties of Archimedean and nilpotent t-norms on $\mathcal{L}^I$

Theorem 6.1 *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$ which satisfies (T.1) and (T.2). Then $\mathcal{T}$ is weakly nilpotent if and only if t-norm T involved in the representation of $\mathcal{T}$ according to Theorem 5.1 is nilpotent.*

Theorem 6.2 *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$ which satisfies (T.1) and (T.2). If the t-norm T involved in the representation of $\mathcal{T}$ according to Theorem 5.1 is nilpotent and if $(\mathcal{T}([0,1],[0,1]))_2 < 1$, then $\mathcal{T}$ is nilpotent.*

Theorem 6.3 *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$ which satisfies (T.1) and (T.2). Then $\mathcal{T}$ is weakly Archimedean if and only if t-norm T involved in the representation of $\mathcal{T}$ according to Theorem 5.1 is Archimedean.*

Theorem 6.4 *Let $\mathcal{T}$ be a continuous t-norm on $\mathcal{L}^I$ which satisfies (T.1) and (T.2). If the t-norm T involved in the representation of $\mathcal{T}$ according to Theorem 5.1 is Archimedean and if $(\mathcal{T}([0,1],[0,1]))_2 < 1$, then $\mathcal{T}$ is Archimedean.*

Note that if $\mathcal{T}$ is t-representable, then $\mathcal{T}([0,1],[0,1]) = [0,1]$. So $[0,1]$ is an idempotent element and thus not a nilpotent element of $\mathcal{T}$. Hence, taking into consideration Theorem 6.2, we find that the only subclass of the class of t-norms represented by Theorem 5.1 that does *not* have nilpotent members is the class of t-representable t-norms. Similarly, since for t-representable t-norms $\mathcal{T}$ it holds that $\mathcal{T}([0,1],[0,1]) = [0,1]$, these t-norms are not Archimedean. Theorem 6.4 shows that the t-representable members of the class of t-norms represented by Theorem 5.1 are the only ones that can never be Archimedean.

Theorem 6.5 *Let $\mathcal{T}$ be a continuous weakly Archimedean t-norm on $\mathcal{L}^I$. Then the following are equivalent:*

(i) $\mathcal{T}$ *is weakly nilpotent;*

(ii) *there exists some nilpotent element of* $\mathcal{T}$ *in* L^I_{12}*;*

(iii) *there exists some zero divisor of* $\mathcal{T}$ *in* L^I_{12}.

Theorem 6.6 *Let* $\mathcal{T}$ *be a continuous strongly Archimedean t-norm on* $\mathcal{L}^I$*. Then the following are equivalent:*

(i) $\mathcal{T}$ *is nilpotent;*

(ii) *there exists some nilpotent element of* $\mathcal{T}$ *in* $L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$*;*

(iii) *there exists some zero divisor of* $\mathcal{T}$ *in* $L^I \setminus \{0_{\mathcal{L}^I}, 1_{\mathcal{L}^I}\}$.

7 Representations of continuous Archimedean t-norms on $\mathcal{L}^I$

In this section we will extend Theorem 3.2 to $\mathcal{L}^I$. Since from Theorem 3.1 it follows that the class of continuous Archimedean t-norms on the unit interval can be splitted into two subclasses (the nilpotent t-norms and the strict t-norms), we also split our discussion on the Archimedean t-norms on $\mathcal{L}^I$ in two parts. First we characterize the weakly Archimedean weakly nilpotent t-norms belonging to the class of continuous t-norms $\mathcal{T}$ on $\mathcal{L}^I$ which satisfy the residuation principle and for which $\mathcal{T}(D, D) \subseteq D$. From Theorem 6.1 we know that these are the t-norms on $\mathcal{L}^I$ for which the t-norm T involved in their representation (according to Theorem 5.1) is nilpotent. After that, we characterize the weakly Archimedean t-norms belonging to the previously mentioned class but which are *not* weakly nilpotent. From Theorem 6.1 we know that the t-norm T involved in their representation (see Theorem 5.1) is not nilpotent, and thus strict.

