Fuzzy Class Theory: Some Advanced Topics

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Abstract

The goal of this paper is to push forward the development of the apparatus of the Fuzzy Class theory. We concentrate on three areas: strengthening the universal quantifier, formalizing the idea that 'similar' fuzzy sets fulfill their properties to 'similar' degrees, and embedding of classical crisp theories into Fuzzy Class theory.

Keywords: Fuzzy Class Theory, fuzzy relations, graded properties, quantifiers, logic MTL

1 Introduction

Fuzzy Class Theory has the aim to axiomatize the notion of fuzzy set. A brief overview of FCT can be found in Appendix A, where also all necessary defined predicates of the theory, freely used in the following sections, are introduced. For a detailed account of the theory we refer the reader to the original paper [1] or a freely available primer [2].

The goal of this paper is to advance the apparatus of FCT in three ways. First, we introduce a class of quantifiers strengthening the universal quantifier, the so-called super-universal quantifiers (note that the classical theory of generalized quantifiers studies just 'weaker' variants of universal quantifier, like 'many' or 'almost all', as in the classical logic the universal quantifier is the only super-universal quantifier). The usual motivations for studies of generalized quantifiers are linguistical, but our motivation is purely 'logical' and 'mathematical'. Recall that for any multiset M of terms we can show:

$$(\forall x)\varphi \to \bigwedge_{t \in M} \varphi(t)$$

The most notable super-universal quantifier is the so-called multiplicative quantifier. It is, roughly speaking, a quantifier $\mathcal{Q}$ such that for each multiset M of terms holds:

$$(\mathcal{Q}x)\varphi \to \mathop{\&}_{t \in M} \varphi(t)$$

Observe that $\forall$ clearly does not satisfy the above formula, but the quantifier $\triangle\forall$ does, and so does some other ones (e.g., the one assigning to each formula a largest idempotent bellow all instances of φ).

Then, in Section 3, we study the so-called weak congruence property, generalizing the work of Bělohlávek ([4, Lemma 4.8]). There the author shows that if two sets are 'equal' in the high degree, they share the same properties, formally written as:

$$X \approx^n Y \to (\varphi(X) \leftrightarrow \varphi(Y))$$

The degree n depends on the complexity of formula (property) φ. The generalization we offer is based on the crucial observation that in FCT we can define the following two different notions of set equality (both used in fuzzy set theory papers):

$$A \approx B \equiv_{\mathrm{df}} (\forall x)(x \in A \leftrightarrow x \in B)$$

$$A \approxeq B \equiv_{\mathrm{df}} (A \subseteq B) \,\&\, (B \subseteq A)$$

By a finer syntactic analysis of the form of formula φ we manage to prove theorem of the form:

$$X \approxeq^{m,n} Y \to (\varphi(X) \leftrightarrow \varphi(Y))$$

note that [4, Lemma 4.8] would give us just:

$$X \approx^{m+n} Y \to (\varphi(X) \leftrightarrow \varphi(Y))$$

We can show the following implication and non-validity of the converse one:

$$A \approx^{m+n} B \to A \approxeq^{m,n} B$$

Thus our result is indeed a non-trivial generalization.

Finally, in Section 4, we study how to embed classical crisp theories into FCT. The results in this section are not surprising, they are just thorough formalizations of the fact that we can translate classical theories formalized in the classical type theory to FCT since FCT 'contains' classical simple type theory. The first step of this formalization, the case of first-order theories, was done already in the original paper [1].

2 Super-universal quantifiers

A unary quantifier is a fuzzy class of the second order and a binary quantifier is a binary fuzzy relation of the second order. We shall use the usual conventions to write $\mathcal{Q}X$ instead of $X \in \mathcal{Q}$. Here we restrict ourselves to unary quantifiers, for simplicity, the more general study of binary ones will be a subject of some future papers. Let us now mention how the quantifiers, as we defined them, relate to the 'usual' ones. We set:

$$(\mathcal{Q}x)\varphi \equiv_{\mathrm{df}} \mathcal{Q}\{x \mid \varphi\} \tag{1}$$

Other way round, if $\mathcal{Q}$ is a 'usual' quantifier of (an extension of) FCT, we can write $\mathcal{Q}X \equiv_{\mathrm{df}} (\mathcal{Q}x)(x \in X)$. In particular we have fuzzy class of fuzzy classes $\forall$ defined as $\forall X \equiv_{\mathrm{df}} (\forall x)(x \in X)$. Now we single out natural properties for "super-universal" quantifiers. Let us, for now, work on the meta-level and understand the following conditions as new axioms or rules added to FCT.

