

# Constructions of aggregation operators that preserve ordering of the data

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## Abstract

We address the issue of identifying various classes of aggregation operators from empirical data, which also preserves the ordering of the outputs. It is argued that the ordering of the outputs is more important than the numerical values, however the usual data fitting methods are only concerned with fitting the values. We will formulate preservation of the ordering problem as a standard mathematical programming problem, solved by standard numerical methods.

**Keywords:** Aggregation operators, preference ordering, decision making.

## 1 Introduction

Construction of aggregation operators from empirical data is very useful in practice, when a specific aggregation operator has to be chosen for a specific application. This work was pioneered in [25], where the authors introduced  $\gamma$ -operators, a convex combination of triangular norms and conorms. Identification of the weights of arithmetic means and OWA operators from the data was treated in [4, 10, 20, 21], identification of the coefficients of fuzzy measures for Choquet integral based aggregation was treated in [4, 13, 14], and identification of additive generators of  $t$ -norms,  $t$ -conorms, uninorms and nullnorms was treated in [4, 5].

In all mentioned studies, the choice of parameters was driven by how well an aggregation function predicted the observed input-output pairs, the data set  $\{(\vec{x}_k, y_k)\}, k = 1, \dots, K$ . The goodness of fit was measured by using the least squares criterion, or the least absolute deviation criterion. In the first case the problem was set up as a standard quadratic programming problem, and in the second case as a linear programming problem.

However, in [16] it was argued that fitting the numerical outputs is not as important as preserving the ordering of the outputs. The empirical data usually comes from human subjective evaluation, and people do not reliably express their preference on a numerical scale. In contrast, people are very good at ranking the alternatives. Therefore, the authors of [16] argued that fitting methods should aim at preserving the order of empirical output values. They showed that various methods of fitting the numerical values do not preserve this ordering. However, a solution which does preserve the ordering of the outputs has never been spelled out.

The aim of this contribution is to show that preservation of outputs ordering can be achieved by a very simple technique of adding  $K - 1$  linear inequalities to the least squares and least absolute deviation problems. Furthermore, in many cases, that cover all major families of aggregation operators, the structure of the resulting quadratic and linear programming problems does not change, which allows one to apply standard numerical optimization methods. We formulate the resulting mathematical programming problems explicitly in these cases. Finally, we present a new formulation of the aggregation operator identification problem, in which a weighted combination of the numerical fitness and ordering preservation criteria is optimized, as well as its solution methods. This problem is particularly useful when the data is contaminated by noise.

## 2 Fitting aggregation operators

Various methods of fitting parameters of an aggregation operator to the data are available [4, 10, 14, 16, 20, 21, 25]. Given a data set  $\{(\vec{x}_k, y_k)\}, k = 1, \dots, K$ , and a class of aggregation operators parameterized by a vector  $\vec{w}$ ,  $f(\vec{x}; \vec{w})$ , the mentioned methods minimize the least squares (LS) criterion

$$\sum_{k=1}^K (f(\vec{x}_k; \vec{w}) - y_k)^2$$

with respect to the parameters  $\vec{w}$ , subject to the conditions that  $f$  is an aggregation operator, i.e., it verifies at least the conditions of monotonicity and  $f(\vec{0}; \vec{w}) = 0, f(\vec{1}; \vec{w}) = 1$ . Of course, other conditions like symmetry, idempotency, existence of a neutral element, annihilator, etc., can also be added.

An alternative is to use the least absolute deviation criterion (LDA) [7], i.e., minimize

$$\sum_{k=1}^K |f(\vec{x}_k; \vec{w}) - y_k|$$

with respect to the weights, subject to the same conditions. The use of LDA is less sensitive to outliers, and importantly, in many cases it allows one to set up an equivalent linear programming problem, which is easily solved by the standard simplex algorithm, even if the number of weights is very large (e.g., when identifying a fuzzy measure).

If  $f$  depends on the weights  $\vec{w}$  linearly, which is the case when  $f$  is an arithmetic mean, OWA, Choquet integral,  $\gamma$ -operator and some other aggregation operators, then minimization of LS criterion becomes a standard quadratic programming problem (QP), and minimization of LDA criterion becomes a linear programming problem (LP) after introducing auxiliary variables. Furthermore, linearization methods allow one to set up QP or LP problems for quasi-arithmetic means, generalized OWA and generalized Choquet integrals, see [4–6].