7.1 Archimedean t-norms which are weakly nilpotent

Lemma 7.1 *Let* $\tilde{g} : [0,1] \to [0,1]$ *be a continuous increasing mapping and* φ *be an automorphism of* $([0,1], \leq)$*. Let* T *be the t-norm on* $([0,1], \leq)$ *defined by* $T(x_1, y_1) = \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)$*, for all* x_1, y_1 *in* $[0,1]$*. Then (T'.1) holds if and only if* $\tilde{g}_{|[\tilde{g}^{(-1)}(0),1]}$ *is a bijection from* $[\tilde{g}^{(-1)}(0), 1]$ *to* $[0,1]$*. Furthermore* $\tilde{g}^{(-1)}(z_1) = \tilde{g}^{-1}(z_1)$*, for all* $z_1 \in]0,1]$.

Lemma 7.2 *Let* $\tilde{g} : [0,1] \to [0,1]$ *be a continuous increasing mapping and* φ *be an automorphism of* $([0,1], \leq)$*. Let* T *be the t-norm on* $([0,1], \leq)$ *defined by* $T(x_1, y_1) = \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)$*, for all* x_1, y_1 *in* $[0,1]$*. Then (T'.2) holds if and only if for all* a, b *in* $[0,1]$ *such that* $a > b > \varphi(\tilde{g}^{(-1)}(0))$*,*

$$\frac{\varphi \circ \tilde{g} \circ \varphi^{-1}(a) - \varphi \circ \tilde{g} \circ \varphi^{-1}(b)}{a - b} \geq 1.$$

Lemma 7.3 *Let* $\tilde{g} : [0,1] \to [0,1]$ *be a continuous increasing mapping and* φ *be an automorphism of* $([0,1], \leq)$*. Let* T *be the t-norm on* $([0,1], \leq)$ *defined by* $T(x_1, y_1) = \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)$*, for all* x_1, y_1 *in* $[0,1]$*, and assume that (T'.1) and (T'.2) hold. Then (T'.3) holds if and only if* $\tilde{g}^{(-1)}(0) \leq \tilde{g}\big(\varphi^{(-1)}\big(I_{T_W}\big(\varphi(\tilde{g}^{(-1)}(t)), \varphi(\tilde{g}^{(-1)}(0))\big)\big)\big)$.

Corollary 7.4 *Let* $\tilde{g} : [0,1] \to [0,1]$ *be a continuous increasing mapping and* φ *be an automorphism of* $([0,1], \leq)$*. Let* T *be the t-norm on* $([0,1], \leq)$ *defined by* $T(x_1, y_1) = \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)$*, for all* x_1, y_1 *in* $[0,1]$*, and assume that (T'.1) and (T'.2) hold. Then (T'.3) holds if and only if*

$$t \leq \tilde{g}\big(\varphi^{(-1)}\big(I_{T_W}\big(\varphi(\tilde{g}^{(-1)}(\tilde{g}^{(-1)}(0))), \varphi(\tilde{g}^{(-1)}(0))\big)\big)\big).$$

Theorem 7.5 *Consider a continuous mapping* $\mathcal{T} : (L^I)^2 \to L^I$*. Then* $\mathcal{T}$ *is a t-norm on* $\mathcal{L}^I$ *such that*

(AT.1) $\mathcal{T}$ *is weakly Archimedean,*

(AT.2) $\mathcal{T}$ *is weakly nilpotent,*

(AT.3) $\mathcal{T}$ *satisfies the residuation principle, and*

(AT.4) $\mathcal{T}(D, D) \subseteq D$,

if and only if there exist an element $t \in [0,1]$*, an increasing permutation* φ *of* $([0,1], \leq)$ *and a continuous increasing mapping* $\tilde{g} : [0,1] \to [0,1]$ *such that*

(AT'.1) $\tilde{g}_{|[\tilde{g}^{(-1)}(0),1]}$ *is a bijection from* $[\tilde{g}^{(-1)}(0), 1]$ *to* $[0,1]$*, and* $\tilde{g}(x_1) = 0$*, for all* $x_1 \in [0, \tilde{g}^{(-1)}(0)]$*,*

(AT'.2) *for all* a, b *in* $[0,1]$ *such that* $a > b > \varphi(\tilde{g}^{(-1)}(0))$*,*

$$\frac{\varphi \circ \tilde{g} \circ \varphi^{-1}(a) - \varphi \circ \tilde{g} \circ \varphi^{-1}(b)}{a - b} \geq 1,$$