1. from $(\varphi \to \psi)$ infer $(\mathcal{Q}x)\varphi \to (\mathcal{Q}x)\psi$
2. from φ infer $(\mathcal{Q}x)\varphi$
3. $(\mathcal{Q}x)\varphi \to \varphi(t)$

Let us rewrite these conditions to the "class" form.

1. $(X \sqsubseteq Y) \to (\mathcal{Q}X \to \mathcal{Q}Y)$
2. $\mathcal{Q}\mathrm{V}$
3. $\mathcal{Q} \subseteq \forall$

Recall the meaning of the super-universal quantifier $(\mathcal{Q}x)\varphi$ can be roughly described as: "closeness of the class $\{x \mid \varphi\}$ to the universal class V". The first condition expresses monotonicity of this concept ("bigger" classes are closer to V than the "smaller" ones), the second one expresses the fact that V is fully close to itself. The final one is the condition of super-universality: in classical logic it would lead to $\mathcal{Q} = \forall$ and so this condition is always omitted when interpreting universal-like quantifiers like 'many', 'almost all', etc. Finally observe that if we would replace condition 1. with:

$$(X \subseteq Y) \to (\mathcal{Q}X \to \mathcal{Q}Y)$$

then by setting $X = \mathrm{V}$ we would get (by condition 2.) that $\forall Y \to \mathcal{Q}Y$; thus together with condition 3. we would get $\forall = \mathcal{Q}$. Other feasible option would be to replace condition 1. by:

$$(\mathcal{Q}x)(\varphi \to \psi) \to ((\mathcal{Q}x)\varphi \to (\mathcal{Q}x)\psi)$$

This condition (together with condition 2.) implies the original condition 1. and so it would constitute an interesting subclass of super-universal quantifiers, we left the study of this class to some upcoming paper.

Let us step down from the meta-level of FCT and start building a theory of super-universal quantifiers *inside* FCT. We define a (third order) class $\mathbb{SUQ}$:

$$\begin{aligned}\mathbb{SUQ}^{m,l,u}(\mathcal{Q}) \equiv_{\mathrm{df}}\\ ((\forall X, Y)((X \sqsubseteq Y) \to (\mathcal{Q}X \to \mathcal{Q}Y)))^m\\ \mathbin{\&} (\mathcal{Q}\mathrm{V})^l \mathbin{\&} \mathcal{Q} \subseteq^u \forall\end{aligned}$$

Let us recall the usual conventions:

$$\mathbb{SUQ}^m(\mathcal{Q}) \equiv_{\mathrm{df}} \mathbb{SUQ}^{m,m,m}(\mathcal{Q}) \quad \mathbb{SUQ}(\mathcal{Q}) \equiv_{\mathrm{df}} \mathbb{SUQ}^1(\mathcal{Q})$$

Lemma 2.1 FCT *proves:*

$$\mathrm{Crisp}(\mathbb{I}) \mathbin{\&} \mathbb{I} \neq \emptyset \mathbin{\&} (\forall \mathcal{Q} \in \mathbb{I})(\mathbb{SUQ}(\mathcal{Q})) \to \mathbb{SUQ}(\textstyle\bigcap \mathbb{I})$$
$$\mathrm{Crisp}(\mathbb{I}) \mathbin{\&} \mathbb{I} \neq \emptyset \mathbin{\&} (\forall \mathcal{Q} \in \mathbb{I})(\mathbb{SUQ}(\mathcal{Q})) \to \mathbb{SUQ}(\textstyle\bigcup \mathbb{I})$$

Obviously, in FCT we can prove, $\triangle\mathbb{SUQ}(\forall)$, $\triangle\mathbb{SUQ}(\mathrm{Ker}(\forall))$, and $\mathrm{Ker}(\forall) = \{V\}$. In fact we can prove more:

Lemma 2.2 FCT *proves:*

$$\forall = \bigcup\{\mathcal{Q} \mid \mathbb{SUQ}(\mathcal{Q})\} = \bigcup\{\mathcal{Q} \mid \triangle\mathbb{SUQ}(\mathcal{Q})\}$$
$$\mathrm{Ker}(\forall) = \bigcap\{\mathcal{Q} \mid \mathbb{SUQ}(\mathcal{Q})\} = \bigcap\{\mathcal{Q} \mid \triangle\mathbb{SUQ}(\mathcal{Q})\}$$

I.e. the universal quantifier is the largest (fully) super-universal quantifier and its kernel is the least (fully) super-universal quantifier.

