Intersections between Basic Families of Fuzzy Implications: (S, N)-, R- and QL-Implications

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Abstract

In this work, our primary focus is to determine the intersections that exist between the family of QL-implications and the families of (S, N)- and R-implications. Toward this end, firstly, we investigate the conditions under which a QL-operation becomes a fuzzy implication. Since the exchange principle and the ordering property of fuzzy implications play an important role in this study, we propose some necessary and/or sufficient conditions on the underlying operations under which QL-operations satisfy them. As part of this attempt some interesting results pertaining to natural negations from t-conorms and the exact intersection between (S, N)- and R-implications have been obtained. We also mention some open problems relating to QL-implications.

Keywords: Fuzzy implication, QL-implication, R-implication, S-implication, (S, N)-implication, Law of excluded middle.

1 Introduction

The natural generalization of the implication in quantum logic to fuzzy logic – QL-operations – has not received as much attention as (S, N)- and R-implications. Perhaps, one of the reasons can be attributed to the fact that not all members of this family satisfy one of the main properties expected of a fuzzy implication, viz., left antitonicity. Also, in the earlier works, some conditions imposed on the fuzzy logic operations employed in the definition of QL-operations restricted both the class of operations from which they could be obtained and the properties these implications satisfied (see Remark 5 in Section 6.2 for details).

In this work, we study the family of QL-operations in fuzzy logic, without any restrictions on the underlying operations. We propose some necessary and/or sufficient conditions on the underlying operations under which QL-operations satisfy some of the most desirable algebraic properties. Finally, we return to the prime focus of this work, viz., the intersections that exist between the family of QL-implications and the families of (S, N)- and R-implications. Toward this end, we have also precisely determined the intersection of the families of (S, N)- and R-implications.

2 Preliminaries

Firstly we briefly mention some of the concepts and results employed in the rest of the work.

Definition 1 (see [5, 7, 8]). A decreasing function $N: [0, 1] \to [0, 1]$ is called a fuzzy negation if $N(0) = 1$ and $N(1) = 0$. A fuzzy negation N is said to be

- strict if it is strictly decreasing and continuous;
- strong if it is an involution, i.e., $N(N(x)) = x$ for all $x \in [0, 1]$;
- non-vanishing if $N(x) = 0 \iff x = 1$;
- non-filling if $N(x) = 1 \iff x = 0$.

The classical negation $N_{\mathbf{C}}(x) = 1 - x$ is a strong negation, while $N_{\mathbf{K}}(x) = 1 - x^2$ is only strict, whereas $N_{\mathbf{D1}}$ and $N_{\mathbf{D2}}$ are non-filling and non-vanishing negations, respectively, where:

$$N_{\mathbf{D1}}(x) = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x > 0, \end{cases} \quad N_{\mathbf{D2}}(x) = \begin{cases} 1, & \text{if } x < 1, \\ 0, & \text{if } x = 1. \end{cases}$$

Definition 2 (see [8]). An associative, commutative, increasing operation $T: [0, 1]^2 \to [0, 1]$ is called t-norm if it has neutral element equal to 1. An associative, commutative, increasing operation $S: [0, 1]^2 \to [0, 1]$ is called t-conorm if it has neutral element equal to 0. A t-norm T is said to be positive if $T(x, y) = 0 \iff x = 0$ or $y = 0$. A t-conorm S is said to be negative if $S(x, y) = 1 \iff x = 1$ or $y = 1$.

Table 1: Examples of t-norms

Name	**Formula**
$T_{\mathbf{M}}$: minimum	$\min(x,y)$
$T_{\mathbf{P}}$: product	$x \cdot y$
$T_{\mathbf{L}}$: Łukasiewicz	$\max(x+y-1,0)$,
$T_{\mathbf{D}}$: drastic product	$\begin{cases} 0, & \text{if } x,y \in [0,1) \\ \min(x,y), & \text{otherwise} \end{cases}$
$T_{\mathbf{nM}}$: nilpotent minimum	$\begin{cases} 0, & \text{if } x+y \leq 1 \\ \min(x,y), & \text{otherwise} \end{cases}$

Table 2: Examples of t-conorms

NAME	**FORMULA**
$S_{\mathbf{M}}$: maximum	$\max(x,y)$
$S_{\mathbf{P}}$: algebraic sum	$x+y-x\cdot y$
$S_{\mathbf{L}}$: Łukasiewicz	$\min(x+y,1)$
$S_{\mathbf{D}}$: drastic sum	$\begin{cases} 1, & \text{if } x,y \in (0,1] \\ \max(x,y), & \text{otherwise} \end{cases}$
$S_{\mathbf{nM}}$: nilpotent maximum	$\begin{cases} 1, & \text{if } x+y \geq 1 \\ \max(x,y), & \text{otherwise} \end{cases}$

Firstly, we show that one can obtain a fuzzy negation from any t-conorm S and discuss its relevance vis-á-vis the law of excluded middle, which in the classical case has the following form: $p \vee \neg p$.

Definition 3 (cf. [8], Definition 5.5.2)**.** Let S be a t-conorm. A function $N_S \colon [0,1] \to [0,1]$ defined as

$$N_S(x) = \inf\{y \in [0,1] \mid S(x,y) = 1\}, \quad x \in [0,1], \quad (1)$$

is called the natural negation of S.

