Valverde-Style Representation Results in a Graded Framework

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Abstract

This paper generalizes the well-known representations of fuzzy preorders and similarities according to Valverde to the graded framework of Fuzzy Class Theory (FCT). The results demonstrate that FCT is a powerful tool and that new results and interesting constructions can be obtained by considering fuzzy relations in the graded framework of FCT.

Keywords: Fuzzy Class Theory, Fuzzy Relations, Graded Properties, MTL Logic.

1 Introduction

This paper aims at generalizing two of the most important and influential theorems in the theory of fuzzy relations—Valverde's representation theorems for fuzzy preorders and similarities [23]:

Consider a fuzzy relation $R\colon U \times U \to [0,1]$. R is a fuzzy preorder with respect to some left-continuous triangular norm $*$ if and only if there exists a family $(f_i)_{i\in I}$ of score functions $U \to [0,1]$ such that R can be represented as (for all $x, y \in U$)

$$R(x,y) = \inf_{i\in I}(f_i(x) \Rightarrow f_i(y)),$$

where $\Rightarrow$ is the residual implication of $*$. Moreover, R is a similarity with respect to $*$ if and only if there exists a family $(f_i)_{i\in I}$ of score functions $U \to [0,1]$ such that R can be represented as (for all $x, y \in U$)

$$R(x,y) = \inf_{i\in I}(f_i(x) \Leftrightarrow f_i(y)),$$

where $\Leftrightarrow$ is the residual bi-implication of $*$. These two results hold for fuzzy preorders and similarities, respectively, but clearly they do not provide us with any insight if the relations fail to fulfill those requirements.

In the early 1990's, Gottwald has introduced what he calls *graded properties of fuzzy relations* [17–19], a framework in which it is possible to deal with partial—*graded*—fulfillment of properties like reflexivity, transitivity, etc. In this framework, it is not only possible to define properties in a graded way, but also to generalize theorems on fuzzy relations in the sense that an assertion about a certain sub-class of fuzzy relations holds to a degree that depends on the degree to which the relations fulfill the necessary properties. Even though these ideas seem obvious and meaningful, Gottwald's approach unfortunately found only little resonance (exceptions are, for instance, [6,21]), mainly because it is not a full-fledged axiomatic framework and is not strictly separating syntax from semantics. For this reason, proofs are complicated and difficult.

With the advent of Fuzzy Class Theory (FCT) [3], a formal axiomatic framework is available in which it is just natural to consider properties of fuzzy relations in a graded manner. Notions are inspired by (and derived from) the corresponding notions of classical mathematics [4]; the syntax of FCT is close to the syntax of classical mathematical theories and the proofs in FCT resemble the proofs of the corresponding classical theorems. Therefore, it is technically easier to handle graded properties of fuzzy relations than in Gottwald's previous works and it is possible to access deeper results than in Gottwald's framework.

This paper is devoted to this advancement, concentrating on Valverde's famous representation theorems for fuzzy preorders and similarities. In the tradition of Cantor [9], Valverde uses score functions to represent relations. As these score functions map to the unit interval, they can also be considered as fuzzy sets, which facilitates a reformulation of these results in FCT.

The paper is organized as follows. After some preliminaries concerning FCT in Section 2, we introduce graded properties in Section 3. Section 4 is devoted to the generalization of Valverde's representation theorem for fuzzy preorders, while Section 5 deals with the corresponding results for similarities. Note that this paper is an excerpt of a larger manuscript that has been submitted for publication [2]. For background information and proof details, readers are referred to this upcoming article.

2 Preliminaries

We aim this paper at researchers in the theory and applications of fuzzy relations to attract their interest in graded theories of fuzzy relations. In the traditional theory of fuzzy relations, it is not usual to separate formal syntax from semantics as it is the case in FCT. So it may be difficult for some readers who are new to FCT to follow the results. Therefore, we would like to provide the readers basically with a dictionary that improves understanding of the results in this paper and that demonstrates how the results would translate to the traditional language of fuzzy relations. For a more formal introduction to FCT, readers are referred to the appendix of this paper and the freely available primer [5].

FCT strictly distinguishes between its syntax and semantics; that is, we distinguish between a formal syntax of formulae and the fuzzy relations modeling them. This feature has two important consequences: (i) To make this distinction clear, the objects of the formal theory are called *fuzzy classes* and not *fuzzy sets*. The name *fuzzy set* is reserved for membership functions in the *models* of the theory (see Appendix). Nevertheless, the theorems of FCT about fuzzy classes are always valid for fuzzy *sets* and fuzzy relations. Thus, whenever we speak of classes, the reader can always safely substitute usual fuzzy sets for "classes". (ii) FCT screens off direct references to truth values; truth degrees belong to the *semantics* of FCT, rather than to its syntax. Thus, there are *no variables for truth degrees* in the language of FCT. The degree to which an element x belongs to a fuzzy class A is expressed simply by the atomic formula $x \in A$ (which can alternatively be written in a more traditional way as Ax).