We note that all mentioned methods approximate, not interpolate, the empirical values  $y_k$  (except some special cases). Empirical data comes with errors, and it is pointless to fit it exactly. If the data were interpolated, then of course the order of the outputs would be preserved automatically. Thus our goal is to ensure that the order is preserved during the solution to LS or LDA problems.

### 3 Preservation of ordering

Without loss of generality, we assume that the outputs are ordered as  $y_1 \leq y_2 \leq \dots \leq y_K$  (the data can always be re-ordered in this way). The condition for order preservation is

$$f(\vec{x}_i; \vec{w}) \leq f(\vec{x}_j; \vec{w}), \text{ for all } i < j. \quad (1)$$

Because the data set is ordered, this condition is implied by a simpler condition

$$f(\vec{x}_i; \vec{w}) \leq f(\vec{x}_{i+1}; \vec{w}), \text{ for all } i = 1, \dots, K-1. \quad (2)$$

In general, this is a system of  $K-1$  nonlinear inequalities, which is very hard to solve. But in many interesting cases  $f$  depends on  $\vec{w}$  linearly, and in this case

we obtain a system of linear inequalities, which does not change the structure of the LS or LDA problem.

Let  $f(\vec{x}; \vec{w}) = \langle \vec{g}(\vec{x}), \vec{w} \rangle = \sum_{i=1}^n w_i g_i(\vec{x})$ ,  $\vec{g}$  being some basis functions. For example  $g_i(\vec{x}) = x_i$  for the arithmetic means,  $g_i(\vec{x}) = x_{(i)}$  for an OWA operator<sup>1</sup>. Then the LS problem becomes

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K (\langle \vec{g}(\vec{x}_k), \vec{w} \rangle - y_k)^2, \quad (3) \\ \text{s.t.} \quad & \langle \vec{g}(\vec{x}_{k+1}), \vec{w} \rangle - \langle \vec{g}(\vec{x}_k), \vec{w} \rangle \geq 0, \\ & k = 1, \dots, K-1, \\ & \text{other (linear) conditions on } \vec{w}. \end{aligned}$$

Problem (3) is QP, which differs from the original LS problem only by additional  $K-1$  linear constraints. Consequently, standard methods of solution of QPs can be employed. In the case of LDA problem, the situation is similar, we have an additional set of linear constraints, and if LDA was converted to LP, then the additional constraints are directly transferred to the LP problem. Thus by using the auxiliary variables  $r_k^+, r_k^- \geq 0$ , such that  $r_k^+ + r_k^- = |\langle \vec{g}(\vec{x}_k), \vec{w} \rangle - y_k|$ , and  $r_k^+ - r_k^- = \langle \vec{g}(\vec{x}_k), \vec{w} \rangle - y_k$ , we have an LP

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K r_k^+ + r_k^-, \quad (4) \\ \text{s.t.} \quad & \langle \vec{g}(\vec{x}_k), \vec{w} \rangle - r_k^+ + r_k^- = y_k, \\ & k = 1, \dots, K, \\ & \langle \vec{g}(\vec{x}_{k+1}), \vec{w} \rangle - \langle \vec{g}(\vec{x}_k), \vec{w} \rangle \geq 0, \\ & k = 1, \dots, K-1, \\ & r_k^+, r_k^- \geq 0, k = 1, \dots, K, \\ & \text{other (linear) conditions on } \vec{w}. \end{aligned}$$

The methods of solution to problems (3) and (4) are well known, see [7, 18].

## 4 Special cases

In this section we will present explicit problem formulations for a number of popular families of aggregation operators, in the case of LS fitting (Problem (3)). The case of LDA (4) is treated very similarly.