Remark 1. (i) It is easy to prove that N_S is a fuzzy negation for every t-conorm S.

(ii) If S is a negative t-conorm, then $N_S = N_{\mathbf{D2}}$.

(iii) Let S be a t-conorm and $x \in [0,1]$ be fixed. Let us denote $A_x = \{y \in [0,1] \mid S(x,y) = 1\}$. Then $1 \in A_x$ and hence $A_x \neq \emptyset$. If $x_0 = \inf A_x$, then, by the monotonicity of S, we have that either $A_x = [x_0, 1]$ or $A_x = (x_0, 1]$.

(iv) If $S(x,y) = 1$ for some $x, y \in [0,1]$, then $y \geq N_S(x)$. On the other hand, since A_x is always an interval for every fixed $x \in [0,1]$, if $y > N_S(x)$, then $S(x,y) = 1$.

Definition 4 (cf. [5])**.** Let S be a t-conorm and N a fuzzy negation. We say that the pair (S,N) satisfies the law of excluded middle if

$$S(N(x),x) = 1, \qquad x \in [0,1]. \qquad \text{(LEM)}$$

A graphical interpretation of (LEM) is as follows: the graph of the negation N demarcates the region on the unit square $[0,1]^2$ above which $S = 1$. It is possible that there are a few more points below the graph of N on whom S assumes the value 1. For example, consider the Łukasiewicz t-conorm $S_{\mathbf{L}}$ and the strict negation $N_{\mathbf{K}}(x) = 1 - x^2$. Then $S_{\mathbf{L}}(0.5, N(0.5)) = S_{\mathbf{L}}(0.5, 0.75) = 1$. Also notice that $S_{\mathbf{L}}(0.5, 0.5) = 1$. On the other hand, it should be emphasized that $S = S_{\mathbf{D}}$ does not satisfy (LEM) with its natural negation $N_{S_{\mathbf{D}}} = N_{\mathbf{D1}}$.

Lemma 1. *If a t-conorm S and a fuzzy negation N satisfy* (LEM)*, then*

(i) $N(x) \geq N_S(x)$, for all $x \in [0,1]$;

(ii) $N_S \circ N(x) \leq x$, for all $x \in [0,1]$.

Proof. (i) On the contrary, if for some $x_0 \in [0,1]$ we have $N(x_0) < N_S(x_0)$, then $S(N(x_0), x_0) < 1$ by Remark 1 (iv).
(ii) From Definition 3 we have, for any $x \in [0,1]$,

$$N_S(N(x)) = \inf\{y \in [0,1] \mid S(N(x), y) = 1\}.$$

Now, since $S(N(x), x) = 1$, we get $x \geq N_S(N(x))$. ☐

It follows from Remark 1 (ii) that if S is a negative t-conorm, then S satisfies (LEM) only with the greatest fuzzy negation $N_{\mathbf{D2}}$.

Proposition 1. *If S is a right-continuous t-conorm with the natural negation N_S, then the following statements are equivalent:*

(i) N_S is continuous.

(ii) N_S is strong.

Proof. By the right-continuity of S we can show that the infimum in (1) reduces to minimum and thus the pair (S, N_S) satisfies (LEM). Hence $x \geq N_S \circ N_S(x)$, so $N_S(x) \leq N_S \circ N_S \circ N_S(x)$. On the other hand, since (LEM) holds for every $x \in [0,1]$ we have

$$S(N_S \circ N_S(x), N_S(x)) = S(N_S(x), N_S \circ N_S(x)) = 1,$$

which implies that $N_S(x) \geq N_S \circ N_S \circ N_S(x)$. Since N_S is continuous, for every $y \in [0,1]$ there exists an $x \in [0,1]$ such that $y = N_S(x)$. Therefore, from the above inequalities, we get that $y = N_S \circ N_S(y)$ for every $y \in [0,1]$, i.e., N_S is a strong negation.
The reverse implication is obvious. ☐

Remark 2. Just as one can obtain the natural negation N_S from a t-conorm, the natural negation of a t-norm T can be obtained as follows:

$$N_T(x) = \sup\{y \in [0,1] \mid T(x,y) = 0\}, \quad x \in [0,1].$$

The counterpart of the law of excluded middle is the law of contradiction

$$T(N(x), x) = 0, \qquad x \in [0,1], \tag{LC}$$

where T is a t-norm and N is a fuzzy negation. For more on the above laws of excluded middle and contradiction, see for example [5].

It should be noted, that the following relation exists between N_T and N_S.

Theorem 1 ([4])**.** *Let T be a left-continuous t-norm with N_T being strong. If (T, N_T, S) form a De Morgan triple, i.e., S is the N_T-dual of T, then S is right-continuous and $N_S = N_T$.*

3 Fuzzy Implications

In this work the following equivalent definition proposed by Fodor and Roubens [5] is used.