The algebraic structure of truth degrees in the semantics of FCT is that of $\mathrm{MTL}_\triangle$-chains [13,20].[1] If the domain of truth values is the unit interval $[0,1]$, $\mathrm{MTL}_\triangle$-chains are characterized as algebras

$$([0,1], *, \Rightarrow, \min, \max, 0, 1, \triangle),$$

where $*$ is a left-continuous t-norm, $\Rightarrow$ is its residual implication, and $\triangle$ is a unary operation defined as

$$\triangle x = \begin{cases} 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

[1]Note that even though in the first paper [3] FCT was based on the logic ŁΠ [14], the logic $\mathrm{MTL}_\triangle$ is sufficient for our present needs.

This means that we can translate the results to the language of fuzzy relations in the following way, where we may specify an arbitrary universe of discourse U and a left-continuous t-norm $*$ (with the residuum $\Rightarrow$):

FCT	**Fuzzy relations**
object variable x	element $x \in U$
(fuzzy) class A	fuzzy set $A \in \mathcal{F}(U)$
2nd-order fuzzy class $\mathcal{A}$	fuzzy set $\mathcal{A} \in \mathcal{F}(\mathcal{F}(U))$
unary predicate	fuzzy subset of U, $\mathcal{F}(U)$, etc.
n-ary predicate	n-ary f. rel. on U^n, $(\mathcal{F}(U))^n$, etc.
strong conjunction &	left-continuous t-norm $*$
implication $\rightarrow$	residual implication $\Rightarrow$
weak conjunction $\wedge$	minimum
weak disjunction $\vee$	maximum
negation $\neg$	the function $\neg x = (x \Rightarrow 0)$
equivalence $\leftrightarrow$	bi-residuum: $\min(x \Rightarrow y, y \Rightarrow x)$
universal quantifier $\forall$	infimum
existential quantifier $\exists$	supremum
predicate $=$	crisp identity
predicate $\in$	evaluation of membership function
class term $\{x \mid \varphi(x)\}$	f. set def. as $Ax = \varphi(x)$, for $x \in U$

For details on the syntax of FCT and defined notions see Appendix A and [3,5].

Let us now shortly consider two examples. For instance, the truth degree of $A \subseteq B$, defined by the formula $(\forall x)(x \in A \rightarrow x \in B)$ in FCT (see Definition A.3) is in an $\mathrm{MTL}_\triangle$-chain computed as

$$\inf_{x \in U} (Ax \Rightarrow Bx),$$

which is a well-known concept of fuzzy inclusion (see [1,6,7,18] and many more). The degree of reflexivity $\mathrm{Refl}(R)$, defined as $(\forall x)Rxx$ in Section 3, is nothing else but

$$\inf_{x \in U} Rxx.$$

We shall now proceed to how the theorems in the following sections should be read in a graded way (although they do not necessarily look graded at first glance). In traditional (fuzzy) logic, a theorem is read as follows:

> *If* a (non-graded) assumption is true
> (i.e., fully true, since non-graded),
> *then* a (non-graded) conclusion is (fully) true.

If an implication is provable in FCT, by soundness, it always holds to degree 1. Now take into account that, in all $\mathrm{MTL}_\triangle$-chains (comprising all standard $\mathrm{MTL}_\triangle$-chains), the following correspondence holds:

$$(x \Rightarrow y) = 1 \text{ if and only if } x \leq y.$$

So an implication that we can prove in FCT can be read as follows:

> *The more* a (graded) assumption is true
> (even if partially),
> *the more* a (graded) conclusion is true
> (i.e., at least as true as the assumption).

In other words, the truth degree of an assumption is a lower bound for the truth degree of the conclusion in provable implications.

Remark: To motivate and illustrate the results in this paper, we will use several examples. In order to make them compact and readable, we will, *in examples,* deviate from our principle to keep formulae separate from their semantics. Instead of mentioning models over some logics, we will simply say that we use some standard logic, for instance, standard Łukasiewicz logic (standing for the standard $\mathrm{MTL}_\triangle$-chain induced by the Łukasiewicz t-norm; analogously for other logics). In examples, we shall furthermore not distinguish between predicate symbols and the fuzzy sets or relations that model them. Instead of saying that a certain model of a fuzzy predicate R fulfills reflexivity to a degree of 0.8, we will simply write $\mathrm{Refl}(R) = 0.8$. This is not the cleanest way of writing it, but it is short and expressive, and it should always be clear to the reader what is meant.

3 Basic Graded Properties of Fuzzy Relations

As an important prerequisite, we first define graded variants of well-known properties of fuzzy relations in the framework of FCT.

Definition 3.1 In FCT, we define basic properties of fuzzy relations as follows:

$$\begin{aligned} \mathrm{Refl}(R) &\equiv_{\mathrm{df}} (\forall x)Rxx \\ \mathrm{Sym}(R) &\equiv_{\mathrm{df}} (\forall x, y)(Rxy \to Ryx) \\ \mathrm{Trans}(R) &\equiv_{\mathrm{df}} (\forall x, y, z)(Rxy \,\&\, Ryz \to Rxz) \end{aligned}$$

Example 3.2 Let us shortly provide a simple example to illustrate the concepts introduced in Definition 3.1. Consider the domain $U = \{1, \ldots, 6\}$ and the following fuzzy relation (for convenience, in matrix notation):

$$P_1 = \begin{pmatrix} 1.0 & 1.0 & 0.5 & 0.4 & 0.3 & 0.0 \\ 0.8 & 1.0 & 0.4 & 0.4 & 0.3 & 0.0 \\ 0.7 & 0.9 & 1.0 & 0.8 & 0.7 & 0.4 \\ 0.9 & 1.0 & 0.7 & 1.0 & 0.9 & 0.6 \\ 0.6 & 0.8 & 0.8 & 0.7 & 1.0 & 0.7 \\ 0.3 & 0.5 & 0.6 & 0.4 & 0.7 & 1.0 \end{pmatrix}$$