Let us define other notable quantifiers, first the largest idempotent super-universal quantifier:

$$\mathcal{S} = \bigcup\{\mathcal{Q} \mid \triangle\mathbb{SUQ}(\mathcal{Q}) \mathbin{\&} \mathcal{Q} = \mathcal{Q} \cap \mathcal{Q}\} \tag{2}$$

It can be easily shown that this quantifier corresponds to the storage quantifier introduced in [8]. Clearly, as $\mathrm{Ker}(\forall) = \mathrm{Ker}(\forall) \cap \mathrm{Ker}(\forall)$ we can use Lemma 2.1 to get:

Lemma 2.3 FCT *proves* $\mathbb{SUQ}(\mathcal{S})$

Semantically speaking we get:

Lemma 2.4 *Let* **A** *be an* MTL$_\triangle$*-algebra and* **M** *an* **A***-model of* FCT. *Then* $\|\mathcal{S}X\|$ *is the largest idempotent smaller than* $\|a \in X\|$ *for each* $a \in M$.

Lemma 2.5 *Let* φ *be a formula,* x *a variable,* M *a multiset of terms. Then in* FCT *we can prove:*

$$(\mathcal{S}x)\varphi \to \mathop{\&}_{t\in M} \varphi(t) \tag{3}$$

$$(\mathcal{S}x)\varphi \to (\mathcal{S}x)\varphi \,\&\, \varphi(t) \tag{4}$$

Now we define another 'multiplicative' quantifier.

$$\mathcal{M} = \bigcup\{\mathcal{Q} \mid \triangle\mathbb{SUQ}(\mathcal{Q}) \,\&\, (\forall X, x)$$
$$\triangle\Big(\big((x \in X \to \mathcal{Q}X) \to \mathcal{Q}X\big) \vee \big(\mathcal{Q}X \leftrightarrow \mathcal{Q}X \,\&\, \mathcal{Q}X\big)\Big)\}$$

As clearly $\mathcal{S}$ fulfills the defining condition, we can use Lemma 2.1 to get:

Lemma 2.6 FCT *proves* $\mathbb{SUQ}(\mathcal{M})$.

Now we show the semantical interpretation of quantifier. Let **A** be an MTL$_\triangle$-algebra. We define for a set $B \subseteq A$ a 'multiplication' of B as:

$$\mathop{\&} B = \inf_{n\in\mathbb{N}, a_1,\dots,a_n\in B} a_1 \& \dots \& a_n\,.$$

Notice that we do not assume that $a_i \neq a_j$ (for $i \neq j$) in the above definition, i.e., $\mathop{\&} B$ is the infimum of products $a_1 \& \dots \& a_n$ for all finite sequences $a_1, \dots, a_n$ of elements from B. Due to the space restriction we have to skip the formal proofs of the following two lemmata, but the second one is a clear semantical consequence of the first one.

Lemma 2.7 *Let* **A** *be an* MTL$_\triangle$*-algebra and* **M** *an* **A***-model of* FCT. *Then*

$$\|\mathcal{M}X\| = \mathop{\&}\{\|a \in X\| \mid a \in M\}$$

Lemma 2.8 *Let* φ *be a formula,* x *a variable, and* M *a finite multiset of terms. Then in* FCT *we can prove:*

$$(\mathcal{S}x)\varphi \longrightarrow (\mathcal{M}x)\varphi \longrightarrow \mathop{\&}_{t\in M} \varphi(t) \tag{5}$$

Thus we can say that both $\mathcal{M}$ and $\mathcal{S}$ are 'multiplicative' quantifiers. As the following lemma shows the first implication cannot be reversed, i.e., $\mathcal{M}$ is 'better' multiplicative quantifier as it gives us a better bound to $\mathop{\&}_{t\in M} \varphi(t)$. In fact its semantics shows that $\mathcal{M}$ gives the best bound.