(AT'.3) $\tilde{g}^{(-1)}(0) \leq \tilde{g}\big(\varphi^{(-1)}\big(I_{T_W}\big(\varphi(\tilde{g}^{(-1)}(t)), \varphi(\tilde{g}^{(-1)}(0))\big)\big)\big)$,

(AT'.4) *for all* x, y *in* $[0,1]$,

$$\begin{aligned}&\mathcal{T}(x,y)\\&= \Big[\varphi^{-1}\big(\max(0,\varphi(x_1)+\varphi(y_1)-1)\big),\\&\max\Big(\tilde{g}\big(\varphi^{-1}\big(\max(0,\varphi(\tilde{g}^{(-1)}(t))+\\&\varphi(\tilde{g}^{(-1)}(x_2))+\varphi(\tilde{g}^{(-1)}(y_2))-2)\big)\big),\\&\tilde{g}\big(\varphi^{-1}\big(\max(0,\varphi(\tilde{g}^{(-1)}(x_2))+\varphi(y_1)-1)\big)\big),\\&\tilde{g}\big(\varphi^{-1}\big(\max(0,\varphi(\tilde{g}^{(-1)}(y_2))+\varphi(x_1)-1)\big)\big),\\&\varphi^{-1}\big(\max(0,\varphi(x_1)+\varphi(y_1)-1)\big)\Big)\Big],\end{aligned}$$

where, for all z_1 *in* $[0,1]$,

$$\tilde{g}^{(-1)}(z_1) = \sup\{y_1 \mid y_1 \in [0,1] \text{ and } \tilde{g}(y_1) = z_1\}.$$

From Theorems 6.1, 6.2, 6.3 and 6.4 it follows that a similar representation theorem holds when we replace (AT.1) and (AT.2) by the conditions: $\mathcal{T}$ is Archimedean and nilpotent, and $\mathcal{T}([0,1],[0,1]) <_{L^I} [0,1]$.

7.2 Archimedean t-norms which are not weakly nilpotent

Lemma 7.6 *Let* $\tilde{g} : [0,1] \to [0,1]$ *be a continuous increasing mapping and* φ *be an automorphism of* $([0,1],\leq)$. *Let* T *be the t-norm on* $([0,1],\leq)$ *defined by* $T(x_1,y_1) = \varphi^{-1}(\varphi(x_1)\varphi(y_1))$, *for all* x_1, y_1 *in* $[0,1]$. *Then (T'.1) holds if and only if* $\tilde{g}_{|[\tilde{g}^{(-1)}(0),1]}$ *is a bijection from* $[\tilde{g}^{(-1)}(0),1]$ *to* $[0,1]$. *Furthermore* $\tilde{g}^{(-1)}(z_1) = \tilde{g}^{-1}(z_1)$, *for all* $z_1 \in \,]0,1]$.

Lemma 7.7 *Let* $\tilde{g} : [0,1] \to [0,1]$ *be a continuous increasing mapping and* φ *be an automorphism of* $([0,1],\leq)$. *Let* T *be the t-norm on* $([0,1],\leq)$ *defined by* $T(x_1,y_1) = \varphi^{-1}(\varphi(x_1)\varphi(y_1))$, *for all* x_1, y_1 *in* $[0,1]$. *Then (T'.2) holds if and only if for all* a, b *in* $[0,1]$ *such that* $a > b > \varphi(\tilde{g}^{(-1)}(0))$,

$$\frac{a}{b} \leq \frac{\varphi\circ\tilde{g}\circ\varphi^{-1}(a)}{\varphi\circ\tilde{g}\circ\varphi^{-1}(b)}.$$

Lemma 7.8 *Let* $\tilde{g} : [0,1] \to [0,1]$ *be a continuous increasing mapping and* φ *be an automorphism of* $([0,1],\leq)$. *Let* T *be the t-norm on* $([0,1],\leq)$ *defined by* $T(x_1,y_1) = \varphi^{-1}(\varphi(x_1)\varphi(y_1))$, *for all* x_1, y_1 *in* $[0,1]$, *and assume that (T'.1) and (T'.2) hold. Then (T'.3) holds if and only if* $\tilde{g}^{(-1)}(0) \leq \tilde{g}\big(\varphi^{(-1)}\big(I_{T_P}\big(\varphi(\tilde{g}^{(-1)}(t)),\varphi(\tilde{g}^{(-1)}(0))\big)\big)\big)$.