It is easy to check that P_1 is a fuzzy preorder with respect to the Łukasiewicz t-norm $\max(x + y - 1, 0)$, hence, taking standard Łukasiewicz logic, we obtain $\mathrm{Refl}(P_1) = 1$ and $\mathrm{Trans}(P_1) = 1$. In this setting, one can easily compute $\mathrm{Sym}(P_1) = 0.4$ (note that, for a finite fuzzy relation R, in standard Łukasiewicz logic, $\neg\,\mathrm{Sym}(R)$ is nothing else but the largest difference between the two values Rxy and Ryx).

Now let us see what happens if we add some disturbances to P_1. Consider the following fuzzy relation:

$$P_2 = \begin{pmatrix} 1.00 & 1.00 & 0.56 & 0.40 & 0.30 & 0.00 \\ 0.87 & 1.00 & 0.33 & 0.44 & 0.26 & 0.02 \\ 0.67 & 0.92 & 0.93 & 0.87 & 0.70 & 0.39 \\ 0.93 & 1.00 & 0.64 & 1.00 & 0.97 & 0.67 \\ 0.52 & 0.79 & 0.82 & 0.71 & 1.00 & 0.59 \\ 0.27 & 0.50 & 0.61 & 0.41 & 0.72 & 1.00 \end{pmatrix}$$

Simple computations give the following results: $\mathrm{Refl}(P_2) = 0.93$, $\mathrm{Sym}(P_2) = 0.41$, $\mathrm{Trans}(P_2) = 0.85$ (for standard Łukasiewicz logic again).

Example 3.3 Consider $U = \mathbb{R}$ and let us define the following parameterized class of fuzzy relations (with $a, c > 0$):

$$E_{a,c}xy = \min(1, \max(0, a - \tfrac{1}{c}|x - y|))$$

It is well known that, for $a = 1$, we obtain fuzzy equivalence relations with respect to the Łukasiewicz t-norm [10, 11, 22, 23]; hence, using standard Łukasiewicz logic again, $\mathrm{Refl}(E_{1,c}) = 1$, $\mathrm{Sym}(E_{1,c}) = 1$, and $\mathrm{Trans}(E_{1,c}) = 1$ for all $c > 0$. On the contrary, it is obvious that reflexivity in the non-graded manner cannot be maintained for $a < 1$. Actually, we obtain

$$\mathrm{Refl}(E_{a,c}) = \min(1, a).$$

for all $a, c > 0$. Similarly, it is a well-known fact that, for $a > 1$, transitivity in the non-graded sense is violated [12]. Regarding graded transitivity, we obtain the following:

$$\mathrm{Trans}(E_{a,c}) = \min(1, \max(0, 2 - a))$$

None of these results depends on the parameter c, as c only corresponds to a re-scaling of the domain. We can conclude that the larger a, the more reflexive, but less transitive, $E_{a,c}$ is. Figure 1 shows two examples.

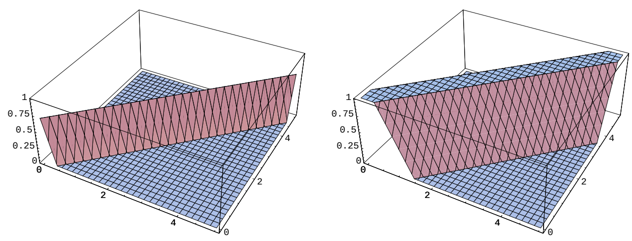

Figure 1: The fuzzy relations $E_{0.7,2}$ (left) and $E_{1.4,1}$ (right). From Example 3.3, we can infer that $\mathrm{Refl}(E_{0.7,2}) = 0.7$, $\mathrm{Trans}(E_{0.7,2}) = \mathrm{Refl}(E_{1.4,1}) = 1$, and $\mathrm{Trans}(E_{1.4,1}) = 0.6$.

In classical mathematics, special properties of relations are rarely studied completely independently of each other. Instead, these properties most often occur in some combinations in the definitions of special classes of relations—with (pre)orders and equivalence relations being two most fundamental examples. The same is true in the theory of fuzzy relations, where fuzzy (pre)orders and fuzzy equivalence relations are the most important classes. Compound properties of this kind are defined as conjunctions of some of the simple properties of Definition 3.1. In the non-graded case, the properties are crisp, so the conjunction we need is the classical Boolean conjunction. In FCT, however, all properties are graded, so it indeed matters which conjunction we take. Thus, besides the (more usual) combinations by strong conjunction & (corresponding to the t-norm in the standard case), we also define their weak variants combined by weak conjunction (corresponding to the minimum). In this paper, we restrict to the investigation of fuzzy preorders and similarities.[2]