Lemma 2.9 *In* FCT *we cannot prove*

$$(\mathcal{M}x)\varphi \to (\mathcal{S}x)\varphi \tag{6}$$

$$(\mathcal{M}x)\varphi \to (\mathcal{M}x)\varphi \,\&\, \varphi(t) \tag{7}$$

Proof: We just show the second claim, as the first one is its simple consequence (taking in account the previous lemma and Lemma 2.5). We just construct an MTL$_\triangle$-chain **A** which is a counterexample to the claim that

$$\mathop{\&} B \leq \Big(\mathop{\&} B\Big) \,\&\, a\,,$$

where $B \subseteq A$ and $a \in B$. The construction of a particular model of FCT which refutes $(\mathcal{M}x)\varphi \to (\mathcal{M}x)\varphi \,\&\, \varphi(t)$ is then simple.

Let $A' = \mathbb{Z}^- \times \mathbb{Z}^-$, where $\mathbb{Z}^-$ is the set of non-positive integers. The set A' is ordered lexicographically, i.e., $\langle k, r\rangle \leq \langle m, s\rangle$ iff $k < m$ or $k = m$ and $r \leq s$. The operations are defined as follows:

$$\langle k, r\rangle \,\&\, \langle m, s\rangle = \langle k + m, r + s\rangle\,,$$

$$\langle k, r\rangle \to \langle m, s\rangle = \begin{cases} \langle 0, 0\rangle \text{ if } \langle k, r\rangle \leq \langle m, s\rangle\,, \\ \langle m - k, \min\{0, s - r\}\rangle \text{ otherwise.} \end{cases}$$

Then $\mathbf{A}' = (A', \&, \to, \leq, \langle 0, 0\rangle)$ is an integral commutative residuated chain and the ordinal sum $\mathbf{A} = \mathbf{2} \oplus \mathbf{A}'$ is an MTL-chain (where **2** is the two-element Boolean algebra). The MTL-chain **A** can be clearly viewed as an MTL$_\triangle$-chain.

Now, let $B = \{\langle 0, -1\rangle\} \subseteq A$. Then

$$\mathop{\&} B = \inf_{n\in\mathbb{N}} \langle 0, -1\rangle^n = \langle -1, 0\rangle\,.$$

However $\langle -1, 0\rangle \,\&\, \langle 0, -1\rangle = \langle -1, -1\rangle < \langle -1, 0\rangle$. Q.E.D.

3 Weak congruence property

We define a notion of positive and negative occurrence of a subformula ψ in a formula φ. Let us denote it by $o^\varphi_\psi \in \{+1, -1\}$. This notation is rather relaxed as the subformula ψ can appear in φ several times. Let us understand ψ is this notion as a subformula together with its fixed occurrence rather than just a subformula. We define it by induction, let φ is a predicate formula in the language of FCT, without the propositional connective $\leftrightarrow$ and ψ its subformula.

- $o^\psi_\psi = +1$
- $o^{\varphi\circ\chi}_\psi = o^{\chi\circ\varphi}_\psi = o^\varphi_\psi$ for $\circ \in \{\&, \wedge, \vee\}$ and any formula χ
- $o^{\triangle\varphi}_\psi = o^{(\forall x)\varphi}_\psi = o^{(\exists x)\varphi}_\psi = o^\varphi_\psi$
- $o^{\chi\to\varphi}_\psi = o^\varphi_\psi$ for any formula χ
- $o^{\neg\varphi}_\psi = o^{\varphi\to\chi}_\psi = -o^\varphi_\psi$ for any formula χ

We say that an occurrence of subformula ψ of φ is $\triangle$-bound if it lies in the scope of $\triangle$ connective.