Definition 5. A function $I\colon [0,1]^2 \to [0,1]$ is called a fuzzy implication if it satisfies the following conditions:

$$I \text{ is decreasing in the first variable,} \tag{I1}$$
$$I \text{ is increasing in the second variable,} \tag{I2}$$
$$I(0,0) = 1, \qquad I(1,1) = 1, \qquad I(1,0) = 0. \tag{I3}$$

Table 3: Examples of some fuzzy implications

Name	Formula
$I_{\mathbf{KD}}$: Kleene-Dienes	$\max(1-x, y)$
$I_{\mathbf{LK}}$: Łukasiewicz	$\min(1, 1-x+y)$
$I_{\mathbf{FD}}$: Fodor	$\begin{cases} 1, & \text{if } x \le y \\ \max(1-x, y), & \text{if } x > y \end{cases}$
$I_{\mathbf{SD}}$	$\begin{cases} y, & \text{if } x = 1 \\ N(x), & \text{if } y = 0 \\ 1, & \text{otherwise} \end{cases}$
$I_{\mathbf{TD}}$	$\begin{cases} 1, & \text{if } x < 1 \\ y, & \text{if } x = 1 \end{cases}$
$I_{\mathbf{TM}}$	$\begin{cases} 1, & \text{if } x \le y \\ S(N(x), y), & \text{if } x > y \end{cases}$
$I_{\mathbf{KP}}$	$\min(1, 1-x^2+xy)$

Directly from Definition 5 we see that each fuzzy implication I satisfies the following left and right boundary condition, respectively:

$$I(0, y) = 1, \qquad y \in [0,1], \tag{LB}$$
$$I(x, 1) = 1, \qquad x \in [0,1]. \tag{RB}$$

Therefore, I satisfies also the normality condition:

$$I(0,1) = 1. \tag{NC}$$

Definition 6. A fuzzy implication I is said to satisfy

- the left neutrality property, if
$$I(1, y) = y, \qquad y \in [0,1]; \tag{NP}$$
- the exchange principle, if for all $x, y, z \in [0,1]$,
$$I(x, I(y,z)) = I(y, I(x,z)); \tag{EP}$$
- the identity principle, if
$$I(x, x) = 1, \qquad x \in [0,1]; \tag{IP}$$
- the ordering property, if
$$I(x,y) = 1 \Longleftrightarrow x \le y, \qquad x, y \in [0,1]. \tag{OP}$$

Definition 7. Let I be a fuzzy implication. The function N_I defined as $N_I(x) := I(x, 0)$ for all $x \in [0,1]$, is called the natural negation of I.

It can be easily shown that N_I is a fuzzy negation.

Proposition 2 (cf. [5], Corollary 1.1; [2], Lemma 14)**.** *If a function $I\colon [0,1]^2 \to [0,1]$ satisfies* (EP) *and* (OP)*, then N_I is either a strong negation or a discontinuous negation.*

4 (S, N)-Implications and R-Implications

In this section, we give a brief introduction to two of the families of fuzzy implications that are very well studied in the literature and present some characterizations and results that will be useful in the sequel.

Definition 8 (see [1, 3, 5, 10])**.** A function $I\colon [0,1]^2 \to [0,1]$ is called an (S, N)-implication if there exist a t-conorm S and a fuzzy negation N such that

$$I(x, y) = S(N(x), y), \qquad x, y \in [0,1].$$

If N is a strong negation, then I is called an S-implication. Moreover, if I is generated from S and N, then we will often write $I_{S,N}$.

The following characterization of (S, N)-implications are from [3], which is an extension of a result in [10].

Theorem 2. *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (S, N)-implication generated from some t-conorm S and some continuous (strict, strong) fuzzy negation N.

(ii) I satisfies (I2), (EP) *and the function N_I is a continuous (strict, strong) fuzzy negation.*

Definition 9 (see [5, 7])**.** A function $I\colon [0,1]^2 \to [0,1]$ is called an R-implication if there exists a t-norm T such that for all $x, y \in [0,1]$,

$$I(x,y) = \sup\{t \in [0,1] : T(x,t) \le y\}. \qquad (2)$$

If I is generated from T, then we will often write I_T.

In this work we only consider I_T from left-continuous t-norms, in which case the supremum in (2) reduces to maximum (see [7]).

Theorem 3 ([5], Theorem 1.14)**.** *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an R-implication based on some left-continuous t-norm T.

(ii) I satisfies (I2)*,* (EP)*,* (OP) *and $I(x,\cdot)$ is right-continuous for any fixed $x \in [0,1]$.*

It can be immediately noted that $N_T(\cdot) = I_T(\cdot, 0)$, where I_T is obtained from a t-norm T. From the above theorem we see that for a left-continuous t-norm T, the fuzzy negation N_T is either strong or discontinuous. Therefore Proposition 1 can also be seen as the dual of above result.

We also have the following connections between a left-continuous t-norm T and the R-implication I_T.