Definition 3.4 In FCT, we define the following compound properties of fuzzy relations:

$$\begin{aligned}
\mathrm{Preord}(R) &\equiv_{\mathrm{df}} \mathrm{Refl}(R) \,\&\, \mathrm{Trans}(R)\\
\mathrm{wPreord}(R) &\equiv_{\mathrm{df}} \mathrm{Refl}(R) \wedge \mathrm{Trans}(R)\\
\mathrm{Sim}(R) &\equiv_{\mathrm{df}} \mathrm{Refl}(R) \,\&\, \mathrm{Sym}(R) \,\&\, \mathrm{Trans}(R)\\
\mathrm{wSim}(R) &\equiv_{\mathrm{df}} \mathrm{Refl}(R) \wedge \mathrm{Sym}(R) \wedge \mathrm{Trans}(R)
\end{aligned}$$

Example 3.5 Let us shortly revisit Example 3.2. We can conclude the following:

$$\begin{aligned}
\mathrm{Preord}(P_1) &= 1 & \mathrm{Preord}(P_2) &= 0.78\\
\mathrm{wPreord}(P_1) &= 1 & \mathrm{wPreord}(P_2) &= 0.85\\
\mathrm{Sim}(P_1) &= 0.4 & \mathrm{Sim}(P_2) &= 0.19\\
\mathrm{wSim}(P_1) &= 0.4 & \mathrm{wSim}(P_2) &= 0.41
\end{aligned}$$

The values in the second column once more demonstrate why it is justified to speak of strong and weak properties—the properties with strong conjunction get smaller truth degrees and thus are harder fulfil. Obviously, the implications $\mathrm{Preord}(R) \to \mathrm{wPreord}(R)$ and $\mathrm{Sim}(R) \to \mathrm{wSim}(R)$ hold.

Example 3.6 For the family of fuzzy relations defined in Example 3.3, we obtain the interesting result

$$\mathrm{Preord}(E_{a,c}) = \mathrm{wPreord}(E_{a,c}) = \max(0, 1 - |1 - a|),$$

from which we can infer that $\mathrm{Preord}(E_{a,c}) = \mathrm{wPreord}(E_{a,c}) = 1$ if and only if $a = 1$. Note that $\mathrm{Sym}(E_{a,c}) = 1$, so $\mathrm{Sim}(E_{a,c}) = \mathrm{Preord}(E_{a,c})$ and $\mathrm{wSim}(E_{a,c}) = \mathrm{wPreord}(E_{a,c})$, which implies that $\mathrm{Sim}(E_{a,c}) = \mathrm{wSim}(E_{a,c}) = 1$ if and only if $a = 1$.

[2] In line with Zadeh's original work [25], we use the term *similarity (relation)* synonymously for fuzzy equivalence (relation).

4 Representation of Fuzzy Preorders

This section aims at generalizing Valverde's representation theorem for fuzzy preorders. We will proceed as follows: we first generalize Fodor's characterization by means of traces and then use this characterization to prove the generalization of Valverde's theorem. Note that Valverde's original proof [23] implicitly follows the same lines.

So given a fuzzy relation R, let us first consider the fuzzy relation R^ℓ defined as

$$R^\ell xy \equiv_{\mathrm{df}} (\forall z)(Rzx \to Rzy)$$

This is called the *left trace* of R [15, 16]. Analogously, we can define the *right trace* as

$$R^r xy \equiv_{\mathrm{df}} (\forall z)(Ryz \to Rxz).$$

Observe the meaning of the following expressions:

$$\begin{aligned}
R^\ell \subseteq R &\leftrightarrow (\forall x, y)[(\forall z)(Rzx \to Rzy) \to Rxy]\\
R \subseteq R^\ell &\leftrightarrow (\forall x, y)[Rxy \to (\forall z)(Rzx \to Rzy)]\\
R \approx R^\ell &\leftrightarrow (\forall x, y)[Rxy \leftrightarrow (\forall z)(Rzx \to Rzy)]
\end{aligned}$$

Now we can formulate characterizations of graded reflexivity and transitivity, which are not difficult to prove in FCT.

Theorem 4.1 *The following properties hold in* FCT*:*

$$\begin{aligned}
\mathrm{Refl}(R) &\leftrightarrow R^\ell \subseteq R\\
\mathrm{Trans}(R) &\leftrightarrow R \subseteq R^\ell
\end{aligned}$$

As a corollary we obtain graded versions of Fodor's characterizations [15, Theorems 4.1, 4.3, and Corollary 4.4]. For the two notions of fuzzy equality $\approx$ and $\approxeq$, see Definition A.3.

Corollary 4.2 *The following is provable in* FCT*:*

$$\mathrm{wPreord}(R) \leftrightarrow R \approx R^\ell$$
$$\mathrm{Preord}(R) \leftrightarrow R \approxeq R^\ell$$
$$R \approx^2 R^\ell \longrightarrow \mathrm{Preord}(R) \longrightarrow R \approx R^\ell$$

Note that, regardless of the symmetry of R, we can replace R^ℓ in the above characterizations by the right trace as well.

Now we have all prerequisites for formulating and proving a graded version of Valverde's representation theorem for preorders. In order to make notations more compact, let us define two graded notions of *Valverde preorder representation* (a strong one and a weak one), for a given fuzzy relation R and a fuzzy class of fuzzy classes $\mathcal{A}$:

$$\mathrm{ValP}(R,\mathcal{A}) \equiv_{\mathrm{df}} R \approxeq \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$$
$$\mathrm{wValP}(R,\mathcal{A}) \equiv_{\mathrm{df}} R \approx \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$$

The predicates ValP and wValP express the degree to which the fuzzy class $\mathcal{A}$ represents the relation R.