Lemma 3.1 *Let $\varphi, \psi, \chi, \delta$ be formulas, δ a subformula of χ, let us pick one occurrence of δ. Let us define formulas $\chi(\delta!\varphi)$ and $\chi(\delta!\psi)$ as formulas resulting from χ by replacing the chosen occurrence of δ by φ (ψ respectively). Finally, let $x_1, \dots x_n$ be free variables of φ and ψ which become bound in $\chi(\delta!\varphi)$ or $\chi(\delta!\psi)$*

In first-order $\mathrm{MTL}_\triangle$ *we can prove:*

- $(\forall x_1, \dots x_n)\triangle(\varphi \to \psi) \to (\chi(\delta!\varphi) \to \chi(\delta!\psi))$ *if that occurrence is positive*
- $(\forall x_1, \dots x_n)\triangle(\psi \to \varphi) \to (\chi(\delta!\varphi) \to \chi(\delta!\psi))$ *if that occurrence is negative*

Furthermore if that occurrence is not $\triangle$-bound we can prove:

- $(\forall x_1, \dots x_n)(\varphi \to \psi) \to (\chi(\delta!\varphi) \to \chi(\delta!\psi))$ *if the chosen occurrence is positive*
- $(\forall x_1, \dots x_n)(\psi \to \varphi) \to (\chi(\delta!\varphi) \to \chi(\delta!\psi))$ *if the chosen occurrence is negative*

The proof is almost straightforward but very tedious, so we skip it.

Let us denote by $\chi(\delta : \varphi)$ the formula resulting from χ by replacing *all* occurrences of its subformula δ by φ. Finally by $+^\varphi_\psi$ we denote the number of positive occurrences of ψ in φ, if any of those occurrences is $\triangle$-bound define $+^\varphi_\psi = \triangle$ (analogously we define $-^\varphi_\psi$ for negative occurrences). Recall the convention that $\varphi^\triangle = \triangle\varphi$.

Corollary 3.2 *Let $\varphi, \psi, \chi, \delta$ be formulas, δ a subformula of χ. Let $x_1, \dots x_n$ be free variables in φ and ψ which become bound in $\chi(\delta : \varphi)$ or $\chi(\delta : \psi)$. In first-order* $\mathrm{MTL}_\triangle$ *we can prove:*

$$((\forall x_1, \dots x_n)(\varphi \to \psi))^{+^\chi_\delta} \,\&\, ((\forall x_1, \dots x_n)(\psi \to \varphi))^{-^\chi_\delta} \to (\chi(\delta : \varphi) \to \chi(\delta : \psi))$$

Corollary 3.3 *Let φ, ψ be sentences and χ, δ be formulas, δ a subformula of χ. In first-order* $\mathrm{MTL}_\triangle$ *we can prove:*

$$(\varphi \to \psi)^{+^\chi_\delta} \,\&\, (\psi \to \varphi)^{-^\chi_\delta} \to (\chi(\delta : \varphi) \to \chi(\delta : \psi))$$

As a corollary we obtain the main theorem of this section. But first we define:

Definition 3.4 *Let φ be a formula of* FCT *and A a free variable of any order of φ. We say that A is purely extensional in φ if it occurs in φ just in subformulas of the form $X_i \in A$ for some object variables X_i. Then by $+^\varphi_A$ we denote the number of positive occurrences of formulas of the form $X_i \in A$ in φ (analogously we define $-^\varphi_A$), again we set it as $\triangle$ if any of those occurrences if $\triangle$-bound.*

Finally by $\varphi(A : X)$ we denote the formula resulting from φ by replacing all occurrences of class term A by term X of the same order as A.

Theorem 3.5 *Let φ be a formula of* FCT *and A a purely extensional free variable of φ. Then we can prove:*

$$X \approxeq^{+^\varphi_A, -^\varphi_A} Y \to (\varphi(A : X) \to \varphi(A : Y))$$

Observe that if no occurrence of A is $\triangle$-bound then $n = +^\varphi_A + -^\varphi_A$ is the number of occurrences of A in φ and as FCT trivially proves

$$X \approx^n Y \to X \approxeq^{+^\varphi_A, -^\varphi_A} Y,$$

we get:

$$X \approx^n Y \to (\varphi(A : X) \to \varphi(A : Y))$$

This form was proven already in [4, Lemma 4.8.] (in a slightly different framework). Notice that our result is indeed stronger as:

Lemma 3.6 FCT *does not prove:*

$$X \approxeq^{+^\varphi_A, -^\varphi_A} \to X \approx^n Y$$

Example 3.7 *In* FCT, *we can prove*

- $R \subseteq S \to (\mathrm{Refl}(R) \to \mathrm{Refl}(S))$
- $R \approxeq S \to (\mathrm{Sym}(R) \to \mathrm{Sym}(S))$
- $R \approxeq^{1,2} S \to (\mathrm{Trans}(R) \to \mathrm{Trans}(S))$