### 4.1 Arithmetic means and OWA

Since  $g_i(\vec{x}) = x_i$  and we have constraints  $w_i \geq 0$ ,  $\sum_{i=1}^n w_i = 1$ ,  $n$  is the dimension of the input vector  $\vec{x}$ , Problem (3) translates into

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K (\langle \vec{x}_k, \vec{w} \rangle - y_k)^2, \quad (5) \\ \text{s.t.} \quad & \langle \vec{x}_{k+1} - \vec{x}_k, \vec{w} \rangle \geq 0, \\ & k = 1, \dots, K-1, \\ & \sum w_i = 1, w_i \geq 0. \end{aligned}$$

<sup>1</sup>As usual,  $x_{(i)}$  denotes the  $i$ -th largest component of  $\vec{x}$ .

For OWA operators [23], let  $\vec{z}_k = (x_{k,(1)}, x_{k,(2)}, \dots, x_{k,(n)})$  be the vector obtained from  $\vec{x}_k$  by arranging its components in non-increasing order. Then LS problem translates into

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K (\langle \vec{z}_k, \vec{w} \rangle - y_k)^2, \\ \text{s.t.} \quad & \langle \vec{z}_{k+1} - \vec{z}_k, \vec{w} \rangle \geq 0, \\ & k = 1, \dots, K-1, \\ & \sum w_i = 1, w_i \geq 0. \end{aligned} \quad (6)$$

For OWA operators, a frequent additional requirement is preservation of a given measure of orness [10, 23], which translates into an additional linear constraint

$$\langle \vec{a}, \vec{w} \rangle = \alpha,$$

where  $a_i = \frac{n-i}{n-1}$ , and  $0 \leq \alpha \leq 1$  is specified by the user.

## 4.2 Quasi-arithmetic means and generalized OWA

Let  $h : [0, 1] \rightarrow [-\infty, \infty]$ , be a given continuous strictly monotone function. A quasi-arithmetic mean is the function

$$f(\vec{x}; \vec{w}) = h^{-1} \left( \sum_{i=1}^n w_i h(x_i) \right).$$

This class includes geometric, harmonic, quadratic means, power means and many others.

A generalized OWA operator (also called Ordered Weighted Quasi-Arithmetic means (OWQA) in [8]) is the function

$$f(\vec{x}; \vec{w}) = h^{-1} \left( \sum_{i=1}^n w_i h(z_i) \right),$$

where  $\vec{z} = (x_{(1)}, \dots, x_{(n)})$ .

Fitting the weights of quasi-arithmetic means and generalized OWA is done by linearizing inputs and outputs, i.e., solving

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K (\langle h(\vec{x}_k), \vec{w} \rangle - h(y_k))^2, \\ \text{s.t.} \quad & \langle h(\vec{x}_{k+1}) - h(\vec{x}_k), \vec{w} \rangle \geq 0, \\ & k = 1, \dots, K-1, \\ & \sum w_i = 1, w_i \geq 0, \end{aligned} \quad (7)$$

where  $h(\vec{x}) = (h(x_1), \dots, h(x_n))$ .

## 4.3 Choquet integrals

Let the set  $N$  be  $N = \{1, 2, \dots, n\}$ . A fuzzy measure is a set function  $v : 2^N \rightarrow [0, 1]$  which is monotonic (i.e.  $v(S) \leq v(T)$  whenever  $S \subseteq T$ ) and satisfies  $v(\emptyset) = 0, v(N) = 1$ . The discrete Choquet integral is defined

with respect to a fuzzy measure, and can be written as [12]

$$C_v(\vec{x}) = \sum_{i=1}^n [x_{(i)} - x_{(i-1)}] v(H_i), \quad (8)$$

where  $x_{(0)} = 0$  by convention, and  $H_i = \{(i), \dots, (n)\}$  is the subset of indices of  $n - i + 1$  largest components of  $\vec{x}$ . Note that here  $x_{(i)}$  denotes the  $i$ -th smallest component of  $\vec{x}$ . A fuzzy measure has  $2^n$  parameters, two of which are fixed:  $v(\emptyset) = 0, v(N) = 1$ .