Lemma 2 (cf. [5, 7])**.** *(i) If T is a left-continuous t-norm, then $T = T_{I_T}$, where, for all $x, y \in [0,1]$,*

$$T_{I_T}(x,y) = \min\{t \in [0,1] \mid I_T(x,t) \ge y\}.$$

(ii) T is a left-continuous t-norm if and only T and I_T form an adjoint (residual) pair, i.e.,

$$T(x,y) \le z \Longleftrightarrow I_T(x,z) \ge y. \qquad \text{(RP)}$$

Theorem 4 ([2], Theorem 15)**.** *If a function $I\colon [0,1]^2 \to [0,1]$ satisfies* (EP)*,* (OP) *and N_I is a strong negation, then $T\colon [0,1]^2 \to [0,1]$ defined as*

$$T(x,y) = N_I(I(x, N_I(y))), \qquad x, y \in [0,1],$$

is a t-norm. Additionally, T and I satisfy (RP)*.*

5 QL-Operations and QL-Implications

While (S,N)- and R-implications are the generalizations of a material and intuitionistic-logic implications, in this section we deal with yet another popular way of obtaining fuzzy implications - as the generalization of the following implication defined in quantum logic:

$$p \to q \equiv \neg p \vee (p \wedge q).$$

Needless to state, when the truth values are restricted to $\{0,1\}$ its truth table coincides with the classical implication. In this section we deal with the generalization of the above implication.

Definition 10 (cf. [5, 9])**.** A function $I\colon [0,1]^2 \to [0,1]$ is called a QL-operation if there exist a t-norm T, a t-conorm S and a fuzzy negation N such that

$$I(x,y) = S(N(x), T(x,y)), \qquad x, y \in [0,1].$$

If I is generated from the triple (T, S, N), then we will often write $I_{T,S,N}$ instead of I.

Firstly, we investigate some properties of QL-operations. We will see that not all QL-operation are fuzzy implications in the sense of Definition 5. The proof of the following proposition can be obtained in a straightforward manner.

Proposition 3. *If $I_{T,S,N}$ is a QL-operation, then $I_{T,S,N}$ satisfies* (I2)*,* (I3)*,* (NC)*,* (LB)*,* (NP) *and $N_{I_{T,S,N}} = N$.*

Remark 3. $I_{T,S,N}$ does not always satisfy (I1). For example, consider the function $I_{\mathbf{Z}}(x,y) = \max(1-x, \min(x,y))$, called the Zadeh implication in the literature (see [5]). Let $x_1 = 0.7 < 0.8 = x_2$ and $y = 0.9$. Then $I_{\mathbf{Z}}(x_1, y) = 0.7 < 0.8 = I_{\mathbf{Z}}(x_2, y)$ and hence $I_{\mathbf{Z}}$ does not satisfy (I1), but it is a QL-operation obtained from the triple $(T_{\mathbf{M}}, S_{\mathbf{M}}, N_{\mathbf{C}})$. On the other hand, the Kleene-Dienes fuzzy implication $I_{\mathbf{KD}}$ is a QL-operation obtained from the triple $(T_{\mathbf{L}}, S_{\mathbf{L}}, N_{\mathbf{C}})$ (see Table 4).

Therefore the first main problem is connected with the characterization of those QL-operations which satisfy (I1). Unfortunately, only partial results are known in the literature. A characterization of QL-operations satisfying (I1) is given in [9] for some continuous cases.

Lemma 3. *If a QL-operation $I_{T,S,N}$ obtained from a triple (T, S, N) is a fuzzy implication, then the pair (S, N) satisfies the law of excluded middle* (LEM)*.*

That the condition in the above Lemma is only necessary but not sufficient can be seen from the $I_{T,S,N}$ obtained from the triple $(T_{\mathbf{P}}, S_{\mathbf{nM}}, N_{\mathbf{C}})$ which is not a fuzzy implication.

The following results are easy to obtain from Lemma 1.

Proposition 4. *If a fuzzy negation N in a triple (T, S, N) is less than N_S, then the pair (S, N) does not satisfy* (LEM) *and hence the QL-operation $I_{T,S,N}$ is not a fuzzy implication.*

Proposition 5. *A QL-operation $I_{T,S,N}$ obtained from a triple (T, S, N), where S is a negative t-conorm, is a fuzzy implication if and only if $N = N_{\mathbf{D2}}$. Moreover, $I_{T,S,N} = I_{\mathbf{TD}}$, in this case (see Tables 3 and 4).*

Following the terminology used by Mas *et al.* [9], only if the QL-operation $I_{T,S,N}$ is a fuzzy implication we use the term QL-implication.

Table 4: Examples of some QL-implications. (NV = Non-Vanishing; P = Positive)

S	**N**	**T**	$\mathbf{I_{T,S,N}}$
$S_{\mathbf{D}}$	NV	P	$I_{\mathbf{SD}}$
any	$N_{\mathbf{D2}}$	any	$I_{\mathbf{TD}}$
any	any	$T_{\mathbf{M}}$	$I_{\mathbf{TM}}$
$S_{\mathbf{L}}$	$N_{\mathbf{K}}$	$T_{\mathbf{P}}$	$I_{\mathbf{KP}}$
$S_{\mathbf{L}}$	$N_{\mathbf{C}}$	$T_{\mathbf{L}}$	$I_{\mathbf{KD}}$
$S_{\mathbf{L}}$	$N_{\mathbf{C}}$	$T_{\mathbf{M}}$	$I_{\mathbf{LK}}$
$S_{\mathbf{nM}}$	$N_{\mathbf{C}}$	$T_{\mathbf{M}}$	$I_{\mathbf{FD}}$

6 QL-Operations and other Properties

Keeping with our main aim of this note, in this section, we investigate the conditions under which QL-operations satisfy the properties introduced in Section 3.

6.1 QL-Operations and the Exchange Property

Next result follows from Theorem 2.