Compare with [4, Lemma 4.8], which would give us only:

- $R \approx S \to (\mathrm{Refl}(R) \to \mathrm{Refl}(S))$
- $R \approx^2 S \to (\mathrm{Sym}(R) \to \mathrm{Sym}(S))$
- $R \approx^3 S \to (\mathrm{Trans}(R) \to \mathrm{Trans}(S))$

4 Containment of crisp theories

We define a notion of hereditary crisp set by induction as:

$$\mathrm{HCrisp}^{(2)}(X) \equiv_{\mathrm{df}} \mathrm{Crisp}(X)$$

$$\mathrm{HCrisp}^{(n+1)}(X) \equiv_{\mathrm{df}} \mathrm{Crisp}(X) \,\&\, (\forall Z \in X)(\mathrm{HCrisp}^{(n)}(Z))$$

Sometimes we write just HCrisp(X) when the type is known. Notice that $\{X \mid \mathrm{HCrisp}^{(n+1)}(X)\}$ is indeed a fuzzy class of the $(n+1)$-st order.

Since FCT contains the classical theory of classes (for classes which are crisp), we can introduce all concepts which are definable in classical class theory (i.e., in classical simple type theory, or Boolean higher-order logic). The only thing we need to do is adding new predicate and functional symbols of the appropriate sorts and axioms saying that all predicates and functions appearing in the theory are crisp.

Definition 4.1 *Let* $\mathbf{\Gamma}$ *be a higher-order language and* T *a* $\mathbf{\Gamma}$*-theory. We define the language* FCT($\mathbf{\Gamma}$) *as the language of* FCT *extended by* $\mathbf{\Gamma}$*. We define translation* $'$ *of* $\mathbf{\Gamma}$*-formulas and terms into* FCT($\mathbf{\Gamma}$)*-formulas and terms in the following way:*

- $t' = t$ *for each term* t
- $\varphi' = \varphi$ *for each atomic formula* φ
- $(\varphi \diamond \psi)' = \varphi' \diamond \psi'$ *for* $\diamond \in \{\&, \to, \wedge\}$
- $((\forall X)(\varphi))' = (\forall X)(\mathrm{HCrisp}(X) \to \varphi')$
- $((\exists X)(\varphi))' = (\exists X)(\mathrm{HCrisp}(X) \mathbin{\&} \varphi')$

We define the theory FCT(T) *in the language* FCT($\mathbf{\Gamma}$) *as the theory with the following axioms:*

- *The axioms of* FCT
- *The axioms of the form* φ' *for each* $\varphi \in T$
- HCrisp(Q) *for each predicate symbol* $Q \in \mathbf{\Gamma}$

The proof of the following lemma is almost straightforward.

Lemma 4.2 *Let* $\mathbf{\Gamma}$ *be a higher-order language,* T *a* $\mathbf{\Gamma}$*-theory,* $\mathbf{L}$ *an* MTL$_\triangle$*-algebra. If* $\mathbf{M}$ *is an* $\mathbf{L}$*-model of* FCT(T)*, then the classical model* $\mathbf{M}^c$ *in the language* $\mathbf{\Gamma}$ *with the domain* M *and* $S_{\mathbf{M}^c} = S_{\mathbf{M}}$ *for each* $S \in \mathbf{\Gamma}$*, is a model (in the sense of classical Henkin style higher-order logic) of the theory* T*. Vice versa, for each classical model* $\mathbf{M}$ *of* T *there is an* $\mathbf{L}$*-model* $\mathbf{N}$ *of* FCT(T) *such that* $\mathbf{N}^c$ *is isomorphic to* $\mathbf{M}$*.*

Therefore, $T \vdash \varphi$ *iff* FCT(T) $\vdash \varphi'$*, for any* $\mathbf{\Gamma}$*-formula* φ *(where the first provability is in classical Henkin style higher-order logic and the second in* FCT*).*

Example 4.3 *Let* τ *be a constant for a class of classes and* T *the theory with the axioms:*