Then we can prove the following essential result for preorders and weak preorders.

Theorem 4.3 FCT *proves the following:*

$$(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \;\&\; \mathrm{ValP}^2(R,\mathcal{A}) \longrightarrow \mathrm{Preord}(R) \longrightarrow (\exists \mathcal{A})(\mathrm{Crisp}(\mathcal{A}) \;\&\; \mathrm{ValP}(R,\mathcal{A}))$$

$$(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \;\&\; \mathrm{wValP}^3(R,\mathcal{A}) \longrightarrow \mathrm{wPreord}(R) \longrightarrow (\exists \mathcal{A})(\mathrm{Crisp}(\mathcal{A}) \;\&\; \mathrm{wValP}(R,\mathcal{A}))$$

It can be shown that the exponents in this theorem cannot be lowered, see [2] for a counterexample. Obviously, this theorem is more complicated than Valverde's original result; it is an example where the graded framework does not provide us with just a plain copy of the non-graded (or crisp) result. The following corollary gives us a result that is comparable with Valverde's original theorem.

Corollary 4.4 *The following equivalences are provable in* FCT*:*

$$\triangle\,\mathrm{Preord}(R) \longleftrightarrow R = R^{\ell} \longleftrightarrow \quad (1)$$
$$(\exists \mathcal{A})(\triangle(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \;\&\; \triangle\,\mathrm{ValP}(R,\mathcal{A})) \quad (2)$$
$$\longleftrightarrow (\exists \mathcal{A})(\mathrm{Crisp}(\mathcal{A}) \;\&\; \triangle\,\mathrm{ValP}(R,\mathcal{A})) \quad (3)$$

Observe that also the analogous formulas with wPreord and wValP are equivalent to those in this corollary.

Example 4.5 Let us shortly revisit Example 3.2 (in which we use standard Łukasiewicz logic). The fuzzy relation P_1 was actually constructed from the following crisp family of three fuzzy sets $\mathcal{A} = \{A_1, A_2, A_3\}$ that are defined as follows (for convenience, in vector notation):

$$A_1 = (0.7, 0.8, 0.2, 0.5, 0.4, 0.6)$$
$$A_2 = (0.3, 0.5, 0.6, 0.4, 0.7, 1.0)$$
$$A_3 = (1.0, 1.0, 0.6, 0.4, 0.3, 0.0)$$

Although the formula (3) is a perfect copy of Valverde's non-graded representation, the corollary still has graded elements—note that in (2), the class $\mathcal{A}$ may still be a fuzzy class of fuzzy classes, if only it satisfies $\triangle(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A})$. Recall that in Gödel logic, this condition is fulfilled by *all* fuzzy classes $\mathcal{A}$, and that in any logic it is satisfied by a system $\mathcal{A}$ in a model if all degrees of membership in $\mathcal{A}$ are idempotent with respect to conjunction.

The degree of $A \in \mathcal{A}$ may be considered as a weighting factor that controls the influence of a specific A on the final result. Corollary 4.4 requires all membership degrees in $\mathcal{A}$ to be idempotent to ensure that the relation represented by $\mathcal{A}$ is a fuzzy preorder, but its graded version in Theorem 4.3 also shows that (loosely speaking) it will *almost* be a fuzzy preorder if $\mathcal{A}$ *almost* satisfies $\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}$ (e.g., in standard Łukasiewicz logic if it is close to crispness).

Example 4.6 Let us consider a $[0,1]$-valued fuzzy logic with the triangular norm

$$x * y = \begin{cases} \max(x+y-\frac{1}{2}, 0) & \text{if } x \in [0,\frac{1}{2}]^2, \\ \min(x,y) & \text{otherwise,} \end{cases}$$

i.e. a simple ordinal sum with a scaled Łukasiewicz t-norm in $[0,\frac{1}{2}]^2$ and the Gödel t-norm anywhere else. It is clear that the set of idempotent elements of this t-norm is $\{0\} \cup [\frac{1}{2}, 1]$ and that the corresponding residual implication is given as

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ \max(y, \frac{1}{2} - x + y) & \text{otherwise.} \end{cases}$$

Now reconsider $U = \{1, \ldots, 6\}$ and the three fuzzy sets A_1, A_2 and A_3 from Example 4.5 and define a fuzzy class of fuzzy classes $\mathcal{A}$ such that $\mathcal{A}A_1 = 0.9$, $\mathcal{A}A_1 = 1.0$, and $\mathcal{A}A_3 = 0.8$. Since all three values are idempotent elements of $*$, we can be sure by Theorem 4.3 that the construction $R_1 =_{\mathrm{df}} \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$ always gives us a fuzzy preorder in the given logic. In this particular example, we obtain the following:

$$R_1 = \begin{pmatrix} 1.0 & 1.0 & 0.2 & 0.4 & 0.3 & 0.0 \\ 0.3 & 1.0 & 0.2 & 0.4 & 0.3 & 0.0 \\ 0.3 & 0.5 & 1.0 & 0.4 & 0.3 & 0.0 \\ 0.4 & 1.0 & 0.2 & 1.0 & 0.4 & 0.1 \\ 0.3 & 0.5 & 0.3 & 0.4 & 1.0 & 0.2 \\ 0.3 & 0.5 & 0.2 & 0.4 & 0.4 & 1.0 \end{pmatrix}$$