Let us represent Choquet integral as a dot product  $\langle \vec{g}(\vec{x}), \vec{v} \rangle$ , where  $\vec{v} \in [0, 1]^{2^n}$  is the vector of coefficients of the fuzzy measure. It is convenient to use the index  $j = 0, \dots, 2^n - 1$  whose binary representation corresponds to the characteristic vector of the set  $J \subseteq N$ ,  $\vec{c} \in \{0, 1\}^n$  defined by  $c_{n-i+1} = 1$  if  $i \in J$  and 0 otherwise. For example, let  $n = 5$ ; for  $j = 101$  (binary),  $\vec{c} = (0, 0, 1, 0, 1)$  and  $v_j = v(\{1, 3\})$ . We shall use letters  $K, J$ , etc., to denote subsets that correspond to indices  $k, j$ , etc.

Let us define the basis functions  $g_j, j = 0, \dots, 2^n - 1$  as  $g_j(\vec{x}) = \max(0, \min_{i \in J} \{x_i\} - \max_{i \in N \setminus J} \{x_i\})$ , where  $J \subseteq N$  whose characteristic vector corresponds to the binary representation of  $j$ . Then  $C_v(\vec{x}) = \langle \vec{g}(\vec{x}), \vec{v} \rangle$ .

Now, identification of the coefficients of the fuzzy measure  $v$  becomes a QP,

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K (\langle \vec{g}(\vec{x}_k), \vec{v} \rangle - y_k)^2, \\ \text{s.t.} \quad & \langle \vec{g}(\vec{x}_{k+1}) - \vec{g}(\vec{x}_k), \vec{v} \rangle \geq 0, \\ & k = 1, \dots, K-1, \\ & v_0 = 0, v_{2^n-1} = 1, \\ & v_k - v_j \geq 0 \text{ for all } k, j \text{ such that } J \subset K. \end{aligned} \quad (9)$$

This is a large scale (even for moderate  $n$ ) QP with a sparse matrix of constraints, and there are numerical methods that exploit such a sparse structure [11]. However when using LDA criterion, it becomes an LP problem with a sparse matrix, which can be solved efficiently for a very large number of parameters.

It is well known that for additive fuzzy measures Choquet integrals become arithmetic means, and for symmetric fuzzy measures, they become OWA operators. To reduce the complexity of the problem, Grabisch introduced  $k$ -additive fuzzy measures [12], in which only combinations of at most  $k$  indices allow for interactions of variables. The condition of  $k$ -additivity is translated into a set of additional linear equality constraints on the coefficients of fuzzy measure, and these are readily included into QP or LP. Furthermore, the same applies to various other indices, such as Shapley index and its generalizations [12].

By applying a non-linear invertible transformation  $h$

to the components of  $\vec{x}$ , one obtains a generalized Choquet integral [8, 22]

$$C_{vh}(\vec{x}) = h^{-1} \left( \sum_{i=1}^n [h(x_{(i)}) - h(x_{(i-1)})] v(H_i) \right). \quad (10)$$

The coefficients of the fuzzy measure can be fitted by linearizing, similarly to the case of quasi-arithmetic means and generalized OWA operators (by applying  $h$  to  $\vec{x}_k$  and to  $y_k$  in (9)).

#### 4.4 $\gamma$ -operators

We consider a generalized version of  $\gamma$ -operators by Zimmermann [24, 25], which are called T-S operators in [19], defined as a linear or log-linear combination of a t-norm  $T$  and t-conorm  $S$ ,

$$f(\vec{x}) = \gamma T(\vec{x}) + (1 - \gamma) S(\vec{x}),$$

$\gamma \in [0, 1]$ , or

$$f(\vec{x}) = T(\vec{x})^\gamma S(\vec{x})^{1-\gamma}.$$

A more general version is obtained by using an invertible strictly monotone function  $h$

$$f(\vec{x}) = h^{-1} (\gamma h(T(\vec{x})) + (1 - \gamma) h(S(\vec{x}))).$$

The linear and log-linear combinations are the special cases corresponding to  $h = Id$  and  $h = \log$ .