Theorem 7.9 *Consider a continuous mapping* $\mathcal{T} : (L^I)^2 \to L^I$. *Then* $\mathcal{T}$ *is a t-norm on* $\mathcal{L}^I$ *such that*

(AT.1) $\mathcal{T}$ *is weakly Archimedean,*

(AT.2) $\mathcal{T}$ *is not weakly nilpotent,*

(AT.3) $\mathcal{T}$ *satisfies the residuation principle, and*

(AT.4) $\mathcal{T}(D,D) \subseteq D$,

if and only if there exist an element $t \in [0,1]$, *a continuous t-norm* T *on* $([0,1],\leq)$ *and a continuous increasing mapping* $\tilde{g} : [0,1] \to [0,1]$ *such that*

(AT".1) $\tilde{g}_{|[\tilde{g}^{(-1)}(0),1]}$ *is a bijection from* $[\tilde{g}^{(-1)}(0),1]$ *to* $[0,1]$, *and* $\tilde{g}(x_1) = 0$, *for all* $x_1 \in [0,\tilde{g}^{(-1)}(0)]$,

(AT".2) *for all* a, b *in* $[0,1]$ *such that* $a > b > \varphi(\tilde{g}^{(-1)}(0))$,

$$\frac{\varphi\circ\tilde{g}\circ\varphi^{-1}(a)}{\varphi\circ\tilde{g}\circ\varphi^{-1}(b)} \geq \frac{a}{b},$$

(AT".3) $\tilde{g}^{(-1)}(0) \leq \tilde{g}\big(\varphi^{(-1)}\big(I_{T_P}\big(\varphi(\tilde{g}^{(-1)}(t)), \varphi(\tilde{g}^{(-1)}(0))\big)\big)\big)$,

(AT".4) *for all* x, y *in* $[0,1]$,

$$\begin{aligned}&\mathcal{T}(x,y)\\&= \Big[\varphi^{-1}\big(\varphi(x_1)\varphi(y_1)\big),\\&\max\Big(\tilde{g}\big(\varphi^{-1}\big(\varphi(\tilde{g}^{(-1)}(t))\\&\varphi(\tilde{g}^{(-1)}(x_2))\varphi(\tilde{g}^{(-1)}(y_2))\big)\big),\\&\tilde{g}\big(\varphi^{-1}\big(\varphi(\tilde{g}^{(-1)}(x_2))\varphi(y_1)\big)\big),\\&\tilde{g}\big(\varphi^{-1}\big(\varphi(\tilde{g}^{(-1)}(y_2))\varphi(x_1)\big)\big),\\&\varphi^{-1}\big(\varphi(x_1)\varphi(y_1)\big)\Big)\Big],\end{aligned}$$

where, for all z_1 *in* $[0,1]$,

$$\tilde{g}^{(-1)}(z_1) = \sup\{y_1 \mid y_1 \in [0,1] \text{ and } \tilde{g}(y_1) = z_1\}.$$

From Theorems 6.1, 6.2, 6.3 and 6.4 it follows that a similar representation theorem holds when we replace (AT.1) and (AT.2) by the conditions: $\mathcal{T}$ is Archimedean and not nilpotent, and $\mathcal{T}([0,1],[0,1]) <_{L^I} [0,1]$. It remains an open problem whether all continuous Archimedean t-norms on $\mathcal{L}^I$ which are not (weakly) nilpotent are strict.

8 Conclusion

In this paper we presented some properties of t-norms on $\mathcal{L}^I$ which are Archimedean and nilpotent, or which satisfy some related properties such as the weak Archimedean property and weak nilpotency. We investigated some properties of the class of t-norms $\mathcal{T}$ on $\mathcal{L}^I$ which are continuous, satisfy the residuation principle and $\mathcal{T}(D,D) \subseteq D$. We gave necessary and sufficient conditions for the members of this class so that they satisfy the Archimedean property, nilpotency, or any of the weak variants of these properties. We also gave a representation of the members of the above mentioned class of t-norms on $\mathcal{L}^I$ which are either (weakly) Archimedean and (weakly) nilpotent, or (weakly) Archimedean but *not* (weakly) nilpotent. Thus we obtained a full characterization of the class of continuous t-norms $\mathcal{T}$ on $\mathcal{L}^I$ which satisfy the residuation principle, $\mathcal{T}(D,D) \subseteq D$, and which are (weakly) Archimedean.

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