If we repeat this construction and define a fuzzy relation $R_2 =_{\mathrm{df}} \{\langle x,y\rangle \mid (\forall A \in \mathcal{A})(Ax \to Ay)\}$ with $\mathcal{A}$ defined as above, but the connectives interpreted in standard Łukasiewicz logic, we obtain the following:

$$R_2 = \begin{pmatrix} 1.0 & 1.0 & 0.6 & 0.6 & 0.5 & 0.2 \\ 0.8 & 1.0 & 0.5 & 0.6 & 0.5 & 0.2 \\ 0.7 & 0.9 & 1.0 & 0.8 & 0.9 & 0.6 \\ 0.9 & 1.0 & 0.8 & 1.0 & 1.0 & 0.8 \\ 0.6 & 0.8 & 0.9 & 0.7 & 1.0 & 0.9 \\ 0.3 & 0.5 & 0.6 & 0.4 & 0.7 & 1.0 \end{pmatrix}.$$

Straightforward calculations show that $\text{Refl}(R_2) = 1$ and $\text{Trans}(R_2) = \text{Preord}(R_2) = \text{wPreord}(R_2) = 0.8$. This is not at all contradicting to Theorem 4.3, as $\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}$ holds only to a degree of 0.6 in standard Łukasiewicz logic.

5 Representations of Similarities

In his landmark paper [23], Valverde not only considers fuzzy preorders, but also similarities (as obvious from the title of this paper). So the question naturally arises how we can modify the above results in the presence of symmetry. As will be seen next, the modifications are not as straightforward as in the non-graded case. We first define a fuzzy relation $R^{\ell s}$ as

$$R^{\ell s}xy \equiv_{\text{df}} (\forall z)(Rzx \leftrightarrow Rzy)$$

(for a given fuzzy relation R). There is no particular name for this fuzzy relation in literature. In analogy to Section 4, let us call it *left symmetric trace* of R.

The following lemma demonstrates how this notion is related to the defining properties of similarities.

Theorem 5.1 *The following are theorems of* FCT*:*

$$R^{\ell s} \subseteq R \leftrightarrow \text{Refl}(R) \quad (4)$$

$$R \subseteq R^{\ell s} \rightarrow \text{Trans}(R) \quad (5)$$

$$R \approxeq R^{\ell s} \rightarrow \text{Sym}(R) \quad (6)$$

$$\text{Sym}(R) \,\&\, \text{Trans}(R) \rightarrow R \subseteq R^{\ell s} \quad (7)$$

The following theorem provides us with an analogue of Corollary 4.2, unfortunately, with looser bounds on the left-hand side.

Corollary 5.2 FCT *proves:*

$$R \approx^4 R^{\ell s} \longrightarrow R \approxeq^2 R^{\ell s} \longrightarrow \text{Sim}(R)$$
$$\text{Sim}(R) \longrightarrow R \approxeq R^{\ell s} \longrightarrow R \approx R^{\ell s}$$

$$R \approx^2 R^{\ell s} \longrightarrow R \approxeq R^{\ell s} \longrightarrow \text{wSim}(R)$$
$$\text{wSim}^2(R) \longrightarrow R \approx R^{\ell s}$$

The question arises whether it is really necessary to require $\approxeq$ rather than $\approx$ in (6). The following example tells us that this is indeed the case. It also implies that $R \approx R^{\ell s} \rightarrow \text{wSim}(R)$ does *not* hold in general.

Example 5.3 Consider $U = \{1, 2\}$, standard Łukasiewicz logic, and the following fuzzy relation:

$$R = \begin{pmatrix} 0.5 & 1.0 \\ 0.0 & 0.5 \end{pmatrix}$$

It is obvious that $\text{Refl}(R) = 0.5$ and $\text{Sym}(R) = 0$. Moreover, routine calculations show $\text{Trans}(R) = 1$ and that $R^{\ell s}$ is given as follows:

$$R^{\ell s} = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 1.0 \end{pmatrix}$$

So, we finally obtain that $(R \approx R^{\ell s}) = 0.5$, while $(R \approxeq R^{\ell s}) = 0$.

Now we can formulate a graded version of Valverde's representation theorem for similarities. Analogously to the above considerations, let us define the graded notion of *Valverde similarity representation* (strong one and weak one) for a given fuzzy relation R and a fuzzy class $\mathcal{A}$:

$$\text{ValS}(R, \mathcal{A}) \equiv_{\text{df}} R \approxeq \{\langle x, y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}$$
$$\text{wValS}(R, \mathcal{A}) \equiv_{\text{df}} R \approx \{\langle x, y\rangle \mid (\forall A \in \mathcal{A})(Ax \leftrightarrow Ay)\}$$

In the same way as for preorders, we can prove Valverde's representation theorem of similarities and weak similarities.