Theorem 5 (cf. [9] Proposition 8)**.** *For a QL-operation $I_{T,S,N}$ obtained from a triple (T,S,N), where N a continuous negation, the following statements are equivalent:*

(i) $I_{T,S,N}$ *satisfies* (EP).

(ii) $I_{T,S,N}$ *is an (S,N)-implication.*

Remark 4. (i) Theorem 5 also gives a sufficient condition for a QL-operation $I_{T,S,N}$ obtained from a triple (T,S,N) with N a continuous negation to be a fuzzy implication. On the other hand, $I_{\mathbf{SD}}$ and $I_{\mathbf{TD}}$ show that the continuity of N is not necessary for an $I_{T,S,N}$ to satisfy (EP).

(ii) It is interesting to note that the QL-implications $I_{\mathbf{SD}}$ and $I_{\mathbf{TD}}$ are also (S,N)-implications - $I_{\mathbf{TD}}$ is an (S,N)-implication obtained from any t-conorm S and $N = N_{\mathbf{D2}}$, i.e., $I_{\mathbf{TD}} = I_{\mathbf{S},\mathbf{N_{D2}}}$, while $I_{\mathbf{SD}}$ is an (S,N)-implication where $S = S_{\mathbf{D}}$ and N is any non-vanishing negation, i.e., $I_{\mathbf{SD}} = I_{\mathbf{S_D},\mathbf{N}}$.

6.2 QL-Operations and the Identity Principle

We start our investigations with the following result.

Theorem 6 (cf. [10], Theorem 3.2)**.** *For a QL-operation $I_{T,S,N}$ obtained from a triple (T,S,N), where S is a right-continuous t-conorm, the following statements are equivalent:*

(i) $I_{T,S,N}$ *satisfies* (IP).

(ii) $T(x,x) \geq N_S \circ N(x)$*, for all* $x \in [0,1]$.

Proof. (i) $\Longrightarrow$ (ii) If $I_{T,S,N}$ satisfies (IP), then for any $x \in [0,1]$ we have $I_{T,S,N}(x,x) = S(N(x),T(x,x)) = 1$. From Remark 1 (iv) we have that $T(x,x) \geq N_S \circ N(x)$ for all $x \in [0,1]$.
(ii) $\Longrightarrow$ (i) By Definition 3 and by right-continuity of S, for any $x \in [0,1]$, $N_S \circ N(x) = \min A_{N(x)}$. From Remark 1 (iv), $T(x,x) \geq N_S \circ N(x)$ for all $x \in [0,1]$ implies that $T(x,x) \in A_{N(x)}$ and hence $S(N(x),T(x,x)) = 1$, i.e., for any $x \in [0,1]$, $I_{T,S,N}(x,x) = S(N(x),T(x,x)) = 1$, so $I_{T,S,N}$ satisfies (IP). □

Example 1. Let us consider the QL-implication $I_{\mathbf{KP}}$ obtained from the triple $(T_{\mathbf{P}}, S_{\mathbf{L}}, N_{\mathbf{K}})$. Since $N_{S_{\mathbf{L}}}(x) = 1-x$, note also that, $N_S \circ N_{\mathbf{K}}(x) = 1 - N_{\mathbf{K}}(x) = 1-(1-x^2) = x^2$ and hence $T_{\mathbf{P}}(x,x) = N_S \circ N_{\mathbf{K}}(x)$ for all $x \in [0,1]$. It is easy to note that $I_{\mathbf{KP}}$ has (IP).

Observe, that if S is negative, then from Proposition 5 we note that the QL-implications $I_{T,S,N}$ obtained, viz., $I_{\mathbf{TD}}$, satisfy (IP). Also if

- T is any t-norm, S a t-conorm and $N = N_{\mathbf{D2}}$, or
- $T = T_{\mathbf{M}}$, S is a t-conorm and N a fuzzy negation such that they satisfy (LEM), or
- T is a positive t-norm, $S = S_{\mathbf{D}}$ and N a non-vanishing negation,

then a QL-implication $I_{T,S,N}$ obtained from the triple (T,S,N) satisfies (IP).

Remark 5. Trillas and Valverde in [10] require the negation N in the definition of a QL-implication to be strong. Also the t-norm T and t-conorm S are continuous, and are expected to form a De Morgan triple with the negation N. In fact, in Theorem 3.2 of the same work, under these restrictions, condition (ii) of Theorem 6 has been obtained. From their proof, it is clear that the considered T and S are both continuous and Archimedean and hence either they are strict or nilpotent, in which case they show that condition (ii) is not satisfied and hence the claim that "QL-implications never satisfy (IP)". Whereas from the QL-implications $I_{\mathbf{TD}}$ and $I_{\mathbf{KP}}$ we see that $I_{T,S,N}$ does have (IP).