We consider the general case with an arbitrary strictly monotone function  $h$ . For a fixed pair of t-norm and t-conorm, the goal is to identify an unknown parameter  $\gamma$  that fits the data best. This is done by using  $w_1 = \gamma$ ,  $w_2 = 1 - \gamma$  and writing

$$h^{-1}(f(\vec{x})) = h^{-1}(w_1 h(T(\vec{x})) + w_2 h(S(\vec{x}))) =$$

$$h^{-1}(\langle \vec{g}(\vec{x}), \vec{w} \rangle),$$

$\vec{g} = (h(T), h(S))$ ,  $w_1 + w_2 = 1$ ,  $w_1, w_2 \geq 0$ , and noticing that after linearization we obtain a QP problem again. In this specific case we get

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K (w_1 h(T(\vec{x}_k)) + w_2 h(S(\vec{x}_k)) - h(y_k))^2, \\ \text{s.t.} \quad & w_1 (h(T(\vec{x}_{k+1})) - h(T(\vec{x}_k))) + \\ & w_2 (h(S(\vec{x}_{k+1})) - h(S(\vec{x}_k))) \geq 0, \\ & k = 1, \dots, K - 1, \\ & w_1 + w_2 = 1, w_1, w_2 \geq 0. \end{aligned} \quad (11)$$

#### 4.5 General aggregation operators

A method of fitting general aggregation operators using tensor-product splines was proposed in [2, 3]. This

method is based on representing  $f$  by means of a linear combination  $f(\vec{x}) = \langle \vec{B}(\vec{x}), \vec{c} \rangle$ , where functions  $\vec{B} = \vec{B}_1(x_1) \vec{B}_2(x_2) \dots \vec{B}_n(x_n)$  are tensor products of univariate B-splines with respect to each variable [9]. For explicit formulae we refer to [2–4]. For our discussion we only need to note that monotone tensor product splines are linear combinations of some well defined basis functions, and that the conditions of monotonicity translate into a system of linear inequalities on spline coefficients. Thus fitting tensor-product splines to the data involves a QP problem (or LP problem if we use LDA criterion).

Preservation of the ordering of the outputs, as we know, is an additional system of linear inequalities, that does not change the structure of QP or LP, thus the methods presented in [3, 4] can be applied with only a minor modification.

#### 4.6 Fitting additive generators of t-norms/t-conorms

A method of fitting continuous Archimedean t-norms/t-conorms to empirical data was presented in [3–5]. It relies on fitting the additive generators, as pointwise convergence of a sequence of additive generators is equivalent to uniform convergence of the corresponding t-norms/t-conorms [15, 17]. In this method an additive generator is represented via a monotone spline

$$h(t) = \langle \vec{B}(t), \vec{c} \rangle,$$

where  $\vec{B}(t)$  is a vector of B-splines, and  $\vec{c}$  is the vector of spline coefficients. The conditions of monotonicity of  $h$  are imposed through linear restrictions on spline coefficients, and the additional conditions  $h(0) = 1$ ,  $h(0.5) = 1$  also translate into linear equality constraints<sup>2</sup>.

Since Archimedean t-norms satisfy

$$T(\vec{x}) = h^{(-1)} \left( \sum_{i=1}^n h(x_i) \right),$$

( $h^{(-1)}$  denotes pseudoinverse), after linearization the least squares criterion translates into

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K \left( \sum_{i=1}^n \langle \vec{B}(x_{kn}), \vec{c} \rangle - \langle \vec{B}(y_k), \vec{c} \rangle \right)^2 \\ \text{s.t.} \quad & \text{linear restrictions on } \vec{c}. \end{aligned} \quad (12)$$

By rearranging the terms of the sum we get

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K \left( \langle \left[ \sum_{i=1}^n \vec{B}(x_{kn}) - \vec{B}(y_k) \right], \vec{c} \rangle \right)^2 \\ \text{s.t.} \quad & \text{linear restrictions on } \vec{c}. \end{aligned} \quad (13)$$

<sup>2</sup>The issue of asymptotic behaviour near  $t = 0$  for strict Archimedean t-norms is solved by using “well-founded” generators [5, 15]

Next we add preservation of outputs ordering conditions, to obtain the following QP (note that the sign of inequality has changed because  $h$  is decreasing)