- HCrisp(τ)
- $\emptyset \in \tau$
- $\mathrm{V} \in \tau$
- $(\forall \mathcal{X})(\mathrm{HCrisp}(\mathcal{X}) \to (\mathcal{X} \subseteq \tau \to \bigcup \mathcal{X} \in \tau))$
- $(\forall X, Y)(\mathrm{HCrisp}(X) \mathbin{\&} \mathrm{HCrisp}(Y) \to (X \in \tau \mathbin{\&} Y \in \tau \to X \cap Y \in \tau))$

Then in each $\mathbf{L}$*-model of the theory* T*, the constant* τ *is represented by a classical topology on the universe of objects.*

A Fuzzy Class Theory

In this section, we present an overview of Fuzzy Class Theory (FCT) in order to provide the reader with the necessary background. Note that this is only a brief introduction to the most basic concepts of FCT with the aim to keep the paper self-contained. Readers who want to understand all details should not expect to find all necessary material in this paper. Instead, they are referred to the freely available primer [2].

In the first paper [1], FCT was based on the logic ŁΠ [6]. In this paper, we use the logic MTL$_\triangle$; obviously all definitions and basic results of [1] can be transferred from ŁΠ to MTL$_\triangle$. For an introduction to MTL$_\triangle$, see [5]; a more detailed treatment on first-order MTL$_\triangle$ with crisp equality can be found in [7]).

Definition A.1 *Fuzzy Class Theory* (over MTL$_\triangle$) is a theory over multi-sorted first-order logic MTL$_\triangle$ with crisp equality. We have sorts for individuals of the zeroth order (i.e., atomic objects) denoted by lowercase variables $a, b, c, x, y, z, \ldots$; individuals of the first order (i.e., fuzzy classes) denoted by uppercase variables $A, B, X, Y, \ldots$; individuals of the second order (i.e., fuzzy classes of fuzzy classes) denoted by calligraphic variables $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}, \ldots$; the individuals of the n-th order denoted by $X^{(n)}$. Individuals $\xi_1, \ldots, \xi_k$ of each order can form k-tuples (for any $k \geq 0$), denoted by $\langle \xi_1, \ldots, \xi_k \rangle$; tuples are governed by the usual axioms known from classical mathematics (e.g., tuples equal if and only if their respective constituents equal). Furthermore, for each variable x of any order n and for each formula φ there is a class term $\{x \mid \varphi\}$ of order $n+1$.

Besides the logical predicate of identity, the only primitive predicate is the membership predicate $\in$ between successive sorts (i.e., between individuals of the n-th order and individuals of the $(n+1)$-st order, for any n). The axioms for $\in$ are the following (for variables of all orders):

($\in$1) $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$, for each formula φ (comprehension axioms)

(∈2) $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \rightarrow A = B$ (extensionality)

Moreover, we use all axioms and deduction rules of first-order $\mathrm{MTL}_\triangle$. Theorems, theories, proofs, etc., can be defined completely analogously.

Observation A.2 Since the language of FCT is the same at each order, defined symbols of any order can be shifted to all higher orders as well. Since furthermore the axioms of FCT have the same form at each order, all theorems on FCT-definable notions are preserved by uniform upward order-shifts.

Convention A.3 For better readability, let us make the following conventions:

- We use $(\forall x \in A)\varphi$, $(\exists x \in A)\varphi$ as abbreviations for $(\forall x)(x \in A \rightarrow \varphi)$ and $(\exists x)(x \in A \,\&\, \varphi)$, respectively.
- We use the notation $\{x \in A \mid \varphi\}$ as abbreviation for $\{x \mid x \in A \,\&\, \varphi\}$.
- We use $\{\langle x_1, \ldots, x_k\rangle \mid \varphi\}$ as abbreviation for $\{x \mid (\exists x_1)\ldots(\exists x_k)(x = \langle x_1, \ldots, x_k\rangle \,\&\, \varphi)\}$.
- The formulae $\varphi \,\&\, \ldots \,\&\, \varphi$ (n times) are abbreviated φ^n; instead of $(x \in A)^n$, we can write $x \in^n A$ (analogously for other predicates).
- Furthermore, $x \notin A$ is shorthand for $\neg(x \in A)$; analogously for other binary predicates.
- We use Ax and $Rx_1 \ldots x_n$ synonymously for $x \in A$ and $\langle x_1, \ldots, x_n\rangle \in R$, respectively.
- A chain of implications of the form $\varphi_1 \rightarrow \varphi_2$, $\varphi_2 \rightarrow \varphi_3, \ldots, \varphi_{n-1} \rightarrow \varphi_n$ will for short be written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \cdots \longrightarrow \varphi_n$; analogously for the equivalence connective.