6.3 QL-Operations and the Ordering Property

Proposition 6. *Let S be a t-conorm and the QL-operation $I_{T,S,N}$ be obtained from a triple (T,S,N). If $I_{T,S,N}$ satisfies* (OP)*, then N is strictly decreasing.*

Proof. To see this, if possible, let there exist $x, y \in [0,1]$ such that $x < y$, but $N(x) = N(y)$. By (OP) we have

$$\begin{aligned} I_{T,S,N}(x,y) = 1 &\Longrightarrow S(N(x), T(x,y)) = 1 \\ &\Longrightarrow S(N(y), T(y,x)) = 1 \\ &\Longrightarrow I_{T,S,N}(y,x) = 1 \\ &\Longrightarrow y \le x, \end{aligned}$$

a contradiction. □

Therefore, from the previous sections, it is clear that if S is a negative t-conorm, then the QL-implication $I_{T,S,N}$ obtained from a triple (T,S,N), i.e., $I_{\mathbf{TD}}$, does not have (OP). Further, if we assume that $I_{T,S,N}$ is a QL-implication, then from Proposition 4 we see that $N \ge N_{\mathbf{S}}$, which implies that a t-conorm S should be such that its natural negation N_S should be non-filling. From Definition 3 this can happen only if every $x \in (0,1)$ has a $y \in (0,1)$ such that $S(x,y) = 1$. Noting that a fuzzy implication that satisfies (OP) also satisfies (IP), using also Theorem 6, we summarize the above discussion in the following result.

Theorem 7. *If a QL-implication $I_{T,S,N}$ obtained from a triple (T,S,N) satisfies* (OP)*, then*

(i) $T(x,x) \ge N_S \circ N(x)$ *for all* $x \in [0,1]$*;*

(ii) N *is a strictly decreasing negation;*

(iii) S *is a non-negative t-conorm such that for every $x \in (0,1)$ there exists a $y \in (0,1)$ such that* $S(x,y) = 1$.

That the above conditions are not sufficient can be seen from $I_{\mathbf{SD}}$ (see Tables 3 and 4).

Example 2. The QL-implication $I_{\mathbf{KP}}$ obtained from the triple $(T_{\mathbf{P}}, S_{\mathbf{L}}, N)$ satisfies (OP).

In the case when T is the minimum t-norm $T_{\mathbf{M}}$, we have the following stronger result.

Theorem 8. *For a QL-implication $I_{T,S,N}$ obtained from a triple (T,S,N) that satisfies* (OP) *the following statements are equivalent:*

(i) T *is the minimum t-norm* $T_{\mathbf{M}}$.

(ii) $N_S \circ N = \mathrm{id}_{[0,1]}$.

7 QL-Implications with (S,N)- and R-Implications

In this section we discuss the intersection of the family of QL-implications with R- and (S,N)-implications. We give some sufficient conditions under which a QL-implication becomes an (S,N)-implication (in the case when the considered N is strong we show that some stronger results can be obtained). On the other hand, we determine precisely the conditions on the underlying operations T,S,N for a QL-implication to be an R-implication.

Theorem 9 (cf. [4])**.** *For a function $I\colon [0,1]^2 \to [0,1]$ the following statements are equivalent:*

(i) I is an (S,N)-implication, which satisfies the ordering property (OP).

(ii) I is an S-implication obtained from the strong negation N_S.

Theorem 10 (cf. [6], Section 2.1)**.** *For a fuzzy implication I the following statements are equivalent:*

(i) I is both an R-implication obtained from a left-continuous t-norm T and also an (S,N)-implication generated from a t-conorm S and a fuzzy negation N.

(ii) (a) $N = N_S = N_T$ is strong;

(b) T is the N_S dual of S;

(c) $S(x,y) = I(N(x), y)$ is a right-continuous t-conorm.

Proof. (i) $\Longrightarrow$ (ii) Let T be a left-continuous t-norm, S a t-conorm and N a fuzzy negation. Without any loss of generality assume that $I = I_T = I_{S,N}$.

(a) Since I is an R-implication, from Theorem 3 it satisfies (OP). Since I is also an (S,N)-implication, by Theorem 9 above, we have that $N = N_S$ is strong. Further, $N(x) = N_{I_{S,N}}(x) = I_{S,N}(x,0) = I_T(x,0) = N_T(x)$, for all $x \in [0,1]$.

(b) Since I satisfies (EP), (OP) and N_I is strong, from Theorem 4 and above facts we have $T(x,y) = N_I(I(x, N_I(y))) = N_S(S(N_S(x), N_S(y))$.

(c) It is easy to see that $S(x,y) = I(N(x),y)$ for all $x, y \in [0,1]$. Since N_S is strong and S is the N_S-dual of a left-continuous T, S is right-continuous.

(ii) $\Longrightarrow$ (i) Let $S(x,y) = I(N(x),y)$ be a right-continuous t-conorm such that $N = N_S$ is strong. Since T is the N_S dual of the right-continuous S it is left-continuous. Let I_T be the R-implication obtained from T and I_{S,N_S} be the (S,N)-implication obtained from S and N_S. For any $x, y \in [0,1]$ we have $I_{S,N_S}(x,y) = S(N_S(x), y) = I(N_S(N_S(x)), y) = I(x,y)$. Since I_T satisfies (I1), (EP) and its natural negation $N_{I_T} = N_T$ is a strong negation, by Theorem 2 we see that I_T is an (S,N)-implication, i.e.,

$I_T = I_{S',N'}$ for an appropriate t-conorm S' and a strong N'. But $N' = N = N_S$ and hence $I_T = I_{S',N_S}$. Finally, from the proof of (i) $\Longrightarrow$ (ii) above we know that T is the N_S dual of S' and by our assumption T is the N_S dual of S. Hence $S = S'$, i.e., $I = I_T = I_{S,N_S}$. □

7.1 QL-Implications and (S,N)-Implications

We divide our investigation into two parts, based on whether the considered t-conorm S is negative or not. The following result is obvious from Proposition 5.