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K \left( \left\langle \left[ \sum_{i=1}^n \vec{B}(x_{kn}) - \vec{B}(y_k) \right], \vec{c} \right\rangle \right)^2 \\ \text{s.t.} \quad & \left\langle \left[ \sum_{i=1}^n \vec{B}(x_{k+1,n}) - \sum_{i=1}^n \vec{B}(x_{k,n}) \right], \vec{c} \right\rangle \leq 0, \\ & \text{linear restrictions on } \vec{c}. \end{aligned} \quad (14)$$

For t-conorms we obtain a similar problem by duality. Furthermore, a very similar procedure works for representable uninorms and nullnorms. An additional issue here is proper dealing with the neutral element/annihilator, and its identification from the data. It was resolved in [3–5], and fortunately, preservation of output ordering does not change the structure of those methods either, it only adds  $K - 1$  additional linear constraints.

## 5 Balancing ordering and fitting numerical values

In the preceding discussion we specified preservation of the output orderings as hard constraints, enforced at the expense of fitting to the data. Since empirical data has an associated noise, it may be impossible to satisfy all these constraints by using a specified class of aggregation operators. The system of constraints is said to be inconsistent. In this section we discuss modifications of the above mentioned optimization problems, that allow one to soften ordering constraints and balance them against fitting numerical data.

Consider a revised version of Problem (3).

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K (\langle \vec{g}(\vec{x}_k), \vec{w} \rangle - y_k)^2 + \\ & P \sum_{k=1}^{K-1} \max\{\langle \vec{g}(\vec{x}_k) - \vec{g}(\vec{x}_{k+1}), \vec{w} \rangle, 0\}, \\ & \text{other linear conditions on } \vec{w}. \end{aligned} \quad (15)$$

Here  $P$  is the penalty parameter, for small values of  $P$  we emphasize fitting the numerical data, while for large values of  $P$  we emphasize preservation of ordering. Of course, the second sum may not be zero at the optimum, which indicates inconsistency of constraints.

Unfortunately, Problem (15) is no longer a quadratic programming problem, it is a nonsmooth but convex optimization problem, and there are efficient numerical methods of its solution, e.g., [1]. However, for LDA criterion, we can preserve the structure of LP, namely we convert Problem (4), by using auxiliary variables

$r_k^+, r_k^-$  and  $q_k = \max\{\langle \vec{g}(\vec{x}_k) - \vec{g}(\vec{x}_{k+1}), \vec{w} \rangle, 0\}$  into

$$\begin{aligned} \text{Minimize} \quad & \sum_{k=1}^K r_k^+ + r_k^- + P \sum_{k=1}^{K-1} q_k \\ \text{s.t.} \quad & \langle \vec{g}(\vec{x}_k), \vec{w} \rangle - r_k^+ + r_k^- = y_k, \\ & k = 1, \dots, K, \\ & q_k + \langle \vec{g}(\vec{x}_{k+1}) - \vec{g}(\vec{x}_k), \vec{w} \rangle \geq 0, \\ & k = 1, \dots, K-1, \\ & q_k, r_k^+, r_k^- \geq 0, k = 1, \dots, K, \\ & \text{other linear conditions on } \vec{w}. \end{aligned} \quad (16)$$

The special cases we considered in Section 4 allow such an LP formulation, and we note that the dimension of the problem increases only by  $K$ , which is not excessively large.

## 6 Conclusion

Fitting various families of aggregation operators to empirical data is useful for identifying the most suitable aggregation operator in practical applications. It was argued in [16] that preserving the ordering of the output values is more important than fitting actual numerical values, as human subjects — sources of such data, are more consistent with ordering the alternatives than numerical values. The authors of [16] examined a number of classes of aggregation operators and established that no class of that group preserved the ordering of outputs. However they did not set up a suitable optimization problem which would force fitted aggregation operators to preserve the outputs ordering.

In this contribution we developed a general mathematical programming problem which includes preservation of ordering as hard and soft constraints. In the first case, unless the constraints are inconsistent, our method guarantees that the ordering of the outputs is preserved. In the second case, the ordering requirement is balanced against fitting the numerical values, and a solution that minimizes discrepancy of the orderings is delivered.

We have presented specific problem formulations applicable to several broad and popular classes of aggregation operators, and in all cases kept the structure of the optimization problem, either a quadratic or linear programming problem. The advantage is that standard efficient methods of solution are applied.

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