Theorem 5.4 FCT *proves the following:*

$$(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \,\&\, \text{ValS}^3(R, \mathcal{A}) \rightarrow \text{Sim}(R) \quad (8)$$

$$\text{Sim}(R) \rightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \,\&\, \text{ValS}(R, \mathcal{A})) \quad (9)$$

$$(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \,\&\, \text{wValS}^3(R, \mathcal{A}) \rightarrow \text{wSim}(R) \quad (10)$$

$$\text{wSim}^2(R) \rightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \,\&\, \text{wValS}(R, \mathcal{A})) \quad (11)$$

Again, this theorem is more complicated than Valverde's original representation of similarities. In the following corollary, analogously to preorders, we can infer a result very similar to Valverde's original theorem in case that the corresponding properties are fulfilled to degree 1.

Corollary 5.5 *The following equivalences are provable in* FCT*:*

$$\triangle\, \text{Sim}(R) \longleftrightarrow R = R^{\ell s} \longleftrightarrow \quad (12)$$

$$(\exists \mathcal{A})(\triangle(\mathcal{A} \subseteq \mathcal{A} \cap \mathcal{A}) \,\&\, \triangle\, \text{ValS}(R, \mathcal{A})) \quad (13)$$

$$\longleftrightarrow (\exists \mathcal{A})(\text{Crisp}(\mathcal{A}) \,\&\, \triangle\, \text{ValS}(R, \mathcal{A})) \quad (14)$$

Again, like in the case of preorders, we can add equivalent lines with wValS instead of ValS, and (13) has a graded ingredient—the class $\mathcal{A}$ may be a fuzzy class of fuzzy classes.

6 Concluding Remarks

The present paper has generalized Valverde's famous representation theorems for fuzzy preorders and similarities to the fully graded framework of Fuzzy Class

Theory (FCT). In the formal setting of FCT, this generalization can be done relatively easily compared to Gottwald's semi-formal framework of graded properties of fuzzy relations. At the same time, we have seen that the results are not just obtained by simply rewriting known theorems. Indeed we obtain new results that even give rise to interesting new constructions (as demonstrated by Example 4.6).

A Appendix: Fuzzy Class Theory

In this section, we present a self-contained list of definitions related to Fuzzy Class Theory (FCT). For a complete and detailed introduction to FCT, the reader is referred to the freely available primer [5].

Definition A.1 *Fuzzy Class Theory* (over $\mathrm{MTL}_\triangle$) is a theory over multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with crisp equality. There are sorts for individuals of the zeroth order (i.e., atomic objects), denoted by lowercase variables $a, b, c, x, y, z, \ldots$; individuals of the first order (i.e., fuzzy classes), denoted by uppercase variables $A, B, X, Y, \ldots$; individuals of the second order (i.e., fuzzy classes of fuzzy classes), denoted by calligraphic variables $\mathcal{A}, \mathcal{B}, \mathcal{X}, \mathcal{Y}, \ldots$; etc. Individuals $\xi_1, \ldots, \xi_k$ of each order can form k-tuples (for any $k \geq 0$), denoted by $\langle \xi_1, \ldots, \xi_k \rangle$; tuples are governed by the usual axioms known from classical mathematics (e.g., that tuples equal if and only if their respective constituents equal). Furthermore, for each variable x of any order n and for each formula φ there is a class term $\{x \mid \varphi\}$ of order $n + 1$.

Besides the logical predicate of identity, the only primitive predicate is the membership predicate $\in$ between successive sorts (i.e., between individuals of the n-th order and individuals of the $(n+1)$-st order, for any n). The axioms for $\in$ are the following (for variables of all orders):

($\in$1) $y \in \{x \mid \varphi(x)\} \leftrightarrow \varphi(y)$, for each formula φ (comprehension axioms)

($\in$2) $(\forall x)\triangle(x \in A \leftrightarrow x \in B) \to A = B$ (extensionality)

Moreover, we use all axioms and deduction rules of multi-sorted first-order logic $\mathrm{MTL}_\triangle$ with crisp identity. Theorems, proofs, etc., are defined completely analogously as in classical logic.

Convention A.2 For better readability, we make the following conventions:

- We use $(\forall x \in A)\varphi$, $(\exists x \in A)\varphi$ as abbreviations for $(\forall x)(x \in A \to \varphi)$ and $(\exists x)(x \in A \mathbin{\&} \varphi)$, respectively.
- $\{x \in A \mid \varphi\}$ is shorthand for $\{x \mid x \in A \mathbin{\&} \varphi\}$.
- We use $\{\langle x_1, \ldots, x_k \rangle \mid \varphi\}$ as abbreviation for $\{x \mid (\exists x_1) \ldots (\exists x_k)(x = \langle x_1, \ldots, x_k \rangle \mathbin{\&} \varphi)\}$.
- The formulae $\varphi \mathbin{\&} \ldots \mathbin{\&} \varphi$ (n times) are abbreviated φ^n; instead of $(x \in A)^n$, we can write $x \in^n A$ (analogously for other predicates).
- We use Ax and $Rx_1 \ldots x_n$ as synonyms for $x \in A$ and $\langle x_1, \ldots, x_n \rangle \in R$, respectively.
- A chain of implications

  $$\varphi_1 \to \varphi_2,\ \varphi_2 \to \varphi_3,\ \ldots,\ \varphi_{n-1} \to \varphi_n$$

  is, for short, written as $\varphi_1 \longrightarrow \varphi_2 \longrightarrow \cdots \longrightarrow \varphi_n$ (and analogously for the equivalence connective).