Theorem 11. *If S is a negative t-conorm, then the obtained QL-implication $I_{T,S,N} = I_{\mathbf{TD}}$ is an (S,N)-implication.*

Proposition 7. *Let $I_{T,S,N}$ be a QL-implication obtained from a triple (T,S,N) where S is a non-negative t-conorm. If $T = T_{\mathbf{M}}$, then $I_{T,S,N}$ is an (S,N)-implication obtained from the same t-conorm S and the same negation N, i.e., $I_{T,S,N} = I_{S,N}$.*

Proof. If $T = T_{\mathbf{M}}$, then $I_{T,S,N}$ is $I_{\mathbf{TM}}$ (see Table 4). Also, since $I_{T,S,N}$ is a fuzzy implication, we have that the pair (S,N) satisfies (LEM) and hence, by Lemma 1(i), $N \geq N_S$. Now, if $x \leq y$, then

$$I_{T,S,N}(x,y) = S(N(x), T_{\mathbf{M}}(x,y)) = S(N(x),x) = 1,$$

and $I_{S,N}(x,y) = S(N(x),y) \geq S(N_S(x),x) = 1$. On the other hand, if $x > y$, then $I_{T,S,N}(x,y) = S(N(x),y) = I_{S,N}(x,y)$. □

Theorem 12. *Let $I_{T,S,N}$ be the QL-implication obtained from a triple (T,S,N) where S is a non-negative t-conorm such that its natural negation N_S is strong. Consider the following statements:*

- *(i) $I_{T,S,N}$ is an (S,N)-implication obtained from the same S and N, i.e., $I_{T,S,N} = I_{S,N}$.*
- *(ii) $T = T_{\mathbf{M}}$.*
- *(iii) $N = N_S$.*

Then the following relationships exist among the above statements:
(i) and (ii) $\Longrightarrow$ (iii).
(ii) and (iii) $\Longrightarrow$ (i).
(iii) and (i) $\Longrightarrow$ (ii).

Proof. (i) and (ii) $\Longrightarrow$ (iii) Since $T = T_{\mathbf{M}}$, $I_{T,S,N}$ is equal to $I_{\mathbf{TM}}$. If $x \in [0,1]$, then $1 = I_{T,S,N}(x,x) = I_{S,N}(x,x) = S(N(x),x)$, i.e., $S(N(x),x) = 1$. Hence from Lemma 1 (ii) we have $x \geq N_S \circ N(x)$ and by the strongness of N_S we have $N_S(x) \leq N(x)$. On the other hand, by Lemma 1 (ii) again, we have $S(N(x),x) = 1 \Longrightarrow N(x) \geq N_S(x)$. From the above inequalities we find that $N(x) = N_S(x)$, for all $x \in [0,1]$.
(ii) and (iii) $\Longrightarrow$ (i) This follows from Proposition 7.
(iii) and (i) $\Longrightarrow$ (ii) Let $I_{T,S,N_S} = I_{S,N_S}$. We know $T(x,x) \leq x$ for all $x \in [0,1]$. Now for any $x \in (0,1)$, we have $N_S(x) \neq 1$ and

$$\begin{aligned} S(N_S(x), T(x,x)) &= S(N_S(x),x) = 1 \\ &\Longrightarrow T(x,x) \geq N_S \circ N_S(x) = x. \end{aligned}$$

from whence we obtain $T(x,x) = x$ for all $x \in [0,1]$, i.e., $T = T_{\mathbf{M}}$. □

Table 5: Some QL-implications that are also (S,N)-implications. See Remark 6 for more details.

S	**T**	**N**	$\mathbf{N_S}$	$\mathbf{I_{T,S,N}}$
Negative	any	$N_{\mathbf{D2}}$	$N_{\mathbf{D2}}$	$I_{\mathbf{TD}}$
$S_{\mathbf{B}}$	$T_{\mathbf{M}}$	$N_{\mathbf{D2}}$	$N_{S_{\mathbf{B}}}$	$I_{\mathbf{TD}}$
$S_{\mathbf{D}}$	$T_{\mathbf{M}}$	$N_{\mathbf{C}}$	$N_{\mathbf{D1}}$	$I_{\mathbf{S_D},\mathbf{N_C}}$
$S_{\mathbf{L}}$	$T_{\mathbf{L}}$	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$	$I_{\mathbf{KD}}$
$S_{\mathbf{L}}$	$T_{\mathbf{P}}$	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$	$I_{\mathbf{RC}}$
$S_{\mathbf{L}}$	$T_{\mathbf{M}}$	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$	$I_{\mathbf{LK}}$
$S_{\mathbf{nM}}$	$T_{\mathbf{M}}$	$N_{\mathbf{C}}$	$N_{\mathbf{C}}$	$I_{\mathbf{FD}}$

Remark 6. Let us consider a t-conorm S whose natural negation N_S is discontinuous. From Proposition 7 we always have that (ii) $\Longrightarrow$ (i). Let us define a lenient version of (i) as follows:

- **(i')** $I_{T,S,N}$ is an (S,N)-implication obtained from (possibly different) t-conorm S' and negation N', i.e., $I_{T,S,N} = I_{S',N'}$.