Definition A.4 In FCT, we define the following elementary fuzzy set operations:

$$\begin{array}{rcl}
\emptyset & =_{\mathrm{df}} & \{x \mid 0\} \\
\mathrm{V} & =_{\mathrm{df}} & \{x \mid 1\} \\
\{a\} & =_{\mathrm{df}} & \{x \mid x = a\} \\
\mathrm{Ker}(A) & =_{\mathrm{df}} & \{x \mid \triangle(x \in A)\} \\
A \cap B & =_{\mathrm{df}} & \{x \mid x \in A \,\&\, x \in B\} \\
A \sqcap B & =_{\mathrm{df}} & \{x \mid x \in A \wedge x \in B\} \\
A \sqcup B & =_{\mathrm{df}} & \{x \mid x \in A \vee x \in B\}
\end{array}$$

Definition A.5 In FCT, we define basic properties of fuzzy relations as follows:

$$\begin{array}{rcl}
\mathrm{Refl}(R) & \equiv_{\mathrm{df}} & (\forall x)Rxx \\
\mathrm{Sym}(R) & \equiv_{\mathrm{df}} & (\forall x, y)(Rxy \rightarrow Ryx) \\
\mathrm{Trans}(R) & \equiv_{\mathrm{df}} & (\forall x, y, z)(Rxy \,\&\, Ryz \rightarrow Rxz)
\end{array}$$

Definition A.6 Further we define in FCT the following elementary relations between fuzzy sets:

$$\begin{array}{rcl}
\mathrm{Crisp}(A) & \equiv_{\mathrm{df}} & (\forall x)\triangle(x \in A \vee x \notin A) \\
A \subseteq B & \equiv_{\mathrm{df}} & (\forall x)(x \in A \rightarrow x \in B) \\
A \approx B & \equiv_{\mathrm{df}} & (\forall x)(x \in A \leftrightarrow x \in B) \\
A \approxeq^{m,n} B & \equiv_{\mathrm{df}} & (A \subseteq^n B) \,\&\, (B \subseteq^m A)
\end{array}$$

Let us recall the standard conventions:

$$A \approxeq^m B \equiv_{\mathrm{df}} A \approxeq^{m,m} B \qquad A \approxeq B \equiv_{\mathrm{df}} A \approxeq^1 B$$

Definition A.7 The union and intersection of a class of classes are functions defined as

$$\begin{array}{rcl}
\bigcup \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\exists A \in \mathcal{A})(x \in A)\} \\
\bigcap \mathcal{A} & =_{\mathrm{df}} & \{x \mid (\forall A \in \mathcal{A})(x \in A)\}
\end{array}$$

The models of FCT are systems (closed under definable operations) of fuzzy sets (and fuzzy relations) of all orders over some crisp universe U, where the membership functions of fuzzy subsets take values in some $\mathrm{MTL}_\triangle$-chain. Intended models are those which contain *all* fuzzy subsets and fuzzy relations over U (of all orders); we call such models *full*. Models in which moreover the $\mathrm{MTL}_\triangle$-chain is standard (i.e., given by a left-continuous t-norm on the unit interval $[0, 1]$) correspond to Zadeh's [10] original notion of fuzzy set; therefore we call them *Zadeh models*.

FCT is sound with respect to Zadeh (or full) models; thus, whatever we prove in FCT is true about real-valued (or $\mathbf{L}$-valued for any $\mathrm{MTL}_\triangle$-chain $\mathbf{L}$) fuzzy sets and relations. Although the theory of Zadeh models is not *completely* axiomatizable, the axiomatic system of FCT approximates it very well: the comprehension axioms ensure the existence of (at least) all fuzzy sets which are *definable* (by a formula of FCT), and the axioms of extensionality ensure that fuzzy sets are determined by their membership functions. This axiomatization is sufficient for almost all practical purposes; it can be characterized as *simple type theory over fuzzy logic* (cf. [9]) or *Henkin-style higher-order fuzzy logic*.

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