Definition A.3 We define the following elementary relations between fuzzy sets in FCT:

$$\begin{array}{rcl} \mathrm{Crisp}(A) & \equiv_{\mathrm{df}} & (\forall x)\triangle(x \in A \vee x \notin A) \\ A \subseteq B & \equiv_{\mathrm{df}} & (\forall x)(x \in A \to x \in B) \\ A \approxeq B & \equiv_{\mathrm{df}} & (A \subseteq B) \mathbin{\&} (B \subseteq A) \\ A \approx B & \equiv_{\mathrm{df}} & (\forall x)(x \in A \leftrightarrow x \in B) \end{array}$$

The *models* of FCT are systems (closed under definable operations) of fuzzy sets (and fuzzy relations) of all orders over some crisp universe U, where the membership functions of fuzzy subsets take values in some $\mathrm{MTL}_\triangle$-chain. *Intended* models are those which contain *all* fuzzy subsets and fuzzy relations over U (of all orders). Models in which moreover the $\mathrm{MTL}_\triangle$-chain is standard (i.e., given by a left-continuous t-norm on the unit interval $[0, 1]$) correspond to Zadeh's [24] original notion of fuzzy set; therefore we call them *Zadeh models*. FCT is sound with respect to Zadeh models, therefore all theorems provable in FCT are true statements about fuzzy sets and relations in the traditional sense.

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References

[1] Wyllis Bandler and Ladislav J. Kohout. Fuzzy power sets and fuzzy implication operators. *Fuzzy Sets and Systems*, 4:183–190, 1980.

[2] Libor Běhounek, Ulrich Bodenhofer, and Petr Cintula. Relations in fuzzy class theory: initial steps. Submitted to Fuzzy Sets and Systems.

[3] Libor Běhounek and Petr Cintula. Fuzzy class theory. *Fuzzy Sets and Systems*, 154(1):34–55, 2005.

[4] Libor Běhounek and Petr Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems*, 157(5):642–646, 2006.

[5] Libor Běhounek and Petr Cintula. Fuzzy Class Theory: A primer v1.0. Technical Report V-939, Institute of Computer Science, Academy of Sciences of the Czech Republic, Prague, 2006. Available at www.cs.cas.cz/research/library/reports_900.shtml.

[6] Radim Bělohlávek. *Fuzzy Relational Systems: Foundations and Principles*, volume 20 of *IFSR International Series on Systems Science and Engineering*. Kluwer Academic/Plenum Press, New York, 2002.

[7] Ulrich Bodenhofer. A similarity-based generalization of fuzzy orderings preserving the classical axioms. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 8(5):593–610, 2000.

[8] Ulrich Bodenhofer, Bernard De Baets, and János Fodor. A compendium of fuzzy weak orders: Representations and constructions. To appear in Fuzzy Sets and Systems, 2007.

[9] Georg Cantor. Beiträge zur Begründung der transfiniten Mengenlehre. *Mathematische Annalen*, 46:481–512, 1895.

[10] Bernard De Baets and Radko Mesiar. Pseudo-metrics and T-equivalences. *Journal of Fuzzy Mathematics*, 5(2):471–481, 1997.

[11] Bernard De Baets and Radko Mesiar. Metrics and T-equalities. *Journal of Mathematical Analysis and Applications*, 267:331–347, 2002.

[12] Martine De Cock and Etienne E. Kerre. On (un)suitable fuzzy relations to model approximate equality. *Fuzzy Sets and Systems*, 133(2):137–153, 2003.

[13] Francesc Esteva and Lluís Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[14] Francesc Esteva, Lluís Godo, and Franco Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[15] János Fodor. Traces of fuzzy binary relations. *Fuzzy Sets and Systems*, 50(3):331–341, 1992.

[16] János Fodor and Marc Roubens. *Fuzzy Preference Modelling and Multicriteria Decision Support*. Kluwer Academic Publishers, Dordrecht, 1994.

[17] Siegfried Gottwald. Fuzzified fuzzy relations. In Robert Lowen and Marc Roubens, editors, *Proceedings of 4th IFSA Congress*, volume Mathematics (ed. P. Wuyts), pages 82–86, Brussels, 1991.

[18] Siegfried Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application—from a Mathematical Point of View*. Vieweg, Wiesbaden, 1993.

[19] Siegfried Gottwald. *A Treatise on Many-Valued Logics*, volume 9 of *Studies in Logic and Computation*. Research Studies Press, Baldock, 2001.

[20] Petr Hájek and Petr Cintula. Triangular norm predicate fuzzy logics. To appear in Proceedings of Linz Seminar 2005, 2006.

[21] Joan Jacas and Jordi Recasens. Fuzzified properties of fuzzy relations. In *Proceedings of the 9th IPMU Conference*, volume 1, pages 157–161, Annecy, 2002.

[22] Enric Trillas and Llorenç Valverde. An inquiry into indistinguishability operators. In Heinz J. Skala, Settimo Termini, and Enric Trillas, editors, *Aspects of Vagueness*, pages 231–256. Reidel, Dordrecht, 1984.

[23] Llorenç Valverde. On the structure of F-indistinguishability operators. *Fuzzy Sets and Systems*, 17(3):313–328, 1985.

[24] Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.

[25] Lotfi A. Zadeh. Similarity relations and fuzzy orderings. *Information Sciences*, 3:177–200, 1971.