Obviously, when $S = S'$, $N = N'$ we have (i') = (i). Then from Table 5 the following observations can be made:

- From the first entry we notice that $N = N_S$ is not strong and $I_{T,S,N} = I_{S,N}$, but $T \neq T_{\mathbf{M}}$, i.e., (iii) & (i) $\not\Longrightarrow$ (ii), if N is not strong.
- From the second and third entries it is clear that even if $I_{T,S,N} = I_{S,N}$ and $T = T_{\mathbf{M}}$ we can have $N_S \neq N$, i.e., (i) & (ii) $\not\Longrightarrow$ (iii), when N is not strong.
- From the fourth and fifth entries it is clear that even if $I_{T,S,N} = I_{S',N'}$ and $N = N' = N_C$ we can have $T \neq T_{\mathbf{M}}$, i.e., (i') & (iii) $\not\Longrightarrow$ (ii).

7.2 QL-Implications and R-Implications

Firstly, if S is a negative t-conorm or if $N = N_{\mathbf{D2}}$, then the QL-implication $I_{T,S,N}$ obtained from the triple (T,S,N) is the R-implication $I_{\mathbf{TD}}$ obtained from the non-left-continuous t-norm $T_{\mathbf{D}}$.

From Theorem 3 we see that an R-implication I_T from a left-continuous t-norm T has both the exchange principle (EP) and the ordering property (OP). Now, if an $I_{T,S,N}$ is also an R-implication obtained from a left-continuous t-norm, then from Proposition 2 we know that $N_I = N$ is either strong or discontinuous. But from Theorem 7 (ii) we have N is strictly decreasing and hence $N_I = N$ is either strong or discontinuous, but strictly decreasing.

In the case when N is strong, since $I_{T,S,N}$ has (EP) we know from Theorem 5 that the $I_{T,S,N}$ is also an S-implication. Hence we have the following result:

Proposition 8. *Let N be a strong negation. If a QL-implication $I_{T,S,N}$ obtained from a triple (T, S, N) is an R-implication obtained from a left-continuous t-norm T, then $I_{T,S,N}$ is also an S-implication.*

The reverse implication of Proposition 8 is not valid. To see this, consider the QL-implication $I_{T,S,N}$ obtained from the triple $(T_{\mathbf{M}}, S_{\mathbf{D}}, N)$, where N is strong and hence is non-vanishing. Then $I_{T,S,N}$ is an S-implication (see Remark 4 (ii)) but not an R-implication since it does not have the ordering property (OP). In fact, this is true even if N is a strict negation.

Theorem 13. *If N is a strong negation, then the following statements are equivalent:*

(i) $I_{T,S,N} = I_{T^*}$, for some left-continuous t-norm T^*.

(ii) $T = T_{\mathbf{M}}$, $N = N_{T^*}$ and S is the right-continuous t-conorm that is the N-dual of T^* with $N = N_S$.

Proof. Let N be a strong negation.
(i) $\Longrightarrow$ (ii) If $I_{T,S,N} = I_{T^*}$, then $I_{T,S,N} = I_{S,N}$ from Proposition 8. Further, since $I_{S,N} = I_{T^*}$, Theorem 10 implies that $N = N_{I_{T^*}} = N_{T^*} = N_S$. Also, S is the right-continuous t-conorm that is the N-dual of T^*. Now, $N = N_S$ is strong and by Theorem 12 we see that $T = T_{\mathbf{M}}$.
(ii) $\Longrightarrow$ (i) On the other hand, let $T = T_{\mathbf{M}}$ and S be a right-continuous t-conorm with $N = N_S$. By virtue of Proposition 7 we get $I_{T,S,N} = I_{S,N}$. Let T^* be the N-dual t-norm of the right-continuous t-conorm S with $N = N_S$, in which case T^* is left-continuous. Now, from Theorem 10 we have that $I_{S,N} = I_{T^*}$ and hence $I_{T,S,N} = I_{T^*}$. □

The QL-implication $I_{\mathbf{KP}}$ obtained from the triple $(T_{\mathbf{P}}, S_{\mathbf{L}}, N_{\mathbf{K}})$ as given in Example 1 is a fuzzy implication that is neither an (S, N)-implication nor an R-implication obtained from a left-continuous t-norm, since it does not have (EP).

8 Conclusion

In this note the intersections that exist between the families of (S, N)-, R- and QL-implications are determined using existing characterization results. As part of this attempt some interesting results pertaining to natural negations from t-conorms, properties of QL-implications and the exact intersection between (S, N)- and R-implications have been obtained (Theorem 10). From this work the following problems arise.

Problem 1. (i) Characterize a triple (T, S, N) such that $I_{T,S,N}$ satisfies (I1). It should be noted that a characterization is already known for some continuous cases (see [9]).

(ii) Prove or give a counter example: Any QL-operation that satisfies (EP) is an (S, N)-implication.

(iii) Give an equivalent condition for an $I_{T,S,N}$ to satisfy the ordering property (OP).

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